Complex Dynamics and Foundational Physics

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Abstract

This work is a top-level summary of several contributions published in the last three decades. It makes the case that complex dynamics of nonlinear systems lies at the heart of foundational physics.

Key words: Complex Dynamics, Ginzburg-Landau (GL) equation, Chaos and Bifurcations, Multifractals, Self-organized Criticality (SOC), Minimal Fractal Manifold, Fractional Field Theory.

1. Introduction

The purpose of this paper is to suggest a top-level account of how foundational physics develops from the universal behavior of *complex dynamics*. We start from the observation that, in general, the underlying dynamics of complex systems may be characterized by the following attributes [11]:

1) Complex systems are open ensembles of many constituents interacting nonlinearly,

2) A complex system has a structure spanning several scales,

3) A complex system is capable of self-organization and emerging behavior,

4) Complex dynamics involves an interplay between order and chaos,

5) Complex dynamics involves an interplay between cooperation and competition.

Taken together, these attributes hint that the high-energy sector of field theory prevents thermalization of quantum fluctuations and creates an environment favoring the onset of complex dynamics. In this type of setting, the underlying principles of classical statistical physics and traditional Quantum Field Theory (QFT) are prone to break down. In particular the ergodic theorem, the fluctuation-dissipation theorem, analyticity, unitarity, locality, finiteness in all orders of perturbation theory and renormalizability are either violated or lose their conventional meaning [12].

From these considerations and in the limited context of our work, we take complex dynamics to represent the *interaction and evolution of large nonlinear systems outside equilibrium*. Framed in these terms, we believe that complex dynamics is likely to offer an adequate picture of how the Universe unfolded in its early stages of evolution.

Below is a condensed picture showing the path from complex dynamics to the conceptual framework of foundational physics. We caution that this chart reflects a personal perspective that may very well stand at odds with mainstream views on foundational topics linked to QFT, Standard Model, Quantum Gravity theories and relativistic Cosmology. Details on the content of the chart are elaborated upon in the text. Given the multifaceted extent and intricacies of complex dynamics, many points are intentionally left out for the sake of clarity and concision.

COMPLEX DYNAMICS

⇓

Far-from-equilibrium nonlinear dynamical systems

⇓

Renormalization Group (RG) flows

⇓

Ginzburg-Landau (GL) equations

⇓

Chaos and Multifractals ⇐ **Self-organization / ε-expansion** ⇒ **Effective Field Theory**

⇓

Minimal Fractal Manifold

⇓

Fractional Field Theory

2. Generic flows in far-from-equilibrium conditions

The Renormalization Group (RG) is a well-established framework for the analysis of complex physical systems at both ends of the energy scale. Over the years, the principles and methods of RG have found a wide range of applications, from critical behavior in statistical physics and condensed matter to perturbative and non-perturbative models in QFT. An appealing feature of RG equations is that they resemble the evolution equations of dynamical systems. In particular, the Callan-Symanzik equation stems from the independence of QFT from its subtraction point, which is on par with *self-similarity of autonomous flows approaching attractors*. In the Wilsonian formulation of the RG, the flow in coupling space is associated with the trajectory of QFT towards a subspace of relevant and marginal operators. Conventional wisdom asserts that the attractors of the RG flow consist of a finite number of isolated fixed points (FP). There is by now mounting evidence that this assumption is too restrictive, that RG flows – echoing the onset of turbulence in fluid mechanics – may evolve towards limit cycles or tori as well as strange attractors, the latter denoting invariant sets having chaotic structure.

The aim of this section is to extrapolate the conventional RG paradigm to a framework which minimizes the potential loss of generality due to simplifying assumptions. To this end, we posit that all trajectories connecting the ultraviolet (UV) and infrared (IR) sectors of a generic field theory are characterized by the following initial conditions:

a) a large count of independent or coupled variables,

b) a large count of independent or coupled control parameters,

c) far-from-equilibrium settings,

d) non-perturbative and non-integrable dynamics.

As alluded to in the Introduction, this framework is likely to enable a more realistic picture of complex dynamics associated with the UV to IR flow. This view is backed up by many examples. For instance, integrable dynamical systems are isomorphic to free, non-interacting theories, which are unable to account for the arrow of time in transient regimes, the physics of self-organization and complex evolution outside equilibrium. Another instance is provided by Sakharov's non-equilibrium conditions for baryogenesis and the observed baryon asymmetry of the Universe [10].

3. RG flows as autonomous dynamical systems

Consider a generic field theory complying with conditions a) - d) introduced above. The RG flow in the space of its couplings $g \in \Gamma$ is a continuous map $\beta = R \times \Gamma \to \Gamma$ called the "*beta function*" and associated with

$$\beta(g) = \mu \frac{dg}{d\mu} = \frac{dg}{d(\log \mu/\mu_0)} \tag{1}$$

such that

$$\beta(0, g) = g \tag{2}$$

$$\beta(\tau, \beta(s, g)) = \beta(\tau + s, g) \tag{3}$$

where the "RG time" is $\tau = \log(\mu/\mu_0)$ and μ is the RG scale. A *fixed point* (FP, equilibrium or conformal point) of (1) is a coupling $g_0 \in \Gamma$ for which $\beta(R, g_0) = g_0$. The FP of the RG flow correspond to zero or infinite correlation lengths and are accordingly classified as "trivial" or "non-trivial". The existence of FP reflects the asymptotic approach towards scale-invariance, and it relates to the *self-similarity of fractal structures*. A subset $\mathrm{I} \subset \Gamma$ is an invariant set of the flow if

$$\beta(R,\mathrm{I}) = \bigcup_{\tau \in \mathbf{R}} \beta(\tau,\mathrm{I}) \subset \Gamma \tag{4}$$

Likewise, the continuous time flow of autonomous dynamical systems is described by the differential equation

$$\frac{dx(\tau)}{d\tau} = f(x(\tau)) \tag{5}$$

where $x \in R^n$ and $f : R^n \to R^n$ is a function on the n-dimensional phase space R^n. There are two ways of relating (5) to a map iteration of the phase space onto itself, namely,

a) Working in discrete "time" ($\tau \to \tau_0$) turns (5) into

$$x_{n+1} = x_n + \tau_0 f(x_n) = F(x_n)\ , \quad x_n = x(n\tau_0) \tag{6}$$

b) If (1) has periodic solutions $x(T) = x(0) = x_0$ for some $T > 0$, one takes a hyperplane R^{n-1} of dimension $n-1$ transverse to the orbit $\tau \to x(\tau)$ through x_0 and evaluates the distribution of neighboring intersections of the orbit with this hyperplane (the method of Poincaré sections).

Many dynamical systems and maps are dependent on a number of control parameters $\lambda \in R^m$. In this case, (5) and (6) take the form

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), \lambda(\tau)) \tag{7}$$

$$x_{n+1} = x_n + \tau_0 f(x_n, \lambda) = F(x_n, \lambda) \tag{8a}$$

Of particular interest is the long-term evolution of (6)-(8), which reflects the behavior of the large k^{th} iterate of the flow in phase-space, $\{F^k(x)\}$, $k >> 1$. By definition, a *period-k* FP of map (6) satisfies the condition

$$x^*_{n+k} = F^{(k)}(\lambda, x^*_n) = x^*_n \tag{8b}$$

Some flows may converge to specific attractors like a FP or a periodic orbit or erratically wander inside a bounded region (I). If all iterates remain "trapped" in (I) for $x \in \mathrm{I}$, then (I) forms an invariant set. Moreover, if (I) has a fine structure, or if there is sensitive dependence on initial conditions (two nearby points get farther apart under a large number of iterates of f), then (I) represents a *strange set*.

4. Reduction to normal form and GL equations

QFT's are known to become scale-invariant at large distances. Viewed in the context of conformal field theory, this property is typically associated with the FP structure of the RG flow. Starting from this observation, and by analogy with (5) and (7), we conjecture below that all field theories evaluated at sufficiently low-energy scales emerge from an underlying system of high-energy entities called *primary variables*. Let the UV sector of field theory be described by a large set of such variables $x \equiv \{x_i\}$, $i = 1, 2, ..., n$, $n >> 1$, whose mutual coupling and dynamics is far-from-equilibrium. The specific nature of the UV variables is irrelevant to our context, as they can take the form of irreducible objects such as, but not limited to, spinors, quaternions, twistors, octonions, strings, branes, loops, knots, bits of information and so on.

The downward flow of $x \equiv \{x_i\}$ may be mapped to a system of ordinary differential equations having the universal form

$$x'_\tau = f(x(\tau), \lambda(\tau), D(\tau)) \tag{9}$$

Here, λ, τ, D denote, respectively, the control parameters vector $\lambda = \{\lambda_j\}$, $j = 1, 2, \ldots m$, the evolution parameter and the dimension of the embedding space. If the dimension of the embedding space is taken to be independent variable or control parameter, the system (9) further reduces to

$$x'_\tau = f(x(\tau), \lambda(\tau)) \tag{10}$$

It is sensible to assume that the flow (9) or (10) occurs in the presence of non-vanishing perturbations induced by far-from-equilibrium conditions. These may surface, for example, from primordial density fluctuations in the early Universe or from unbalanced vacuum fluctuations in the UV regime of QFT.

To make explicit the effect of perturbations, we resolve $x(\tau)$ into a reference stable state $x_s(\tau)$ and a deviation generated by perturbations, i.e.,

$$x(\tau) = x_s(\tau) + y(\tau) \tag{11}$$

Direct substitution in (10) yields the set of homogeneous equations

$$y'_\tau = f(\{x_s + y\}, \lambda) - f(\{x_s\}, \lambda) \tag{12}$$

Further expanding around the reference state leads to

$$y'_{\tau} = \sum_j L_{ij}(x_s, \lambda) y_j + h_i(\{y_j\}, \lambda) \tag{13}$$

where L_{ij} and h_i denote, respectively, the coefficients of the linear and nonlinear contributions induced by departures from the reference state. Here, L_{ij} represents a $n \times n$ matrix dependent on the reference state and on the control parameters vector. Under the assumption that parameters λ stay close to their critical values $(\lambda = \lambda_c)$, it can be shown that (13) undergoes bifurcations and its behavior can be mapped to a closed set of universal equations referred to as normal forms [1, 6]. If, at $\lambda = \lambda_c$ perturbations are non-oscillatory (steady-state), the *normal form equations* are

$$z'_{\tau} = (\lambda - \lambda_c) - uz^2 \tag{14a}$$

$$z'_{\tau} = (\lambda - \lambda_c)z - uz^3 \tag{14b}$$

$$z'_{\tau} = (\lambda - \lambda_c)z - uz^2 \tag{14c}$$

Instead, if perturbations are oscillatory at $\lambda = \lambda_c$, the normal form equation is given by

$$\boxed{z'_{\tau} = [(\lambda - \lambda_c) + i\omega_0]z - uz|z|^2} \tag{15}$$

where ω_0 is the frequency of perturbations at the bifurcation point and both u and z are complex-valued. It can be shown that (15) is a particular embodiment of the *complex Ginzburg-Landau equation* (CGLE), a universal model that holds for all pattern forming systems undergoing a Hopf bifurcation [3, 7]. Furthermore, if $\omega_0 << 1$ and u assumes real values, (15) reduces to the more familiar *real Ginzburg-Landau equation* (RGLE). In turn, RGLE with a driving term h represents a universal model for critical behavior in statistical physics and derives from the most general free energy functional in $D-$ dimensions [2-3]

$$F[z] = \int d^D \varphi[z] \tag{16a}$$

where

$$\varphi[z] = a|z|^2 + b|z|^4 + c|\nabla z|^2 - hz + ... \tag{16b}$$

In summary, the outcome of this analysis is that the multivariable dynamics (9) and (10) reduces in the long-run to a lower dimensional system of universal equations with the emerging variable z playing the role of an effective order parameter. When perturbations are oscillatory, the end result is the GL-type equation in either complex or real-valued form.

5. From GL equations to self-organization and the ε - expansion

As unified model of behavior in the IR sector, GL equations lead to a number of remarkable consequences in statistical physics, condensed matter, field theories and the Standard Model of particle physics (SM). In this section we briefly focus on two of them, namely, the emergence of a *spacetime continuum with minimal fractality* via the $\varepsilon-$ expansion method and the onset of *self-organized criticality* (SOC).

5.1) The $\varepsilon-$expansion method is an explicit perturbative technique linked to the RG analysis for spacetime dimensions close to $D=4$ [2-3]. The Gaussian model of (16) corresponds to the case $b=h=0$. It arises in a natural way upon expanding the free energy (16a) about the most probable field and keeping only the lowest-order (quadratic) fluctuations of the order parameter z. Analysis shows that the Gaussian FP is *stable* for $D=4$ but it becomes *unstable* in $D<4$ dimensions upon adding a quartic correction to (16a) defined by $b\neq 0$. As a result, the Gaussian FP cannot describe critical behavior in less than four dimensions. The $\varepsilon-$expansion method of Wilson and Fisher resorts to an analytic continuation from integer D to continuous D defined by the dimensional parameter

$$\varepsilon = 4 - D << 1 \tag{17}$$

which implies the existence of a *non-trivial FP* besides the Gaussian FP at $\varepsilon=0$. The RG analysis is subsequently carried out in the neighborhood of this new FP, whose coordinates in parameter space are given by

$$a^* \sim O(\varepsilon) \tag{18a}$$

$$b^* \sim O(\varepsilon) \tag{18b}$$

5.2) It can be shown that GL equations lie at the heart of the so-called *sandpile model*, which is considered the prototypical example of SOC. The reader is directed to [4], for an in-depth discussion of this formal connection.

6. Emergence of chaos, multifractals, and effective field theory

Proceeding further, extensive analysis reveals that the onset of multifractals and multifractal measures, as well as the formal structure of effective field theories, consistently follow from SOC [8-9]. As alluded to in section 3, the close relationship between chaos, multifractals and effective field theory reflects the universal flow of complex dynamics towards *scale invariance* and *self-similarity* in the IR limit of field theory [see, e.g., 5].

7. Emergence of the minimal fractal manifold and Fractional Field Theory

As shown in [13] and in many other related contributions, the $\varepsilon-$expansion conjecture, along with the procedure of *dimensional regularization* of QFT, imply the existence of the *minimal fractal manifold* (MFM) structure of spacetime near the Fermi scale. Using fractional derivative and integral operators to characterize the onset of complex dynamics near this scale, necessarily leads to the framework of *Fractional Field Theory*.

Rather surprisingly, we have found over the years that complex dynamics provides helpful insights on the many unsolved challenges confronting SM and beyond. The interested reader is invited to peruse [14] for technical details and clarifications.

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