THE SUPER-GENERALISED FERMAT EQUATION $Pa^x + Qb^y = Rc^z$ AND FIVE RELATED PROOFS

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ABSTRACT. In this paper, we consider five proofs related to the super-generalised Fermat equation, $Pa^x + Qb^y = Rc^z$. All proofs depend on a new identity for $a^x + b^y$ which can be expressed as a binomial sum to an indeterminate power, z. We begin with the Generalised Fermat Conjecture, for the case $P, Q, R = 1$, also known as the Tijdeman-Zagier Conjecture and Beal Conjecture. We then show how the method applies to its famous corollary Fermat's Last Theorem, where $x, y, z = n$. We then return to the title equation, considered by Henri Darmon and Andrew Granville and extend the proof for the case $P, Q, R > 1$ and $x, y, z > 2$. Finally, we use the results to prove Catalan's Conjecture, and from this a weak proof that under certain conditions only one solution exists for equations of the form $a^4 - c^2 = b^y$.

Introduction

In this paper, we consider the generalised Fermat equation, $Pa^x + Qb^y = Rc^z$ and propose five related proofs where $gcd(a, b, c) = 1$. All proofs depend on a new identity for $a^x + b^y$ which can be expressed as a binomial sum to an indeterminate power, z.

First, we begin with the case for $P, Q, R = 1$ where $a^x + b^y = c^z$. This forms the basis for the Generalised Fermat Conjecture (GFC)[1], also known as the Tijdeman-Zagier Conjecture and Beal Conjecture[2]. Closely related to this is the Fermat-Catalan Conjecture, which states that when $1/x + 1y + 1/z < 1$, this equation has only finite solutions. Only 10 solutions are known, namely:

$$1 + 2^3 = 3^2,$$
$$2^5 + 7^2 = 3^4,$$
$$7^3 + 13^2 = 2^9,$$
$$2^7 + 17^3 = 71^2,$$
$$3^5 + 11^4 = 122^2,$$
$$17^7 + 76271^3 = 21063928^2,$$
$$1414^3 + 2213459^2 = 65^7$$

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[1] Samuele Anni and Samir Siksek, *Modular elliptic curves over real abelian fields and the generalized Fermat equation* $x^{2\ell} + y^{2m} = z^p$, Algebra & Number Theory, 10 (6), (2016), 1147-1172.

[2] See www.bealconjecture.com. Last accessed 14.12.17.

$$9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2,$$

$$33^8 + 1549034^2 = 15613^3.$$

Here, however we prove GFC which states that no integer solutions exist for values of $x, y, z > 2$, where x, y, z are square-free integers, and $gcd(x, y, z) = 1$.

Secondly, we apply the same method to demonstrate the famous corollary of this conjecture, Fermat's Last Theorem (FLT), for the case $x, y, z = n$. This more simply states that for the equation $a^n + b^n = c^n$, where $gcd(a, b, c) = 1$, no integer solutions exist for the values of $n > 2$. A proof for FLT was first discovered by Sir Andrew Wiles in 1993. Here, we give just a brief recapitulation of the proof showing only the salient points.

Thirdly, we return to consider whether our method extends to the title equation, $Pa^x + Qb^y = Rc^z$, where $P, Q, R \in \mathbb{Z}_{>1}$. In 1994, using Faltings Theorem, Henri Darmon and Andrew Granville proved that where a, b, c, P, Q, R are non-zero square-free integers, $\gcd(a, b, c, P, Q, R) = 1$, for the hyperbolic case $1/x+1y+1/z < 1$, there are only finitely many integral solutions. However, we wish to go one step further and use our method to prove that no integer solutions exist for the values $x, y, z > 2$ and $gcd(x, y, z) = 1$. We call this the Super-Generalised Fermat Conjecture (SGFC).

Fourthly, we use the results of GFC to prove Catalan's Conjecture (CC). This conjecture was first made by Belgian mathematician Eugne Charles Catalan in 1844, which states that the only solution in the natural numbers of $a^x + 1^y = c^z$ for $a, c > 0$, $x, z, y > 1$ is $a = 2$, $x = 3$, $c = 3$, $z = 2$. In other words, Catalan conjectured that $2^3 + 1 = 3^2$ is the only nontrivial solution. It was finally proved in 2002 by number theorist Preda Mihailescu making extensive use of the theory of cyclotomic fields and Galois modules.

Fifthly, using the results of CC, we give a weak proof that, when $(a^2 + c)$ and $(a^2 - c)$ are divisible by b, the only solution that exists for the case $a^4 - c^2 = b^y$ is $3^4 - 7^2 = 2^5$. This is one of the 10 known solutions of the Fermat-Catalan Conjecture mentioned above.

Theorem 0.1. *Generalised Fermat Conjecture. To prove that the equation $a^x + b^y = c^z$, in $a, b, c \in \mathbb{Z}$ with $gcd(a, b, c) = 1$, in $x, y, z \in \mathbb{Z}_{\geq 3}$, has no solutions.*

We first observe the following identity for $a^x + b^y$ as a binomial expansion (where the upper index n is an indeterminate integer):

$$a^x + b^y = \sum_{k=0}^{n} \binom{n}{k} (a + b)^{n-k} (-ab)^k (a^{x-n-k} + b^{y-n-k}). \tag{0.1}$$

Note how this new identity includes *standard* factors for a binomial expansion, i.e. $(a + b)^{n-k}(-ab)^k$, but also a *non-standard* factor, i.e. $(a^{x-n-k} + b^{y-n-k})$.

Note, further, that regardless of the value of n, the right hand side always equals $a^x + b^y$. This allows us to fix n to any value we choose. So let $n = z$, such that:

$$a^x + b^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} + b^{y-z-k}). \tag{0.2}$$

Proof. We now assume that a solution exists for the equation $a^x + b^y = c^z$ for values of $x, y, z > 2$.

Now let $[(a + b)(\pm s) - ab(\pm t)] = c$ for all $s, t \in \mathbb{Z}$ (although $gcd(s, ab) = 1$ since a shared factor would mean that c would no longer be coprime with ab), such that:

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} + b^{y-z-k}) = [(a+b)(\pm s) - ab(\pm t)]^z. \tag{0.3}$$

We now rearrange c^z as $[(a + b)(\pm s) - (ab)(\pm t)]^z$ and use the binomial theorem to expand (0.3) as:

(0.4)

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k}+b^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(\pm s)^{z-k}(\pm t)^k.$$

Comment: We know that the right hand side is a perfect power (it has the correct exponential form for a standard binomial expansion of a single power); the left hand side may or may not be. The question now becomes how can the left hand side be a perfect power greater than 2? Answer: only if $(a^{x-z-k} + b^{y-z-k}) = (\pm s)^{z-k}(\pm t)^k$ for any given value of $z > 2$ and for every value of k (from $k = 0$ to $k = z$). If so, then solutions will exist and GFC will be proved false. But if just one term (i.e. one value of k) exists where $(a^{x-z-k} + b^{y-z-k}) \neq (\pm s)^{z-k}(\pm t)^k$ then not only will the integrity of that particular term be compromised as a standard binomial term for a power to z, but also the whole expansion of $a^x - b^y$, and GFC will be true.

For example, if $(a^{x-z-k} + b^{y-z-k}) = f^{2z+k}g^{k+1}$, say, then since f^{2z+k} and g^{k+1} do not have the correct standard exponents for a perfect power, no solutions could exist for (0.4). Essentially, therefore, we are proving that $(a^{x-z-k} + b^{y-z-k})$ can never have the correct exponents for a perfect power when $x, y, z > 2$.

Therefore, continuing our assumption, we can simplify the problem by stating that for any value of $z > 2$ *and* for every value of k (from 0 to z):

$$(\pm s)^{z-k}(\pm t)^k = (a^{x-z-k} + b^{y-z-k}). \tag{0.5}$$

At this point, we can derive a contradiction from just the second and penultimate terms for any given value of z. This means that however large z becomes, we need not test every term *ad infinitum.*

First, we calculate the second term *directly* from (0.5). So when $k = 1$, the second term is:

$$(\pm s)^{z-1}(\pm t) = \pm(a^{x-z-1} + b^{y-z-1}). \tag{0.6}$$

We can also deduce the second term *indirectly* by manipulating the first term and the last term for any given z. Thus the first term, when $k = 0$, will be $(\pm s)^z =$

$\pm(a^{x-z}+b^{y-z})$, and the last term, when $k=z$, will be $(\pm t)^z = \pm(a^{x-2z}+b^{y-2z})$. Now we can raise the powers accordingly and multiply together to get:

$$(\pm s)^{z-1}(\pm t) = \pm[(a^{x-z}+b^{y-z})^{1/z}]^{(z-1)}[(a^{x-2z}+b^{y-2z})^{1/z}]. \tag{0.7}$$

Now we subtract (0.6) from (0.7) and rearrange to get:

$$(a^{x-z-1}+b^{y-z-1}) = [(a^{x-z}+b^{y-z})^{1/z}]^{(z-1)}[(a^{x-2z}+b^{y-2z})^{1/z}]. \tag{0.8}$$

Now we raise both sides by z and divide both sides by $(a^{x-z}+b^{y-z})^{(z-2)}$ and rearrange to get:

$$\frac{(a^{x-z-1}+b^{y-z-1})^z}{(a^{x-z}+b^{y-z})^{(z-2)}} = (a^{x-z}+b^{y-z})(a^{x-2z}+b^{y-2z}). \tag{0.9}$$

The procedure for the penultimate term is exactly the same. First, *directly* from (0.5), when $k=z-1$, the penultimate term is:

$$(\pm s)(\pm t)^{z-1} = \pm(a^{x-2z+1}+b^{y-2z+1}). \tag{0.10}$$

And again *indirectly* from the first and last terms, we raise the powers accordingly and multiply together to get the penultimate term:

$$(\pm s)(\pm t)^{z-1} = \pm[(a^{x-z}+b^{y-z})^{1/z}][(a^{x-2z}+b^{y-2z})^{1/z}]^{(z-1)}. \tag{0.11}$$

Now we subtract (0.10) from (0.11) and rearrange to get:

$$(a^{x-2z+1}+b^{y-2z+1}) = (a^{x-z}+b^{y-z})^{1/z}(a^{x-2z}+b^{y-2z})^{1/z}]^{(z-1)}. \tag{0.12}$$

This time, we raise both sides by z and divide both sides of by $(a^{x-2z}+b^{y-2z})^{(z-2)}$ and rearrange to get:

$$\frac{(a^{x-2z+1}+b^{y-2z+1})^z}{(a^{x-2z}+b^{y-2z})^{(z-2)}} = (a^{x-z}+b^{y-z})(a^{x-2z}+b^{y-2z}). \tag{0.13}$$

But now observe that in (0.9) and (0.13) the right hand sides are exactly the same. This means we can subtract (0.9) from (0.13) and rearrange to get:

$$\left(\frac{a^{x-z-1}+b^{y-z-1}}{a^{x-2z+1}+b^{y-2z+1}}\right)^z = \left(\frac{a^{x-z}+b^{y-z}}{a^{x-2z}+b^{y-2z}}\right)^{(z-2)}. \tag{0.14}$$

Solutions will exist to this equation
a) *either* if the large bracketed fractions on each side of have a value of 1 (since the outer exponents are not equal),
b) *or* if the numerators (to their respective outer exponents) on both sides are equal, *and* simultaneously if the denominators (to their respective outer exponents) on both sides are equal.

Taking these two options in turn (still when $x, y, z > 2$):

a) since $z \neq 2z$, $(a^{x-z-1}+b^{y-z-1}) \neq (a^{x-2z+1}+b^{y-2z+1})$ and $(a^{x-z}+b^{y-z}) \neq (a^{x-2z}+b^{y-2z})$. So neither side in (0.14) has a value of 1, eliminating this option;

b) beginning with denominators, even without its outer exponent the left hand denominator $(a^{x-2z+1}+b^{y-2z+1})$ is greater than its right hand counterpart $(a^{x-2z}+b^{y-2z})$; but when the outer exponent is also greater, (i.e. $z > (z-2)$), then the inequality is even greater. So it follows that $(a^{x-2z+1} + b^{y-2z+1})^z \neq (a^{x-2z} + b^{y-2z})^{(z-2)}$. We do not even need to bother with the numerators.

Having now eliminated both options it follows that, for all values of $x, y, z > 2$ and all values of k:

$$(\pm s)^{z-k}(\pm t)^k \neq (a^{x-z-k} + b^{y-z-k}). \tag{0.15}$$

However, this contradicts our equation in (0.5). In turn, therefore, the left hand side of the equation in (0.4) cannot be a perfect power (as we assumed it was). And so our initial assumption that solutions exist for the equation $c^z = a^x + b^y$ for values of $x, y, z > 2$ is false. Therefore GFC is true. □

However, it leaves us with an important final question. What happens for the cases for $z = 1, 2$? Well, from (0.14), when $z = 1$ it follows that:

$$\left(\frac{a^{x-2} + b^{y-2}}{a^{x-1} + b^{y-1}}\right)^1 = \left(\frac{a^{x-1} + b^{y-1}}{a^{x-2} + b^{y-2}}\right)^{-1}, \tag{0.16}$$

$$\Rightarrow \left(\frac{a^{x-2} + b^{y-2}}{a^{x-1} + b^{y-1}}\right) = \left(\frac{a^{x-2} + b^{y-2}}{a^{x-1} + b^{y-1}}\right). \tag{0.17}$$

No contradiction.

And again from (0.14), when $z = 2$, it follows that:

$$\left(\frac{a^{x-3} + b^{y-3}}{a^{x-3} + b^{y-3}}\right)^2 = \left(\frac{a^{x-2} + b^{y-2}}{a^{x-4} + b^{y-4}}\right)^0, \tag{0.18}$$

$$\Rightarrow 1 = 1. \tag{0.19}$$

Again, no contradiction.

So in both cases, when $z = 1$ and when $z = 2$, there is no contradiction. Our non-standard binomial factor, $(a^{x-z-k}+b^{y-z-k})$ is equal to $(\pm s)^{z-k}(\pm t)^k$ for every value of k, which, in turns, means that solutions to both Fermat equations exist, a fact already known.

0.1. **Comment.** It is worth noting that when $z = 1$ the two fractions themselves in (0.17) are equal. This suggests that there is *always* a solution for the sum of two powers. But when $z = 2$, the two fractions in (0.18) are not equal. The equality only exists by virtue of the zero exponent. This suggests that the sum (or difference) of two powers may be a perfect square (as in the case $3^5 + 11^4 = 122^2$), but may not be (e.g. FRTT where $a^4 - b^4 \neq c^2$). Indeed, when we apply this method directly to FRTT, it produces a false positive. So this method is limited. When $z = 2$ we cannot directly prove that solutions do *not* exist. We can only indirectly prove whether solutions exist or not, and even then only under certain conditions.

Theorem 0.2. *Fermat's Last Theorem. To prove that, for the equation $a^n + b^n = c^n$, for all $a, b, c, n \in \mathbb{Z}$ and where $gcd(a, b, c) = 1$, integer solutions only exist for the values of $n = 1, 2$, but not for values of $n > 2$.*

Proof. FLT is a corollary of GFC for cases $x, y, z = n$. So let $x, y, z = n$. And without retracing each step, here we will simply outline the proof, highlighting some of the key equations of GFC. So, parallel to (0.4), using our new binomial identity the equation $a^n + b^n = c^n$ can be reconfigured as:
(0.20)

$$\sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k(a^{-k}+b^{-k}) = \sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k(\pm s)^{n-k}(\pm t)^k.$$

Parallel to (0.5), the problem can be simplified to this equation:

$$(\pm s)^{n-k}(\pm t)^k = (a^{-k}+b^{-k}). \tag{0.21}$$

We assume that this equation has solutions for some value of $n > 2$ and for every value of k for any given value of n. From the second and penultimate terms, we reach the following equation, parallel to (0.14):

$$\left(\frac{a^{-1}+b^{-1}}{a^{-n+1}+b^{-n+1}}\right)^n = \left(\frac{a^0+b^0}{a^{-n}+b^{-n}}\right)^{(n-2)}. \tag{0.22}$$

As before, solutions to this equation will only exist:
a) *either* if the large bracketed fractions on each side of have a value of 1 (since the outer exponents are not equal),
b) *or* if the numerators (to their respective outer exponents) on both sides are equal, *and* simultaneously if the denominators (to their respective outer exponents) on both sides are equal.
Taking these two options in turn (still when $n > 2$):

a) as before, $(a^{-1}+b^{-1}) \neq (a^{-n+1}+b^{-n+1})$ and $(a^0+b^0) \neq (a^{-n}+b^{-n})$. So neither fraction in (0.22) has a value of 1, eliminating this option;

b) beginning with denominators, even without its outer exponent the left hand denominator $(a^{-n+1}+b^{-n+1})$ is greater than its right hand counterpart $(a^{-n}+b^{-n})$; but when the outer exponent is also greater, (i.e. $n > (n-2)$), then the inequality is even greater. So it follows that $(a^{-n+1}+b^{-n+1})^n \neq (a^{-n}+b^{-n})^{(n-2)}$. We do not even need to bother with the numerators.

Having now eliminated both options it follows that, for all values of $n > 2$ and all values of k:

$$(\pm s)^{n-k}(\pm t)^k \neq (a^{-k}+b^{-k}). \tag{0.23}$$

However, this contradicts the equation in (0.21). In turn, therefore, the left hand side of the equation in (0.20) cannot be a perfect power greater than 2 (as we assumed it was). And so our initial assumption that solutions exist for the equation $c^n = a^n + b^n$ for values of $n > 2$ is false. Therefore FLT is true.

□

Again, we want to see what happens when $n = 1, 2$? Well, from (0.22), when $n = 1$ it follows that:

$$\left(\frac{a^{-1}+b^{-1}}{a^0+b^0}\right)^1 = \left(\frac{a^0+b^0}{a^{-1}+b^{-1}}\right)^{-1}. \tag{0.24}$$

$$\Rightarrow \left(\frac{a^{-1}+b^{-1}}{a^0+b^0}\right)^1 = \left(\frac{a^{-1}+b^{-1}}{a^0+b^0}\right)^1. \tag{0.25}$$

No contradiction.

And again from (0.22), when $n = 2$, it follows that:

$$\left(\frac{a^{-1}+b^{-1}}{a^{-1}+b^{-1}}\right)^2 = \left(\frac{a^0+b^0}{a^{-2}+b^{-2}}\right)^0, \tag{0.26}$$

$$\Rightarrow 1 = 1. \tag{0.27}$$

Again, as expected, no contradiction.

We now extend the proof to SFGC.

Theorem 0.3. *The Super-Generalised Fermat Conjecture. To demonstrate that for the Fermat equation $Pa^x + Qb^y = Rc^z$, where a, b, c, P, Q, R are square-free positive integers (of which one of Pa, Qb, Rc at most must be even), and* $\gcd(a, b, c, P, Q, R) = 1$*, no integer solutions exist for the values of $x, y, z > 2$.*

Proof. We note first that,

$$Pa^x + Qb^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(Pa^{x-z-k} + Qb^{y-z-k}).$$

Then, parallel to (0.4):

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(Pa^{x-z-k}+Qb^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} R(\pm s)^{z-k}(\pm t)^k. \tag{0.28}$$

And assuming that $Pa^x + Qb^y = Rc^z$ has solutions, the problem can again be reduced to the following equation, parallel to (0.5):

$$(Pa^{x-z-k} + Qb^{y-z-k}) = R(\pm s)^{z-k}(\pm t)^k \tag{0.29}$$

for *any* value of $z > 2$ and for *every* value of k for any given value of z. From the second and penultimate terms, as we did in Theorems (0.1) and (0.2), we reach the following equation, parallel to (0.14), noting that R has been cancelled out:

$$\left(\frac{Pa^{x-z-1}+Qb^{y-z-1}}{Pa^{x-2z+1}+Qb^{y-2z+1}}\right)^z = \left(\frac{Pa^{x-z}+Qb^{y-z}}{Pa^{x-2z}+Qb^{y-2z}}\right)^{(z-2)}. \tag{0.30}$$

As before, solutions will exist to this equation
a) *either* if the large bracketed fractions on each side of have a value of 1 (since the outer exponents are not equal), b) *or* if the numerators (to their respective outer exponents) on both sides are equal, *and* simultaneously if the denominators (to their respective outer exponents) on both sides are equal. Taking these two options in turn (still when $x, y, z > 2$):

a) Since $z \neq 2z$, $Pa^{x-z-1} + Qb^{y-z-1} \neq Pa^{x-2z+1} + Qb^{y-2z+1}$, eliminating this option.

b) beginning with denominators, even without its outer exponent the left hand denominator $(Pa^{x-2z+1} + Qb^{y-2z+1})$ is greater than its right hand counterpart $(Pa^{x-2z} + Qb^{y-2z})$; but when the outer exponent is also greater, (i.e. $z > (z-2)$), then the inequality is even greater. So it follows that $(Pa^{x-2z+1} + Qb^{y-2z+1})^z \neq (Pa^{x-2z} + Qb^{y-2z})^{(z-2)}$. We do not even need to bother with the numerators.
Having now eliminated both options it follows that, for all values of $x, y, z > 2$ and all values of k:

$$(\pm s)^{z-k}(\pm t)^k \neq (Pa^{x-z-k} + Qb^{y-z-k}). \tag{0.31}$$

However, this contradicts our equation in (0.29). In turn, therefore, the left hand side of the equation in (0.28) cannot be a perfect power (as we assumed it was). And so our initial assumption that solutions exist for the equation $c^z = Pa^x + Qb^y$ for values of $x, y, z > 2$ is false. Therefore the DG is true. □

However, it leaves us with an important final question. What happens for the cases for $z = 1, 2$? Well, from (0.30), when $z = 1$ it follows that:

$$\left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right)^1 = \left(\frac{Pa^{x-1} + Qb^{y-1}}{Pa^{x-2} + Qb^{y-2}}\right)^{-1}, \tag{0.32}$$

$$\Rightarrow \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right) = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right). \tag{0.33}$$

No contradiction.
And again from (0.30), when $z = 2$, it follows that:

$$\left(\frac{Pa^{x-3} + Qb^{y-3}}{Pa^{x-3} + Qb^{y-3}}\right)^2 = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-4} + Qb^{y-4}}\right)^0, \tag{0.34}$$

$$\Rightarrow 1 = 1. \tag{0.35}$$

Again, no contradiction. Thus solutions will always exist for $z = 1$ and solutions may exist for particular cases when $z = 2$.

Theorem 0.4. *Catalan's Conjecture. To prove that the only solution for the equation $Pa^x + Qb^y = Rc^z$, in $a, b, c, P, Q, R \in \mathbb{Z}$ with $gcd(a, b, c) = 1$, where $P, Q, R, b = 1$, $x, y, z \in \mathbb{Z}_{>1}$, is $a = 2$, $x = 3$, $c = 3$, $z = 2$.*

Proof. So for the equation, $a^x + 1^y = c^z$, let $y = 7$, say, and rearrange such that:

$$a^x = c^z - 1^7. \tag{0.36}$$

Finding the value of z.
In GFC we concluded that there can be no solutions for $z > 2$, and since $z > 1$, it follows that $z = 2$.

Finding the value of a.
Since $z = 2$, it follows that:

$$a^x = c^2 - 1. \tag{0.37}$$

$$\Rightarrow a^x = (c+1)(c-1). \tag{0.38}$$

This gives us two options.

1) If $(c+1)$ and $(c-1)$ are *not* divisible by a, they must themselves be powers to x. So let $(c+1) = p^x$ and $(c-1) = q^x$ for all integers p, q $(p > q)$. But since the difference between two identical powers is always greater than 2, and since the difference between $(c+1)$ and $(c-1)$ is only 2, they cannot be themselves be powers to x. Therefore they must be divisible by a.

2) If $(c+1)$ and $(c-1)$ *are* divisible by a, then let $(c+1) = a.j$ and $(c-1) = a.k$ for integers j, k $(j > k, j \neq k)$. We subtract one from the other, such that:

$$(c+1) - (c-1) = a(j-k) = 2. \tag{0.39}$$

From this, either $a = 1$ and $(j-k) = 2$, or $a = 2$ and $(j-k) = 1$. But since $a > 1$, it follows that $a = 2$ and $(j-k) = 1$.

Finding the value of x.
If we multiply aj by ak, we get:

$$a^2(jk) = c^2 - 1. \tag{0.40}$$

But since $c^2 - 1 = a^x$ from our original equation, it follows that:

$$a^2(jk) = a^x, \tag{0.41}$$

$$\Rightarrow jk = a^{x-2}. \tag{0.42}$$

Since $a = 2$, it follows that:

$$jk = 2^{x-2}. \tag{0.43}$$

And since $j - k = 1$, j and k must have opposite polarity. And since their product is a pure power of 2, the odd variable must have a value of 1 and the even variable a value of 2. And since $j > k$, it follows that $j = 2, k = 1$, such that from (0.43):

$$2 = 2^{x-2}, \tag{0.44}$$

$$\Rightarrow x = 3. \tag{0.45}$$

Finding the value of c.
Now since $(c+1) = a.j$ it follows that:

$$(c+1) = 2 \times 2, \tag{0.46}$$

$$\Rightarrow c = 3. \tag{0.47}$$

We now know that $z = 2$, $a = 2, x = 3, c = 3$, from which we can conclude that the only solution for the equation $a^x + 1 = c^z$ is:

$$2^3 + 1 = 3^2. \tag{0.48}$$

Thus CC is true. □

Theorem 0.5. *To prove that when (a^2+c) and (a^2-c) are divisible by b, the only solution that exists for the case $a^4 - c^2 = b^y$ is $3^4 - 7^2 = 2^5$.*

Proof. We begin with the equation:

$$a^4 - c^2 = b^y. \tag{0.49}$$

Finding the value of b.
From (0.49) it follows that:

$$(a^2 + c)(a^2 - c) = b^y. \tag{0.50}$$

Assuming that $(a^2 + c)$ and $(a^2 - c)$ are divisible by b, then let $(a^2 + c) = b.j$ and $(a^2 - c) = b.k$ for integers j, k $(j > k, j \neq k)$. We subtract one from the other, such that:

$$(a^2 + c) - (a^2 - c) = b(j - k), \tag{0.51}$$

$$\Rightarrow 2c = b(j - k). \tag{0.52}$$

Since $gcd(b, c) = 1$ it follows that $c = (j - k)$ and $b = 2$.

Finding the values of a.
Since $b = 2$, it follows that $a^2 + c = 2j$ and $a^2 - c = 2k$. So multiplying these together, we get:

$$a^4 - c^2 = 4jk. \tag{0.53}$$

But since $a^4 - c^2 = b^y$ from our original equation, it follows that:

$$4jk = 2^y, \tag{0.54}$$

$$\Rightarrow jk = 2^{y-2}. \tag{0.55}$$

Now since b is even, it follows that a and c must both be odd. And since $j - k = c$ it follows that j and k must have opposite polarity. But from (0.55) since jk is a pure power of 2, and since $j > k$, then $k = 1$.

But since $j + k = a^2$, it follows that $a^2 - j = 1$. But we also know, from CC, that the only case of two consecutive powers is $3^2 - 2^3 = 1$. Therefore $j = 2^3$ and $a = 3$.

Finding the values of y.
Now if $j = 2^3$, then $jk = 8$. Therefore from (0.55) $y = 5$.

Finding the values of c.
From (0.49) we can now say that

$$3^4 - c^2 = 2^5, \tag{0.56}$$

$$\Rightarrow c^2 = 81 - 32 = 49, \tag{0.57}$$

$$\Rightarrow c = 7. \tag{0.58}$$

We have now shown that the only solution that exists for the case $a^4 - c^2 = b^y$ is $3^4 - 7^2 = 5^y$. $\square$

Unfortunately, this weak proof does not account for circumstances when (a^2+c) and (a^2-c) are *not* divisible by b. For example, when $(a^2-c) = 1$, then $3^5 = 122^2 - 11^4$. If we could prove that there are also no solutions when (a^2+c) and (a^2-c) are *not* divisible by b, then we have also proved Fermat's Right Triangle Theorem. In his life, Pierre de Fermat only left one proof in relation to number theory. He used his method of infinite descent to show that the area of a right triangle cannot be a square within the domain of whole numbers. The proof itself was found after his death in his notes on Diophantus' Arithmetica. He wrote: "If the area of a right-angled triangle were a square, there would exist two biquadrates the difference of which would be a square number." (A biquadrate is a value to the fourth-power. So, the biquadrate of 3 is $3^4 = 81$.). This is equivalent to stating that there are no solutions to the equation $p^4 - q^4 = z^2$.

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