Additive-multiplicative Functions and the Reimann Zita Function

Mohammed Zohal

Abstract

This article [studies] the sum ∑_d/n f(d), with f [being] an arithmetical function which not only multiplicative, but, of another form which will be additive_multiplicative, you will see what is it about, in fact, it has generated incredible formulas, which can give insight into the random arithmetic world of numbers.

Table of contents

V. References

I. Notations .

n=$\prod_{i=1}^{\omega(n)}$ p$^{\alpha_p}$

$\mathcal{P}$: the set of primes

$d \wedge n =$PGCD(d,n)

$\mu\,(n)$ is the Mobius function

$\omega(n)$=$\sum_{p/n,p\epsilon\mathcal{P}} 1$ the number of distinct primes factors of n;

Ω(n)=$\sum_{p/n,,p\epsilon\mathcal{P}} \alpha_p$ the number of prime power factors of n;

τ(n)=$\sum_{d/n} 1$ the number of divisors of n;

σ(n)=$\sum_{d/n}$ d the sum of the divisors of n

φ(n)=$\sum_{d\wedge n,d\leq n} 1$=n$\sum_{d/n} \frac{\mu(d)}{d}$ the Euler totient function;

$\lambda(n)$=(-1)$^{\Omega(n)}$ the liouville function

2.Definitions:

we say that a function g is additive_multiplicative iff g=h×$\prod_{i=1}^{n} f_i$

such that f$_i$ is additive for every i in $\{1,2,......n\}$, with h is multiplicative

3. Lemma 1.

let a,b $\epsilon\mathcal{N}^{\star 2}$ and a∧b=1 and $d/$ab ⇒ ∃ e,e'$\epsilon\mathcal{N}^{\star 2}$ such that e∧e'=1 and $e/a, e'/b$ and d=ee'

II. theoreme 1:

let g be a additive_multiplicative function such that g=f×h.

f and h are respectively additive multiplicative

let G(n)=$\sum_{d/n}$ g(d) and n=$\prod_{i=1}^{w(n)} p_i^{\alpha_i}$

then G(n) = $\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})$H($\frac{n}{p_i^{\alpha_i}}$)

and if H($p_i^{\alpha_i}$) ≠0 for every i in $[1,w(n)]$ then G(n)=H(n)$\sum_{i=1}^{w(n)} \frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

with $\quad H(n)=\sum_{d/n} h(d)$

proof :

let $a \wedge b = 1$ and $H(n)=\sum_{d/n} h(d)$ and $G(n)=\sum_{d/n} g(d)$

with h is multiplicaive and g additve

from lemma1 we have $G(ab)=\sum_{d/ab} f(d)\times h(d)$

$$= \sum_{e/a, e'/b} f(ee')\times h(ee')$$

$$= \sum_{e/a, e'/b} (f(e)+f(e'))\times h(e)h(e')$$

$$=\sum_{e/a}\sum_{e'/b} (f(e)h(e)h(e')+f(e')h(e)h(e'))$$

$$=\sum_{e/a} f(e)h(e)\sum_{e'/b} h(e')+\sum_{e/a} h(e)\sum_{e'/b} f(e')h(e')$$

$$= \quad G(a)H(b)+G(b)H(a)$$

then $G(ab) = G(a)H(b)+G(b)H(a)$ if $a \wedge b = 1$. ($\star$)

let us now complet the proof by reccurence .

1. if $w(n)=1$ then $n= p^{\alpha}$ and $G(p^{\alpha})=G(p^{\alpha})H(1)=G(p^{\alpha})H(\frac{p^{\alpha}}{p^{\alpha}})$

(H(1)=1 it comes from the fuct that if h is multiplicative then
$H(n)=\sum_{d/n} h(d)$ is also multiplicative)

2. let $w(n)>1$, we suppose that $G(n) = \sum_{i=1}^{w(n)} G(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}})$ is true for w(n)

and let prove that it still true for $w'(n) = w(n) + 1$

we set $n =\prod_{i=1}^{w'(n)} p_i^{\alpha_i}$ then $n=\prod_{i=1}^{w(n)} p_i^{\alpha_i} \times p_{w'(n)}^{\alpha_{w'(n)}}$

we set $a=\prod_{i=1}^{w(n)} p_i^{\alpha_i}$ and $b=p_{w'(n)}^{\alpha_{w'(n)}}$ then n=ab from ($\star$) we have $G(n=\text{ab})=\text{G(a)H(b)+G(b)H(a)}$ because $\text{a}\wedge b=1$

$$G(n)=\text{G}(\text{b}=p_{w'(n)}^{\alpha_{w'(n)}})\text{H}(\text{a}=\prod_{i=1}^{w(n)} p_i^{\alpha_i})+\text{G}(\text{a}=\prod_{i=1}^{w(n)} p_i^{\alpha_i})\text{H}(\text{b}=p_{w'(n)}^{\alpha_{w'(n)}})$$

$$=\text{G}(p_{w'(n)}^{\alpha_{w'(n)}})\text{H}(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})+(\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})\text{H}(\frac{a}{p_i^{\alpha_i}}))\text{H}(p_{w'(n)}^{\alpha_{w'(n)}})$$

$$=\text{G}(p_{w'(n)}^{\alpha_{w'(n)}})\text{H}(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})+\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})\text{H}(\frac{a\times p_{w'(n)}^{\alpha_{w'(n)}}}{p_i^{\alpha_i}})$$

$$=\text{G}(p_{w'(n)}^{\alpha_{w'(n)}})\text{H}(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})+\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})\text{H}(\frac{n}{p_i^{\alpha_i}})$$

$$=\sum_{i=1}^{w'(n)} G(p_i^{\alpha_i})\text{H}(\frac{n}{p_i^{\alpha_i}})$$

then for every $w(n)\epsilon\mathcal{N}^{\star 2}$ $\quad \text{G(n)} = \sum_{i=1}^{w(n)} G(p_i^{\alpha_i})\text{H}(\frac{n}{p_i^{\alpha_i}})$

we suppose that $\text{H}(p_i^{\alpha_i}) \neq 0$ for every i in $[1, w(n)]$.

we have H is multiplicative from lemma 2 . then $\text{H}(\text{n}=\frac{n}{p_i^{\alpha_i}}p_i^{\alpha_i})=\text{H}(p_i^{\alpha_i})\text{H}(\frac{n}{p_i^{\alpha_i}})$

then $\quad \text{H}(\frac{n}{p_i^{\alpha_i}})=\frac{H(n)}{H(p_i^{\alpha_i})}$

then $\text{G(n)}=\text{H(n)}\sum_{i=1}^{w(n)} \frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

2.Exemples

exemple 1 : $\sum_{d/n} w(d)=\tau(n)\sum_{i=1}^{w(n)} \frac{\alpha_i}{\alpha_i+1}$.

in particular case if h(n)=1 and f(n)=$w(n)$

we have $\text{H(n)}=\sum_{d/n} 1 \ = \tau(n)$ then $\tau(p_i^{\alpha_i}) = \alpha_i+1$

and $\text{G}(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}} w(d)$

$$=w(1)+w(p_i)\ldots\ldots\ldots+w(p_i^{\alpha_i})$$

$$= 0+1+1+........+1$$

$= \alpha_i$

therfore :

$\sum_{d/n} w(d) = \tau(n) \sum_{i=1}^{w(n)} \frac{\alpha_i}{\alpha_i+1}$

exemple 1 : $\sum_{d/n} w(d)\mu(d)$= -1 if $w(n)=1$

= 0 if not

for h(n)=$\mu(n)$ and f(n)=$w(n)$

we have H(n)=$\sum_{d/n} \mu(d)$ = 1 if n=1

= 0 if not

G($p_i^{\alpha_i}$) =$\sum_{d/p_i^{\alpha_i}} w(d)\mu(d)$

$= w(1)\mu(1)+w(p_i)\mu(p_i)+$..........$+w(p_i^{\alpha_i})\mu(p_i^{\alpha_i})$

= 0-1 +0+.......+0

= -1

then from theoreme 1 we have G(n)=$\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})$H($\frac{n}{p_i^{\alpha_i}}$)

=-$\sum_{i=1}^{w(n)}$ H($\frac{n}{p_i^{\alpha_i}}$)

if $w(n) > 1$ then $\frac{n}{p_i^{\alpha_i}} \neq 1$ then H($\frac{n}{p_i^{\alpha_i}}$)=0 for every i

if $w(n)=1$ then $\frac{n}{p_i^{\alpha_i}}=1$ then H($\frac{n}{p_i^{\alpha_i}}$)=1

therfore $\sum_{d/n} w(d)\mu(d)$= -1 if $w(n)=1$

= 0 if not

Corollaire :

let f be an arithmetic function

then $\sum_{w(d\wedge n)=1, 1\leq d\leq n}$ f(d)=$-\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{j=1}^{\frac{n}{d_1}}$ f(jd$_1$)

if f(n)=1 then $\sum_{w(d\wedge n)=1,1\leq d\leq n} 1=\varphi(n)\sum_{i=1}^{w(n)}\frac{1}{p-1}$

with $\varphi(n)$=n$\sum_{d_1/n}\frac{\mu(d_1)}{d_1}$

proof :

if $w(d\wedge n)=1$ then from exemple 2 we have $\sum_{d_1/d\wedge n} w(d_1)\mu(d_1)$= -1

then $\sum_{w(d\wedge n)=1,1\leq d\leq n}$f(d)$=-\sum_{1\leq d\leq n}$f(d)$\sum_{d_1/d\wedge n} w(d_1)\mu(d_1)$

$$=-\sum_{d_1/d\wedge n} w(d_1)\mu(d_1)\sum_{1\leq d\leq n}\text{f(d)}$$

$$=-\sum_{d_1/d,d/n} w(d_1)\mu(d_1)\sum_{1\leq d\leq n}\text{f(d)}$$

$$=-\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{1\leq d\leq n,d_1/d}\text{f(d)}$$

$$=-\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{1\leq d\leq n,d=\text{jd}_1}\text{f(d)}$$

$$= -\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{1\leq \text{jd}_1\leq n}\text{f(jd}_1)$$

$$=-\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{1\leq j\leq \frac{n}{d_1}}\text{f(jd}_1)$$

if f(n)=1 we have $\sum_{w(d\wedge n)=1,1\leq d\leq n} 1=-\sum_{d_1/n} w(d_1)\mu(d_1)\sum_{1\leq j\leq \frac{n}{d_1}} 1$

$$=-\sum_{d_1/n} w(d_1)\mu(d_1)\frac{n}{d_1}$$

$$=-n\sum_{d_1/n} w(d_1)\frac{\mu(d_1)}{d_1}$$

and from the theoreme1 we have $\sum_{d_1/n} w(d_1)\frac{\mu(d_1)}{d_1}=\sum_{d_1/n}\frac{\mu(d_1)}{d_1}\sum_{i=1}^{w(n)}\frac{\left(0-\frac{1}{p}\right)}{1-\frac{1}{p}}$

$$=-\sum_{d_1/n}\frac{\mu(d_1)}{d_1}\sum_{i=1}^{w(n)}\frac{1}{p-1}$$

then $\sum_{w(d\wedge n)=1,1\leq d\leq n} 1$ = n$\sum_{d_1/n}\frac{\mu(d_1)}{d_1}\sum_{i=1}^{w(n)}\frac{1}{p-1}$

$$= \varphi(n)\sum_{i=1}^{w(n)}\frac{1}{p-1}$$

with $\varphi(n)$=n$\sum_{d_1/n}\frac{\mu(d_1)}{d_1}$ is the Euler totient function

III. theoreme 2:

let g be a additive_multiplicative function such as g=h$\times f\times k$

and f,k both of them additives and h is multiplicative

let G(n)=$\sum_{d/n}$g(d) , $G_1(n)=\sum_{d/n}$h(d)f(d), $G_2(n)=\sum_{d/n}$h(d)k(d) and n=$\prod_{i=1}^{w(n)}p_i^{\alpha_i}$

then G(n) = $\sum_{i=1}^{w(n)}G(p_i^{\alpha_i})$H($\frac{n}{p_i^{\alpha_i}}$) +$\sum_{1\leqslant i\neq j\leq w(n)}G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})$H($\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}}$)

and if H(n) $\neq$0 for every i in $\mathcal{N}^{\star}$

then G(n)=H(n)$\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$+H(n)$\sum_{1\leqslant i\neq j\leq w(n)}\frac{G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})}{\mathrm{H}(p_i^{\alpha_i}p_j^{\alpha_j})}$

with H(n)=$\sum_{d/n}$h(d)

proof :

let g be a additive_multiplicative function such as g=h$\times f\times k$

and f,k both of them additives and h is multiplicative

and n =$\prod_{i=1}^{w(n)}p_i^{\alpha_i}$

let $a\wedge b=1$ G(n=ab)=$\sum_{d/\mathrm{ab}}$h(d)f(d)k(d)

if we apply lemma 1 , G(ab)=$\sum_{e/a}\sum_{e'/b}$h(ee')f(ee')k(ee')

=$\sum_{e/a}\sum_{e'/b}$h(ee')(f(e)+f(e'))(k(e)+k(e'))

=$\sum_{e/a}\sum_{e'/b}$h(ee')(f(e)k(e)+f(e)k(e')+f(e')k(e)+f(e')k(e'))

=$\sum_{e/a}h(e)$f(e)k(e)$\sum_{e'/b}$h(e')+$\sum_{e/a}h(e)$f(e)$\sum_{e'/b}$h(e')k(e')

+$\sum_{e/a}h(e)k(e)\sum_{e'/b}$h(e')f(e')+$\sum_{e/a}h(e)\sum_{e'/b}$h(e')f(e')k(e')

=H(b)G(a)+G_1(a)G_2(b)+G_1(b)G_2(a)+H(a)G(b)

such that G_1(n)=$\sum_{d/n}$h(d)f(d), G_2(n)=$\sum_{d/n}$h(d)k(d) and H(n)=$\sum_{d/n}$h(d) then G(ab)=H(b)G(a)+H(a)G(b)+G_1(a)G_2(b)+G_1(b)G_2(a) if $a \wedge b = 1$ (**)

now we will complet the proof by reccurence

1. if $w(n)=1$ then $n=p^{\alpha_p}$,

$$\text{and } G(n)=G(p^{\alpha_p})H(1)+0=G(p^{\alpha_p})H(\frac{p^{\alpha_p}}{p^{\alpha_p}})+\sum_{1\leqslant i\neq j\leq 1} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$$

2. let $w(n)>1$, we suppose that .

$$G(n) = \sum_{i=1}^{w(n)} G(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}}) + \sum_{1\leqslant i\neq j\leq w(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$$

is true for $w(n)$, and let us repprove it for $w'(n)=w(n)+1$.

we have then $n=\prod_{i=1}^{w'(n)} p_i^{\alpha_i} = \prod_{i=1}^{w(n)} p_i^{\alpha_i} \times p_{w'(n)}^{\alpha_{w'(n)}}=ab$

with $a=\prod_{i=1}^{w(n)} p_i^{\alpha_i}$ and $b=p_{w'(n)}^{\alpha_{w'(n)}}$

then from (**) ;

$$G(ab)=H(\prod_{i=1}^{w(n)} p_i^{\alpha_i})G(p_{w'(n)}^{\alpha_{w'(n)}})+H(p_{w'(n)}^{\alpha_{w'(n)}})G(\prod_{i=1}^{w(n)} p_i^{\alpha_i})+G_1(p_{w'(n)}^{\alpha_{w'(n)}})G_2(\prod_{i=1}^{w(n)} p_i^{\alpha_i})+G_1(\prod_{i=1}^{w(n)} p_i^{\alpha_i})G_2(p_{w'(n)}^{\alpha_{w'(n)}})$$

$$=H(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})G(p_{w'(n)}^{\alpha_{w'(n)}})+H(p_{w'(n)}^{\alpha_{w'(n)}})(\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})H(\frac{b}{p_i^{\alpha_i}})+\sum_{1\leqslant i\neq j\leq w(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{b}{p_i^{\alpha_i}p_j^{\alpha_j}}))+$$

$$G_1(p_{w'(n)}^{\alpha_{w'(n)}})\sum_{i=1}^{w(n)} G_2(p_i^{\alpha_i})H(\frac{b}{p_i^{\alpha_i}})+\sum_{i=1}^{w(n)} G_1(p_i^{\alpha_i})H(\frac{b}{p_i^{\alpha_i}})G_2(p_{w'(n)}^{\alpha_{w'(n)}})$$

$$=H(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})G(p_{w'(n)}^{\alpha_{w'(n)}})+\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})H(\frac{b}{p_i^{\alpha_i}})H(p_{w'(n)}^{\alpha_{w'(n)}})+\sum_{1\leqslant i\neq j\leq w(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{b}{p_i^{\alpha_i}p_j^{\alpha_j}})H(p_{w'(n)}^{\alpha_{w'(n)}})+$$

$$\sum_{i=1}^{w(n)} G_1(p_{w'(n)}^{\alpha_{w'(n)}})G_2(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}p_{w'(n)}^{\alpha_{w'(n)}}})+\sum_{i=1}^{w(n)} G_1(p_i^{\alpha_i})G_2(p_{w'(n)}^{\alpha_{w'(n)}})H(\frac{n}{p_i^{\alpha_i}p_{w'(n)}^{\alpha_{w'(n)}}})$$

$$= H(\frac{n}{p_{w'(n)}^{\alpha_{w'(n)}}})G(p_{w'(n)}^{\alpha_{w'(n)}})+\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}})+\sum_{1\leqslant i\neq j\leq w(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})+$$

$$\sum_{i=1}^{w(n)} G_1(p_{w'(n)}^{\alpha_{w'(n)}})G_2(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}p_{w'(n)}^{\alpha_{w'(n)}}})+\sum_{i=1}^{w(n)} G_1(p_i^{\alpha_i})G_2(p_{w'(n)}^{\alpha_{w'(n)}})H(\frac{n}{p_i^{\alpha_i}p_{w'(n)}^{\alpha_{w'(n)}}})$$

$$= \sum_{i=1}^{w'(n)} G(p_i^{\alpha_i})H(\frac{n}{p_i^{\alpha_i}})+\sum_{1\leqslant i\neq j\leq w'(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})H(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$$

then $\text{G}(\text{n}=\prod_{i=1}^{w'(n)} p_i^{\alpha_i})=\sum_{i=1}^{w'(n)} G(p_i^{\alpha_i})\text{H}(\frac{n}{p_i^{\alpha_i}})+\sum_{1\leqslant i\neq j\leq w'(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})\text{H}(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$

for $w'(n)=w(n)+1$

if H(n)≠0 for every n in $\mathcal{N}^*$.

since H is multiplicative and $p_i^{\alpha_i}p_j^{\alpha_j} \wedge \frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}}=1$ and $p_i^{\alpha_i} \wedge \frac{n}{p_i^{\alpha_i}}=1$ then $\text{H}(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})=\frac{H(n)}{H(p_i^{\alpha_i}p_j^{\alpha_j})}$

and $\text{H}(\frac{n}{p_i^{\alpha_i}})=\frac{H(n)}{H(p_i^{\alpha_i})}$

then $\text{G}(\text{n})=\text{H}(\text{n})\sum_{i=1}^{w(n)} \frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}+\text{H}(\text{n})\sum_{1\leqslant i\neq j\leq w(n)} \frac{G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})}{\text{H}(p_i^{\alpha_i}p_j^{\alpha_j})}$

Exemple 1 :

let f(n)=1 and h(n)=$w(n)$ and k(n)=$w(n)$

then from theorem 2 it follows that

$$\text{G}(\text{n})=\sum_{d/n} w(d)^2 = \tau(n)\sum_{i=1}^{w(n)} \frac{G(p_i^{\alpha_i})}{\tau(p_i^{\alpha_i})}+\tau(n)\sum_{1\leqslant i\neq j\leq w(n)} \frac{G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})}{\tau(p_i^{\alpha_i}p_j^{\alpha_j})}$$

with $\tau(n)=\sum_{d/n} 1$

$$G(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}} w(d)^2$$

$$=0+1+1+\ldots\ldots 1$$

$$=\alpha_i$$

$$G_1(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}} w(d)$$

$$=\alpha_i$$

$$G_2(p_j^{\alpha_j})=\sum_{d/p_j^{\alpha_j}} w(d)$$

$$=\alpha_j$$

then $\sum_{d/n} w(d)^2=\tau(n)\sum_{i=1}^{w(n)} \frac{\alpha_i}{\alpha_i+1}+\tau(n)\sum_{1\leqslant i\neq j\leq w(n)} \frac{\alpha_i\alpha_j}{(\alpha_i+1)(\alpha_j+1)}$

$=\sum_{d/n} w(d)+\tau(n)\sum_{1\leqslant i\leq w(n)} \sum_{1\leqslant j\neq i\leq w(n)} \frac{\alpha_i\alpha_j}{(\alpha_i+1)(\alpha_j+1)}$

$=\sum_{d/n} w(d)+\tau(n)\sum_{1\leqslant i\leq w(n)} \frac{\alpha_i}{\alpha_i+1}\sum_{1\leqslant j\leq w(n)} (\frac{\alpha_j}{(\alpha_j+1)} - \frac{\alpha_i}{\alpha_i+1})$

$=\sum_{d/n} w(d)+\tau(n)(\sum_{1\leqslant i\leq w(n)} \frac{\alpha_i}{\alpha_i+1})^2 -\tau(n)\sum_{1\leqslant i\leq w(n)} (\frac{\alpha_i}{\alpha_i+1})^2$

then $\sum_{d/n} w(d)^2=\sum_{d/n} w(d)+\tau(n)(\sum_{1\leqslant i\leq w(n)} \frac{\alpha_i}{\alpha_i+1})^2 -\tau(n)\sum_{1\leqslant i\leq w(n)} (\frac{\alpha_i}{\alpha_i+1})^2$

Exemple 2:

let h(n)=$\mu(n)$ and f(n)=k(n)=$w(n)$

then from theorem 2 it follows that :

G(n)=$\sum_{d/n} \mu(n)w(d)^2$

$=\sum_{i=1}^{w(n)} G(p_i^{\alpha_i})\mathrm{H}(\frac{n}{p_i^{\alpha_i}}) +\sum_{1\leqslant i\neq j\leq w(n)} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})\mathrm{H}(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$

$G(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}} \mu(n)w(d)^2$

= 0-1

=-1

$G_1(p_i^{\alpha_i}) =\sum_{d/p_i^{\alpha_i}} \mu(n)w(d)$ = -1

$G_2(p_j^{\alpha_j})=\sum_{d/p_j^{\alpha_j}} \mu(n)w(d)$=-1

if $w(n)>2$ then H$(\frac{n}{p_i^{\alpha_i}}) =\sum_{d/\frac{n}{p_i^{\alpha_i}}} \mu(n)$=0

and H$(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}}) =\sum_{d/\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}}} \mu(n)$=0

then $G(n)=\sum_{d/n} \mu(n)w(d)^2$=0

$w(n)=1$ then G(n=$p_i^{\alpha_i}$)=-1

$w(n)=2$ then G(n=$p_i^{\alpha_i}p_j^{\alpha_j}$)= $\sum_{1\leqslant i\neq j\leq 2} G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})\mathrm{H}(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}})$

$=\sum_{1\leqslant i\neq j\leq 2} 1$ =2

therfore $\sum_{d/n} \mu(n)w(d)^2$ = -1 if $w(n)=1$

$= 2$ if $w(n)=2$

$= 0$ if not

IV. Application of those theorems on Reimann Zita function

let $\mathfrak{T}(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Reimann zita function

and $\mathcal{P}(s)=\sum_{p\epsilon\mathcal{P}}\frac{1}{p^s}$ is the prime zita function

from thoerem 1 and theorem 2 we can obtain a lot of formulas like those:

1. $\sum_{d=1}^{\infty}\frac{w(d)}{d^s}=\mathfrak{T}(s)\mathcal{P}(s)$
2. $\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}=\mathfrak{T}(s)\sum_{p\epsilon\mathcal{P}}\frac{1}{p^s-1}$
3. $\sum_{d=1}^{\infty}\frac{\mu(d)w(d)}{d^s}=-\mathfrak{T}(s)^{-1}\sum_{p\epsilon\mathcal{P}}\frac{1}{p-1}$
4. $\sum_{d=1}^{\infty}\frac{w(d)\Omega(d)}{d^s}=\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}+\mathfrak{T}(s)\sum_{p\neq q,p,q\epsilon\mathcal{P}^2}\frac{1}{p^s-1}\times\frac{1}{q^s}$
5. $\sum_{d=1}^{\infty}\frac{w(d)\lambda(d)}{d^s}$=- $\frac{\mathfrak{T}(2s)}{\mathfrak{T}(s)}\sum_{p\epsilon\mathcal{P}}\frac{1}{{p_i}^s}$
6. $\sum_{d=1}^{\infty}\frac{w(d)(-1)^{w(d)}}{d^s}=-\sum_{d=1}^{\infty}\frac{(-1)^{w(d)}}{d^s}\sum_{p\epsilon\mathcal{P}}\frac{1}{{p_i}^s-2}$
7. $\sum_{d=1}^{\infty}\frac{\tau(d)w(d)}{d^s}=\mathfrak{T}^2(s)\sum_{p\epsilon\mathcal{P}}(\frac{2}{p^s}$ - $\frac{1}{p^{2s}})=\mathfrak{T}^2(s)(2\mathcal{P}(s)-\mathcal{P}(2s))$

let $s=\sigma+it$ a complex number and $\mathfrak{T}(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Reimann zita function

$\mathcal{P}(s)=\sum_{p\epsilon\mathcal{P}}\frac{1}{p^s}$ is the prime zita function

1. we set $f(n)=w(n)$ and $h(n)=\frac{1}{n^s}$

we have from theorem 1 $G(n)=\sum_{d/n}\frac{w(d)}{d^s}=H(n)\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

with $H(n)=\sum_{d/n}\frac{1}{d^s}$ and $G(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}}\frac{w(d)}{d^s}$

$$=0+\frac{1}{p_i^s}+\frac{1}{p_i^{2s}}\ldots\ldots\ldots\ldots+\frac{1}{p_i^{\alpha_i s}}$$

$$=\frac{1}{p_i^s}(1+\frac{1}{p_i^s}+\cdots\ldots+\frac{1}{p_i^{\alpha_i s-s}})$$

$$=\frac{1}{{p_i}^s}\frac{1-\frac{1}{p_i^{\alpha_i s}}}{1-\frac{1}{{p_i}^s}}$$

and $\mathrm{H}(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}}\frac{1}{d^s}$

$$=1+\frac{1}{{p_i}^s}+\ldots\ldots.+\frac{1}{p_i^{\alpha_i s}}$$

$$=\frac{1-\frac{1}{p_i^{\alpha \mathrm{is}+s}}}{1-\frac{1}{{p_i}^s}}$$

then $\mathrm{G(n)}=\sum_{d/n}\frac{w(d)}{d^s}=\mathrm{H(n)}\sum_{i=1}^{w(n)}\frac{\frac{1}{{p_i}^s}\frac{1-\frac{1}{p_i^{\alpha_i s}}}{1-\frac{1}{{p_i}^s}}}{\frac{1-\frac{1}{p_i^{\alpha \mathrm{is}+s}}}{1-\frac{1}{{p_i}^s}}}$

if $n\rightarrow\infty$ then $\alpha_i\rightarrow\infty$ for every i

(like we suppose that ∞ is the product of all integer positive numbers)

$\{\mathrm{d}/n\rightarrow\infty\}=\{1,2,\ldots\ldots\ldots\ldots\ldots\rightarrow\infty\}$

$\lim_{n\rightarrow\infty}\sum_{d/n}\frac{w(d)}{d^s}=\sum_{d=1}^{\infty}\frac{w(d)}{d^s}$ and $\lim_{n\rightarrow\infty}\sum_{d/n}\frac{1}{d^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}=\mathfrak{T}(\mathrm{s})$

$$=\lim_{n\rightarrow\infty}\mathrm{H(n)}\sum_{i=1}^{w(n)}\frac{\frac{1}{{p_i}^s}\frac{1-\frac{1}{p_i^{\alpha_i s}}}{1-\frac{1}{{p_i}^s}}}{\frac{1-\frac{1}{p_i^{\alpha \mathrm{is}+s}}}{1-\frac{1}{{p_i}^s}}}$$

$$=\mathfrak{T}(\mathrm{s})\sum_{i=1}^{w(n\rightarrow\infty)}\lim_{\alpha_i\rightarrow\infty}\frac{\frac{1}{{p_i}^s}\frac{1-\frac{1}{p_i^{\alpha_i s}}}{1-\frac{1}{{p_i}^s}}}{\frac{1-\frac{1}{p_i^{\alpha \mathrm{is}+s}}}{1-\frac{1}{{p_i}^s}}}$$

$$=\mathfrak{T}(\mathrm{s})\sum_{i=1}^{\infty}\frac{\frac{1}{{p_i}^s}\frac{1}{1-\frac{1}{{p_i}^s}}}{\frac{1.}{1-\frac{1}{{p_i}^s}}}$$

$$=\mathfrak{T}(\mathrm{s})\sum_{i=1}^{\infty}\frac{1}{{p_i}^s}$$

$$=\mathfrak{T}(\mathrm{s})\mathcal{P}(s)$$

then $\sum_{d=1}^{\infty}\frac{w(d)}{d^s}$=$\mathfrak{T}(s)\mathcal{P}(s)$

2. let h(n)=$\frac{1}{n^s}$ and f(n)=$\Omega(n)$ (Ω is additive)

then G(n)=$\sum_{d/n}\frac{\Omega(d)}{d^s} = \sum_{d/n}\frac{1}{d^s}\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

let n→∞ ;

then $\lim_{\alpha_i\to\infty}G(p_i^{\alpha_i}) = \lim_{\alpha_i\to\infty}\sum_{d/p_i^{\alpha_i}}\frac{\Omega(d)}{d^s} = 0+\frac{1}{{p_i}^s}+\ldots\ldots\ldots\ldots = \frac{1}{{p_i}^s}\left(1+\frac{1}{{p_i}^s}+\cdots\ldots\ldots\right)$

$=\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}$

and $\lim_{\alpha_i\to\infty}H(p_i^{\alpha_i}) = \lim_{\alpha_i\to\infty}\sum_{d/p_i^{\alpha_i}}\frac{1}{d^s} = 1+\frac{1}{{p_i}^s}+\ldots\ldots\ldots\ldots = \frac{1}{1-\frac{1}{{p_i}^s}}$

then $\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}$=$\mathfrak{T}(s)\sum_{i=1}^{\infty}\frac{\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}}{\frac{1}{1-\frac{1}{{p_i}^s}}}$ =$\mathfrak{T}$(s)$\sum_{i=1}^{\infty}\frac{1}{{p_i}^s-1}$

then $\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}$=$\mathfrak{T}(s)\sum_{p\epsilon\mathcal{P}}^{\infty}\frac{1}{p^s-1}$

3. let h(n)=$\frac{\mu(n)}{n^s}$ and f(n)=$w(n)$ then from theorem 1 we have .

G(n)=$\sum_{d/n}\frac{\mu(d)w(d)}{d^s} = \sum_{d/n}\frac{\mu(n)}{d^s}\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

we have $G(p_i^{\alpha_i}) = \sum_{d/p_i^{\alpha_i}}\frac{\mu(d)w(d)}{d^s}$=0-$\frac{1}{p_i}$ and $H(p_i^{\alpha_i}) = \sum_{d/p_i^{\alpha_i}}\frac{\mu(n)}{d^s}$=1 -$\frac{1}{p_i}$

then $\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})} = \sum_{i=1}^{w(n)}\frac{-\frac{1}{p_i}}{1-\frac{1}{p_i}} = -\sum_{i=1}^{w(n)}\frac{1}{p_i-1}$

and if we tend n→∞ we will obtain $\sum_{d=1}^{\infty}\frac{\mu(d)w(d)}{d^s}$ = -$\sum_{d=1}^{\infty}\frac{\mu(n)}{d^s}\sum_{i=1}^{\infty}\frac{1}{p_i-1}$

but we know that $\sum_{d=1}^{\infty}\frac{\mu(n)}{d^s}$=$\mathfrak{T}(s)^{-1}$

Then $\sum_{d=1}^{\infty}\frac{\mu(d)w(d)}{d^s}$=$-\mathfrak{T}(s)^{-1}\sum_{p\epsilon\mathcal{P}}\frac{1}{p-1}$

4. let h(n)=$\frac{1}{n^s}$ and f(n)=$\Omega(n)$ and k(n)=$w(n)$ then from the theorem 2 we will obtain

$$G(n)=\sum_{d/n}\frac{w(d)\Omega(d)}{d^s}=H(n)\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}+H(n)\sum_{1\leqslant i\neq j\leq w(n)}\frac{G_1(p_i^{\alpha_i})G_2(p_j^{\alpha_j})}{H(p_i^{\alpha_i}p_j^{\alpha_j})}$$

with $H(n)=\sum_{d/n}\frac{1}{d^s}$ and $G_1(p_i^{\alpha_i})=\sum_{d/p_i^{\alpha_i}}\frac{\Omega(d)}{d^s}$ and $G_2(p_j^{\alpha_j})=\sum_{d/p_j^{\alpha_j}}\frac{w(d)}{d^s}$

if we tend $n\to\infty$ we will obtain $\lim_{n\to\infty}H(n)=\mathfrak{T}(s)$

and $\lim_{n\to\infty}G_2(p_j^{\alpha_j})=0+\frac{1}{{p_i}^s}+\frac{1}{p_i^{2s}}+\cdots\ldots=\frac{1}{1-\frac{1}{{p_i}^s}}-1=\frac{\frac{1}{{p_i}^s}}{1-\frac{1}{{p_i}^s}}=\frac{1}{{p_i}^s-1}$

and $\lim_{n\to\infty}G_1(p_j^{\alpha_j})=0+\frac{1}{{p_i}^s}+\frac{2}{p_i^{2s}}+\cdots\ldots=\frac{1}{{p_i}^s}\left(1+\frac{2}{{p_i}^s}+\cdots\ldots\right)=\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}$

and $\lim_{n\to\infty}H(p_i^{\alpha_i})=1+\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}\ldots\ldots\ldots=\frac{1}{1-\frac{1}{{p_i}^s}}$

and $\lim_{n\to\infty}G(p_i^{\alpha_i})=0+\frac{1}{{p_i}^s}+\frac{2}{p_i^{2s}}+\cdots\ldots=\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}$

then $\sum_{d=1}^{\infty}\frac{w(d)\Omega(d)}{d^s}=\mathfrak{T}(s)\sum_{i=1}^{\infty}\frac{\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}}{\frac{1}{1-\frac{1}{{p_i}^s}}}+\mathfrak{T}(s)\sum_{1\leq i\neq j\leq\infty}\frac{\frac{1}{{p_i}^s}\frac{1}{\left(1-\frac{1}{{p_i}^s}\right)^2}}{\frac{1}{1-\frac{1}{{p_i}^s}}}\times\frac{\frac{1}{{p_j}^s-1}}{\frac{1}{1-\frac{1}{{p_j}^s}}}$

$$=\mathfrak{T}(s)\sum_{i=1}^{\infty}\frac{1}{{p_i}^s-1}+\mathfrak{T}(s)\sum_{1\leq i\neq j\leq\infty}\frac{1}{{p_i}^s-1}\times\frac{1}{{p_j}^s}$$

then $\sum_{d=1}^{\infty}\frac{w(d)\Omega(d)}{d^s}=\mathfrak{T}(s)\sum_{p\epsilon\mathcal{P}}\frac{1}{p^s-1}+\mathfrak{T}(s)\sum_{p\neq q}\frac{1}{p^s-1}\times\frac{1}{q^s}$

but we know from 2 that $\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}=\mathfrak{T}(s)\sum_{p\epsilon\mathcal{P}}\frac{1}{p^s-1}$

then $\sum_{d=1}^{\infty}\frac{w(d)\Omega(d)}{d^s}=\sum_{d=1}^{\infty}\frac{\Omega(d)}{d^s}+\mathfrak{T}(s)\sum_{p\neq q}\frac{1}{p^s-1}\times\frac{1}{q^s}$

5. let $h(n)=\frac{\lambda(n)}{n^s}$ and $f(n)=w(n)$ then if we apply theorem1 we will obtain

$$G(n)=\sum_{d/n}\frac{w(d)\lambda(d)}{d^s}=H(n)\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$$

if we tend $n\to\infty$ we will obtain $\lim_{n\to\infty}H(n)=\sum_{d=1}^{\infty}\frac{\lambda(d)}{d^s}$

and $\lim_{n\to\infty}G(p_i^{\alpha_i})=0-\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}-\frac{1}{{p_i}^{3s}}\ldots\ldots\ldots=-\frac{1}{{p_i}^s}(1-\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}\ldots..)=-\frac{1}{{p_i}^s}\times\frac{1}{1+\frac{1}{{p_i}^s}}=-\frac{1}{{p_i}^s+1}$

and $\lim_{n\to\infty}H(p_i^{\alpha_i})=1-\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}\ldots\ldots\ldots=\frac{1}{1+\frac{1}{{p_i}^s}}$

but we know that $\sum_{d=1}^{\infty}\frac{\lambda(d)}{d^s}=\frac{\mathfrak{T}(2s)}{\mathfrak{T}(s)}$

then $\sum_{d=1}^{\infty}\frac{w(d)\lambda(d)}{d^s}$= - $\frac{\mathfrak{T}(2s)}{\mathfrak{T}(s)}\sum_{i=1}^{\infty}\frac{\frac{1}{{p_i}^s+1}}{\frac{1}{1+\frac{1}{{p_i}^s}}}$=- $\frac{\mathfrak{T}(2s)}{\mathfrak{T}(s)}\sum_{i=1}^{\infty}\frac{1}{{p_i}^s}$

then $\sum_{d=1}^{\infty}\frac{w(d)\lambda(d)}{d^s}$=- $\frac{\mathfrak{T}(2s)}{\mathfrak{T}(s)}\sum_{p\epsilon\mathcal{P}}^{\infty}\frac{1}{p^s}$

6. let h(n)=$\frac{(-1)^{w(n)}}{n^s}$ and f(n)=$w(n)$ then if we apply theorem1 we will obtain

G(n)=$\sum_{d/n}\frac{w(d)(-1)^{w(d)}}{d^s}$=H(n)$\sum_{i=1}^{w(n)}\frac{G(p_i^{\alpha_i})}{H(p_i^{\alpha_i})}$

if we tend n→∞ we will obtain $\lim_{n\to\infty}$H(n)=$\sum_{d=1}^{\infty}\frac{(-1)^{w(d)}}{d^s}$

and $\lim_{n\to\infty}G(p_i^{\alpha_i})=0$-$\frac{1}{{p_i}^s}-\frac{1}{p_i^{2s}}$.........= -$\frac{1}{{p_i}^s}$(1+$\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}$.........)=- $\frac{1}{{p_i}^s}\frac{1}{1-\frac{1}{{p_i}^s}}$= - $\frac{1}{{p_i}^s-1}$

and $\lim_{n\to\infty}H(p_i^{\alpha_i})=1$-$\frac{1}{{p_i}^s}-\frac{1}{{p_i}^s}$.........=-(1+$\frac{1}{{p_i}^s}+\frac{1}{{p_i}^{2s}}$.........)+2=2-$\frac{1}{1-\frac{1}{{p_i}^s}}$=2-$\frac{{p_i}^s}{{p_i}^s-1}$

then $\sum_{d=1}^{\infty}\frac{w(d)(-1)^{w(d)}}{d^s}=-\sum_{d=1}^{\infty}\frac{(-1)^{w(d)}}{d^s}\sum_{i=1}^{\infty}\frac{\frac{1}{{p_i}^s-1}}{2-\frac{{p_i}^s}{{p_i}^s-1}}=-\sum_{d=1}^{\infty}\frac{(-1)^{w(d)}}{d^s}\sum_{i=1}^{\infty}\frac{1}{2({p_i}^s-1)-{p_i}^s}$

then $\sum_{d=1}^{\infty}\frac{w(d)(-1)^{w(d)}}{d^s}=-\sum_{d=1}^{\infty}\frac{(-1)^{w(d)}}{d^s}\sum_{p\epsilon\mathcal{P}}\frac{1}{{p_i}^s-2}$

V. References ;

1.Elementary Methods in Number Theory - Nathanson M.B

2.The Theory Of The Riemann Zeta-Function -Titshmarch

3.AN INTRODUCTION TO THE THEORY OF NUMBERS - hardy & wright