Mnll, Psll and Prll

Three rules for democratic decision making

RICARDO ALVIRA BAEZA

Abstract

This is an -slightly updated- excerpt of the book 'Mnll, Psll and Prll. Three rules for democratic decision making', published in 2016 (currently undergoing review). Minor adaptations have been incorporated. It was reviewed by vitoriano Ramirez in septembrer 2019, as member of my PhD Dissertation Committee.

ACKNOWLEDGEMENTS. This project is the result of an extensive work which has been possible thanks to the support and patience of Monica Vilhelm. It has also benefitted from the helpful comments of Nicolaus Tideman, Hannu Nurmi and Markus Schulze. Besides those mentioned, this work would not have been possible without the previous work of all those listed in References. However, the author is to be considered the sole responsible for the proposal/statements included in the text.

SUMMARY

This book explains a family of three Condorcet-consistent rules which allow us to make most frequent and important collective decisions [to better explain how each rule works, examples are provided]:

- Mnll, allows us complete ordering of eligible options. For illustrating its use, a review of Catalonia independence referendum is provided which shows many interesting issues.

- Psll, builds on Mnll. By introducing a numeric valuation of the collective preference over each option, it allows us locating them in the range 0-1. This is highly interesting for a class of very important collective decisions: referendums [it enables substituting turnout thresholds by preference thresholds]. For illustrating its use, a review of Brexit referendum is provided.

- Prll, builds on Psll. By introducing proportionality, it allows us building representative bodies. This becomes highly interesting for a class of important collective decisions: elections to parliaments. By shifting the proportionality from first-votes [most preferred choices] to aggregated preference [complete orderings], Prll enables assigning seats in Parliaments in a Condorcet consistent manner, ensuring the most preferred party wins more seats.

We believe many of our societies' most important current problems come from the use of incorrect collective choice rules leading to choices which are not the most preferred by citizens, or to representative chambers, which decisions do not match those alleged citizens would make.

These three rules are our proposal for advancing towards more democratic political systems, which most likely are the first step towards better societies.

TABLE OF CONTENTS

2 MNLL: A RULE FOR DETECTING COLLECTIVE PREFERENCE ORDERINGS

2.2.1 COMPUTING PROCEDURE

Mnll algorithm operates this way:

First, we build a pairwise comparison matrix, i.e., a matrix comparing, for every pair of options, the number of individuals supporting one option against the other.

Secondly, we draw a Matrix of Tie-margins [MoT], which tells us for each pairwise comparison how many votes each option needs to tie with every other option.

Third, we calculate for each option:

- its worst defeat
- its biggest victory

Fourth, we compare the worst defeat of every option with the biggest victory for every option, and we select the option with the lowest value[1]:

- it may be an option with the Lowest Worst Defeat [LWD], therefore it is an option more preferred than any other option
- It may be an option with the Smallest Biggest Victory [SBV], and then it is an option less preferred than any other option.

We locate the selected option in relation to the rest of eligible options; we remove it from the choice set and apply the algorithm again until every eligible option's position in the ordering has been set. A perfectly linear ordering comprising every option is obtained consistent with every non-rebuttable Voting criterion.

Let us see it by an example of a set of preferences

2.2.2 ORDERING A SET OF OPTIONS USING MNLL

Let us consider the following set of preferences [Tideman, 1987, Example 5]. Since the choice involves five options, we can solve it using the simplified procedure.

$7 \quad x_1 \; P \; x_2 \; P \; x_3 \; P \; x_4 \; P \; x_5$

$3 \quad x_5 \; P \; x_4 \; P \; x_1 \; P \; x_2 \; P \; x_3$

$6 \quad x_4 \; P \; x_5 \; P \; x_2 \; P \; x_3 \; P \; x_1$

$3 \quad x_2 \; P \; x_3 \; P \; x_1 \; P \; x_5 \; P \; x_4$

$5 \quad x_5 \; P \; x_3 \; P \; x_1 \; P \; x_2 \; P \; x_4$

$3 \quad x_4 \; P \; x_3 \; P \; x_1 \; P \; x_2 \; P \; x_5$

If we pairwise compare the relative preference for each option, we obtain the following Pairwise Comparison Matrix PCM [light pink cells are defeats, and green cells are victories]

[1] If there are two options tied for the same position, then we review their individual confrontations, since both options may be equally preferred or not.

	$d[\ldots,x_1]$	$d[\ldots,x_2]$	$d[\ldots,x_3]$	$d[\ldots,x_4]$	$d[\ldots,x_5]$
$d[x_1,\ldots]$	-	**18**	10	**15**	13
$d[x_2,\ldots]$	9	-	**19**	**15**	13
$d[x_3,\ldots]$	**17**	8	-	**15**	13
$d[x_4,\ldots]$	12	12	12	-	**16**
$d[x_5,\ldots]$	**14**	**14**	**14**	11	-

From above PCM, we obtain the MoT by the following transformation. Only the votes each option requires to tie each other option are accounted. Hence, when an option beats another option [green cell] its score is 0, while the beaten option [pink cell] has a positive score equal to the difference in votes. If two options tie, both their scores are 0. Every cell that was a defeat will have a positive score, while every cell which was a victory/tie will have a zero score[2]:

	$v[\ldots,x_1]$	$v[\ldots,x_2]$	$v[\ldots,x_3]$	$v[\ldots,x_4]$	$v[\ldots,x_5]$
$v[x_1,\ldots]$	-	**0**	7	**0**	1
$v[x_2,\ldots]$	9	-	**0**	**0**	1
$v[x_3,\ldots]$	**0**	11	-	**0**	1
$v[x_4,\ldots]$	3	3	3		**0**
$v[x_5,\ldots]$	**0**	**0**	**0**	5	

From the MoT, we can obtain a value summarizing how much an option is preferred and how much an option is rejected:

- The maximum value of each row informs us of how many votes each x_i option needs to tie each other option; it is a measure of each option's Worst Defeat. It is a measure of how much each x_i option is 'desired' [blue color cells in table below]. The lower the number of votes, the greater the option is desired.
- The maximum value of each column informs us of how many votes needs every option to tie one specific x_i option; it is a measure of each option's Biggest Victory. It is a measure of how much each x_i option is rejected [red color cells in table below]. The lower the number of votes, the greater the rejection.

	$v[\ldots,x_1]$	$v[\ldots,x_2]$	$v[\ldots,x_3]$	$v[\ldots,x_4]$	$v[\ldots,x_5]$	WORST DEFEATS
$v[x_1,\ldots]$	-	**0**	7	**0**	1	7
$v[x_2,\ldots]$	9	-	**0**	**0**	1	9
$v[x_3,\ldots]$	**0**	11	-	**0**	1	11
$v[x_4,\ldots]$	3	3	3		**0**	3
$v[x_5,\ldots]$	**0**	**0**	**0**	5		5
BIGGEST VICTORIES	9	11	7	5	1	

[2] As a criterion to differentiate victories from ties, we highlight ties in yellow [however, there are no ties in this example]

And the minimum value of all [minim-all] of them [Least Worst Defeat/Smallest Biggest Victory] informs us of individuals' highest intensity in preference/rejection over the set of options. For clarity, it is useful to indicate it in a cell [Least Worst Defeat in the cell on top of them, and Smallest Biggest Victory in the cell at the right of them], so we can directly compare them both.

We highlight both cells in light grey in table below:

	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	v[…,x_5]	3	LEAST WORST DEFEAT
v[x_1,…]	-	**0**	7	**0**	1	7	
v[x_2,…]	9	-	**0**	**0**	1	9	
v[x_3,…]	**0**	11	-	**0**	1	11	WORST DEFEATS
v[x_4,…]	3	3	3		**0**	3	
v[x_5,…]	**0**	**0**	**0**	5		5	
	9	11	7	5	1	1	
	BIGGEST VICTORIES					SMALLEST BIGGEST VICTORY	

We see in the example Smallest Biggest Victory is 1 [for x_5 column] and Least Worst Defeat is 3 [for x_4 row], so the highest preference is rejection/lack of preference for option x_5, which therefore is less desired than any other option. Hence, we can position it in relation to the rest of the options, locating it in a position of lower preference:

$$\{x_1, x_2, x_3, x_4\} \succ x_5 \tag{1}$$

We remove x_5 from the set, and we proceed again. To continue ordering the options we just copy the above table, remove the row and column belonging to x_5 and recalculate the maximum of each row/column, and minimum values of them all:

	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	3
v[x_1,…]	-	**0**	7	**0**	7
v[x_2,…]	9	-	**0**	**0**	9
v[x_3,…]	**0**	11	-	**0**	11
v[x_4,…]	3	3	3		3
	9	11	7	0	0

We see the SBV is 0 [for x_4 column] and LWD is 3 [for x_4 row], so the highest preference is rejection for option x_4, which therefore is less desired than any other remaining option. Hence, we can order it in relation to the rest of the options, locating it in a position of lower preference:

$$\{x_1, x_2, x_3\} \succ x_4 \succ x_5 \tag{2}$$

We remove x_4 and proceed again….

	v[…,x_1]	v[…,x_2]	v[…,x_3]	7
v[x_1,…]	-	**0**	7	7
v[x_2,…]	9	-	**0**	9
v[x_3,…]	**0**	11	-	11
	9	11	7	7

When we only have three options, we can determine the ordering at ones. We determine the most preferred option in the subset as that with the LWD [in this case is x_1] and the least preferred option in the subset as that with the SBV [in this case is x_3]. Therefore:

$$x_1 \succ x_2 \succ x_3 \tag{3}$$

And we can write the complete ordering of the options as:

$$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \tag{4}$$

3 Psll: A RULE FOR COMPUTING OPTIONS' COLLECTIVE PREFERENCE

3.2 PSLL AS A RULE FOR ASSIGNING COLLECTIVE PREFERENCE TO OPTIONS

Psll can be understood as a 'plug-in' to be used after Mnll has been used; i.e., after the complete ordering of the options has been already established.

The most complicate case is when cyclic relative preference relations appear, since positioning the options involved in the cycle requires some rules not needed for positioning orderings without cycles. For clarity, we first review the case without cycles using an example for easier understanding.

3.2.1 POSITIONING A CONDORCET LINEAR ORDERING

Mnll is an iterative procedure. It operates by successive steps, detecting the most/least preferred option at each step and generating as consequence a reduced choice subset with the remaining eligible options. The procedure ends when every option in the initial choice set has been ordered, i.e., at the step where the choice subset that would be generated would be empty.

We can differentiate two types of measures we need to calculate:

- First, we need to measure the difference in collective preference between all options
- Secondly, we need to measure the difference in collective preference from the highest possible value [1] to the most preferred option, and from the lowest possible value [0] to the least preferred option. In other terms, we need to measure the distance of most/least preferred options to the limiting values [1 and 0].

Let us explain how we measure each of them.

When we detect there is a most preferred or least preferred option in a subset, we calculate how much it is more or less preferred that the rest of eligible options in the subset:

- If it is a *most preferred option*, the value of its row [Worst Defeat] is the minimum value of the rows of all eligible options. We calculate the minimum value of the other rows [next most preferred option in the subset] and subtract the value of the row of the selected option. The result represents the desirability differential between said option and the next most desired option in such subset.

Most Preferred

$$\Delta d[x_i - x_{\neg i}] = min\big[rows[x_{\neg i}]\big] - row[x_i] \qquad (0)$$

- If it is a *least preferred option*, the value of its column [Best Victory] is the minimum value of the columns of all eligible options. We calculate the minimum value of the other columns [next least preferred option in the subset] and subtract the value of the column of the selected option. The value is the desirability differential between said option and the next least desired option in the subset.

Least Preferred

$$\Delta d\big[x_{\neg j} - x_j\big] = min\left[columns\big[x_{\neg j}\big]\right] - column\big[x_j\big] \qquad (1)$$

We proceed iteratively until the desirability difference between all options has been calculated. For clarity, let us review as example the following individuals' preferences [Dodgson, 1884: 31]:

21.840	x_1	P	x_2	P	x_4
10.160	x_1	P	x_3	P	x_2
7.999	x_5				

First, we draw the pairwise comparison matrix:

	d[…,x_1]	d[…,x_2]	d[…,x_3]	d[…,x_4]	d[…,x_5]
d[x_1,…]	-	**32.000**	**32.000**	**32.000**	**32.000**
d[x_2,…]	0	-	**21.840**	**32.000**	**32.000**
d[x_3,…]	0	10.160	-	10.160	**10.160**
d[x_4,…]	0	0	**21.840**	-	**21.840**
d[x_5,…]	7.999	7.999	7.999	7.999	-

From this matrix, we calculate the MoT:

	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	v[…,x_5]	0	There is Condorcet Winner
v[x_1,…]	-	**0**	**0**	**0**	**0**	0	x_1
v[x_2,…]	32.000	-	**0**	**0**	**0**	32.000	
v[x_3,…]	32.000	11.680	-	11.680	**0**	32.000	
v[x_4,…]	32.000	32.000	**0**		**0**	32.000	
v[x_5,…]	24.001	24.001	2.161	13.841		24.001	
	32.000	32.000	2.161	13.841	0	**0**	There is Condorcet Loser
					x_5		

We see option x_1 is a Condorcet winner in the subset [it is individually preferred to any other option belonging to the choice subset at this step], and x_5 option is a Condorcet loser in the subset [it is individually less preferred than any other option belonging to the choice subset at this step]. But besides ordering the options, we want to locate them in the 0-1 range. In order to do so, first we need to calculate how much is x_1 more desired and x_5 less desired than the other eligible options at this step.

- As higher x_1's collective desirability in relation to the rest of eligible options, we calculate the difference between x_1's WD and the LWD of the other eligible options [x_2-x_5 options]

x_1

$$\Delta d[x_1 - x_{\neg 1}] = min\left[WD[x_{\neg 1} = x_2, x_3, x_4, x_5]\right] - WD[x_1] \tag{2}$$

$$\Delta d[x_1 - x_{\neg 1}] = 24.001 - 0 = 24.001 \tag{3}$$

- As lower x_5's collective desirability in relation to the rest of eligible options, we calculate the difference between x_5's BV and the LBV of the other eligible options [x_1-x_4 options].

x_1

$$\Delta d[x_{\neg 5} - x_5] = min\left[BV[x_{\neg 5} = x_1, x_2, x_3, x_4]\right] - BV[x_5] \tag{4}$$

$$\Delta d[x_{\neg 5} - x_5] = 2.161 - 0 = 2.161 \tag{5}$$

Hence:

x_1	$\succ$	x_2,x_3,x_4	$\succ$	x_5
	24.001		2.161	

We remove options x_1 and x_5 from the choice space and review again:

	v[…,x_2]	v[…,x_3]	v[…,x_4]	0	There is Condorcet Winner
v[x_2,…]	-	**0**	**0**	0	x_2
v[x_3,…]	11.680	-	11.680	11.680	
v[x_4,…]	32.000	**0**		32.000	
	32.000	0	11.680	**0**	There is Condorcet Loser
		x_3			

We see option x_2 is a Condorcet winner in the subset [it is individually preferred to any other option in the choice subset at this step], and x_3 option is a Condorcet loser in the subset [it is individually less preferred to any other option in the choice subset at this step]. Let us calculate the desirability differentials:

- As higher x_2's collective desirability in relation to the rest of eligible options, we calculate the difference between x_2's WD and the LWD of the other eligible options [x_3-x_4 options]

x_1
$$\Delta d[x_2 - x_{\neg 2}] = min\big[WD[x_{\neg 2} = x_3, x_4]\big] - WD[x_2] \tag{6}$$

$$\Delta d[x_2 - x_{\neg 2}] = 11.680 - 0 = 11.680 \tag{7}$$

- As lower x_3's collective desirability in relation to the rest of eligible options, we calculate the difference between x_3's BV and the LBV of the other eligible options [x_2 & x_4 options].

x_1
$$\Delta d[x_{\neg 3} - x_3] = min\big[BV[x_{\neg 3} = x_2, x_4]\big] - BV[x_3] \tag{8}$$

$$\Delta d[x_{\neg 3} - x_3] = 11.680 - 0 = 11.680 \tag{9}$$

Hence:

x_2	$\succ$	x_4	$\succ$	x_3
	11.680		11.680	

Therefore, we already know the complete ordering of the options according to their collective preference, and the collective desirability differentials among them:

x_1	$\succ$	x_2	$\succ$	x_4	$\succ$	x_3	$\succ$	x_5
	24.001		11.680		11.680		2.161	

However, we do not yet have all the information necessary to know how much each option is collectively desired/non-desired. We need to calculate two parameters:

- How much not desired is the most preferred option [in this example x_1]. This value represents the degree of collective's *lack of consensus* for the decision.
- How much desired is the least preferred option [in the example x_5]. This value represents the degree to which the least preferred option is more desired than any other option not assessed for the choice [i.e. which does not belong to X].

We calculate the first parameter as the number of individuals expressing preferences about any option minus the maximum number of individuals supporting the most preferred option [x_1] in any pairwise confrontation with any other eligible option[3]. In this case we obtain

Distance to 1
$$\Delta d[1 - x_1] = N - max\left[n[x_1 \succ x_j]_{j=2-5}\right] = 39.999 - 32.000 = 7.999 \quad (10)$$

We calculate the second parameter as the maximum number of individuals who support the least preferred option [x_5] against any eligible option at the step [choice subset] where the least preferred option is selected. In this case we obtain:

Distance to 0
$$\Delta d[x_5 - 0] = max\left[n[x_5 \succ x_j]_{j=1-4}\right] = 7.999 \quad (11)$$

Therefore, desirability differentials are:

	x_1	$\succ$	x_2	$\succ$	x_4	$\succ$	x_3	$\succ$	x_5	$\succ$	¬X
7.999		24.001		11.680		11.680		2.161		7.999	

Now we can position each option in the 0-1 range by simply dividing the sum of their desirability differentials starting from ¬X [equivalent to 0] up to the option whose collective desirability we want to calculate by the sum of every desirability differential [from 1 to 0]

$$x_1 = \frac{24.001 + 11.680 + 11.680 + 2.161 + 7.999}{7.999 + 24.001 + 11.680 + 11.680 + 2.161 + 7.999} = 0{,}88$$

$$x_2 = \frac{11.680 + 11.680 + 2.161 + 7.999}{7.999 + 24.001 + 11.680 + 11.680 + 2.161 + 7.999} = 0{,}51$$

$$x_3 = \frac{11.680 + 2.161 + 7.999}{7.999 + 24.001 + 11.680 + 11.680 + 2.161 + 7.999} = 0{,}33$$

$$x_4 = \frac{2.161 + 7.999}{7.999 + 24.001 + 11.680 + 11.680 + 2.161 + 7.999} = 0{,}16$$

$$x_5 = \frac{7.999}{7.999 + 24.001 + 11.680 + 11.680 + 2.161 + 7.999} = 0{,}12$$

[3] If two options tied for the highest preference, we calculate their average value.

We can summarize above values in the table:

	x_1	P	x_2	P	x_4	P	x_3	P	x_5	P	¬X
PREFERENCE INTENSITY	0,878		0,512		0,333		0,155		0,122		

We have calculated the position of each option, and **to provide some contrast of above values, let us calculate each option's collective desirability according Borda's rule.** For clarity, it is convenient to explicit every option's position in each ordering [also those options whose position is not explicitly stated by the individuals], as well as stating the corresponding Borda points for each position in each ordering. Since the last ordering explicit a lower number of options, its 'preference' scale differs:

	3		2		1		0
21.840	x_1	P	x_2	P	x_4	P	x_3I x_5
10.160	x_1	P	x_3	P	x_2	P	x_4I x_5
7.999	x_5	P					x_1I x_2I x_3I x_4
39.999							

Then we calculate the Borda Score for each option:

	BORDA SCORE	
	POINTS	TOTAL
x_1	*=21.840*3+10.160*3+7.999*0=*	96.000
x_2	*=21.840*2+10.160*1+7.999*0=*	53.840
x_3	*=21.840*0+10.160*2+7.999*0=*	20.320
x_4	*=21.840*1+10.160*0+7.999*0=*	21.840
x_5	*=21.840*0+10.160*0+7.999*1=*	7.999

We calculate the maximum possible score any option could obtain; i.e., its score if it was declared most preferred choice by each individual:

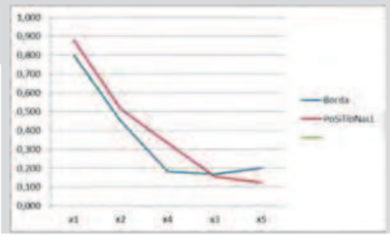
Maximum borda score $$= 39.999 * 3 \ = 119.997 \tag{12}$$

And we can now easily calculate each option's position in the range 0-1 by dividing its score by the maximum possible score:

	BORDA RULE	
	SCORE	POSITION
x_1	*=96.000/119.997=*	0,800
x_2	*=53.840/119.997=*	0,449
x_3	*=20.320/119.997=*	0,169
x_4	*=21.840/119.997=*	0,182
x_5	*=7.999/119.997=*	0,200

Now we can compare the values obtained using Borda rule with those obtained using Psll:

PREFERENCE INTENSITY /COLLECTIVE UTILITY			
	Borda	Psll	Standard Deviation
x_1	0,800	0,878	0,039
x_2	0,449	0,512	0,031
x_4	0,169	0,155	0,007
x_3	0,182	0,333	0,076
x_5	0,200	0,122	0,039



We arrive at an average standard deviation of 0.038 and a Pearson Correlation of 0.96. From a statistical perspective, both rules are measuring the same phenomenon/variable [collective utility].

We see positioning a linear ordering can be easily done. Let us now review the procedure for orderings containing cycles, which requires explaining some additional criteria[4].

3.2.2 POSITIONING A CONDORCET CYCLIC ORDERING

Positioning options involved in a cycle may present higher difficulty. The reason is that if when applying Mnll at some point there is neither Condorcet Winner nor Condorcet Loser, from that point on, the way to measure options' collective utility should be slightly modified.

The collective preference over any option which is selected after a cycle has been detected shall be measured on the step where the cycle was first detected, yet not taking into account options already chosen before the option whose collective preference is being calculated.

In addition, we can find some 'anomalies' which is convenient to review. Mnll builds on two ideas:

- If we arrange options from the one with Lowest Worst Defeat to the one with the Highest Worst Defeat, the ordering should match their ordering according to collective preference ordering.
- If we arrange options from the one with Highest Biggest Victory to the one with Lowest Biggest Victory, the ordering should match their ordering according to collective preference.

Yet when for some cases where there are cycles inside cycles or in some specific situations where there are Condorcet Losers and Smith Sets, we may find these three orderings do not completely match. Some options may appear not matching the collective ordering, and these options shall not be considered for calculating the collective utility assigned to the other options.

For better understanding it, let us review some examples, from easiest to more difficult, carefully explaining them step-by-step so there is no doubt regarding how the procedure is to be applied.

[4] In a strict sense, the procedure we explain below is the general procedure, while the procedure we revised above is a particular case where there is Condorcet Winner and Loser at every step.

A FAMOUS CYCLE: CONDORCET'S PARADOX

Let us start by reviewing the set of individuals' preferences Condorcet used to illustrate the incorrectness of usual conceptualization of majority rule [Condorcet, 1785:lxj]:

23	x_1	P	x_2	P	x_3
17	x_2	P	x_3	P	x_1
2	x_2	P	x_1	P	x_3
10	x_3	P	x_1	P	x_2
8	x_3	P	x_2	P	x_1
60					

If we review the relative preference between each pair of options:

	n[…,x_1]	n[…,x_2]	n[…,x_3]
n[x_1,…]	-	**33**	25
n[x_2,…]	27	-	**42**
n[x_3,…]	**35**	18	-

Therefore, using Condorcet Winner criterion, we arrive to a cyclic preference relation:

$$\dots x_1 \succ x_2 \succ x_3 \dots \tag{13}$$

If we use Mnll :

	v[…,x_1]	v[…,x_2]	v[…,x_3]	7	There is no Condorcet Winner
v[x_1,…]	-	**0**	11	11	
v[x_2,…]	7	-	**0**	7	x_2 Most preferred option
v[x_3,…]	**0**	25	-	25	
	7	25	11	7	There is no Condorcet Loser
	x_1		Least preferred option		

We have only three options so both criteria are dominant, being x_2 the most preferred option, x_1 the least preferred option, and x_3 being located at the intermediate position:

$$x_2 \succ x_3 \succ x_1 \tag{14}$$

Let us now position the three options over a 1-0 scale:

- the most preferred option is x_2, and we can calculate its higher desirability in relation to the rest of eligible options [¬x_2] as:

x_2

$$\Delta d[x_2 - x_{\neg 2}] = min\big[WD[x_{\neg 2} = x_1, x_3]\big] - WD[x_2] \tag{15}$$

$$\Delta d[x_2 - x_{\neg 2}] = 11 - 7 = 4 \tag{16}$$

- the least preferred option is x_1, and we can calculate its lower desirability in relation to the rest of eligible options [$\neg x_1$] as:

x_1
$$\Delta d[x_{\neg 1} - x_1] = min\big[BV[x_{\neg 1} = x_2, x_3]\big] - BV[x_1] \tag{17}$$

$$\Delta d[x_{\neg 1} - x_1] = 11 - 7 = 4 \tag{18}$$

Therefore:

x_2	$\succ$	x_3	$\succ$	x_1
	4,0		4,0	

Now we need to calculate the distance to the limiting points 1 and 0, which we do in relation to most/least preferred options:

- Distance to 1. We calculate it as the number of individuals expressing preferences about any option [including blank votes if they were] minus the maximum number of individuals supporting the most preferred option [x_2] in any pairwise confrontation with any other eligible option. In this case we obtain:

Distance to 1
$$\Delta d[1 - x_2] = N - max\left[n[x_2 \succ x_j]_{j=1,3}\right] = 60 - 42 = 18 \tag{19}$$

- Distance to 0. We calculate it as the maximum number of individuals who support the least preferred option [x_1] against any eligible option [including those individuals who explicit indifference among all eligible options].

Distance to 0
$$\Delta d[x_1 - 0] = max\left[n[x_1 \succ x_j]_{j=2,3}\right] = 33 \tag{20}$$

Therefore desirability differentials are:

	x_2	P	x_3	P	x_1		$\neg X$
18,0		4,0		4,0		33,0	

And we can position options in the 0-1 range by simply dividing the sum of their desirability differentials starting from ¬X [equivalent to 0] up to the option which position we want to calculate by the sum of every desirability differential [from 1 to 0]

$$x_2 = \frac{4 + 4 + 33}{18 + 4 + 4 + 33} = 0{,}695$$

$$x_3 = \frac{4 + 33}{18 + 4 + 4 + 33} = 0{,}627$$

$$x_1 = \frac{33}{18+4+4+33} = 0{,}559$$

The review of this set of individuals' preferences, which gave rise to Condorcet's Paradox, allows us to see something interesting. If every individual expresses a strict preference ordering comprising every option, the emergence of a cycle comprising every eligible option implies every option has a collective preference above 0.5; i.e., *a top cycle implies every option is collectively more preferred than non-preferred.*

Let us again provide some contrast by calculating each option's collective desirability using Borda's rule. First, we state our measuring scale:

	2		1		0
23	x_1	P	x_2	P	x_3
17	x_2	P	x_3	P	x_1
2	x_2	P	x_1	P	x_3
10	x_3	P	x_1	P	x_2
8	x_3	P	x_2	P	x_1
60					

Then we calculate the Borda Score for each option:

	BORDA SCORE	
		TOTAL
x_1	*=23*2+17*0+2*1+10*1+8*0=*	58
x_2	*=23*1+17*2+2*2+10*0+8*1=*	69
x_3	*=23*0+17*1+2*0+10*2+8*2=*	53

We calculate the maximum possible score that any option could obtain; i.e., its score if it was declared most preferred choice by each individual:

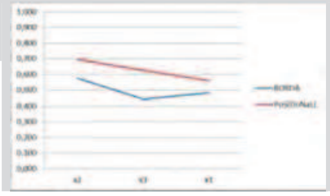
$$\text{Maximum Borda score} = 60 * 2 = 120 \tag{21}$$

And we can now easily calculate each option's position in the range 0-1 by dividing its score by the maximum possible score:

	BORDA RULE	
	SCORE	POSITION
x_1	*=58/120=*	0,483
x_2	*=69/120=*	0,575
x_3	*=53/120=*	0,442

Now we can compare the values obtained using Borda rule with those obtained using PsII, we see:

PREFERENCE INTENSITY /COLLECTIVE UTILITY			
	Borda	Psll	Standard Deviation
x_2	0,575	0,695	0,08
x_3	0,442	0,627	0,13
x_1	0,483	0,559	0,05



We arrive at an average deviation of 0.09 while the Pearson Correlation reduces to 0.67 [Borda switches x_3's and x_1's position in relation to their ordering under Mnll /Psll].

Let us now review two more complicated cycles, where Mnll does not match Ranked Pairs and Condorcet Hare. This review allows us to arrive to some interesting conclusions we state later.

TWO EXAMPLES WHERE MNLL DOES NOT MATCH RANKED PAIRS AND CONDORCET HARE

EXAMPLE 01: MNLL DOES NOT MATCH RANKED PAIRS

Let us consider the following set of individuals' preferences [Schulze, 2016:15. Example 1]:

8	x_1	P	x_3	P	x_4	P	x_2
2	x_2	P	x_1	P	x_4	P	x_3
4	x_3	P	x_4	P	x_2	P	x_1
4	x_4	P	x_2	P	x_1	P	x_3
3	x_4	P	x_3	P	x_2	P	x_1

If we pairwise compare the relative preference between options:

	n[…,x_1]	n[…,x_2]	n[…,x_3]	n[…,x_4]
n[x_1,…]	-	8	**14**	10
n[x_2,…]	**13**	-	6	2
n[x_3,…]	7	**15**	-	**12**
n[x_4,…]	**11**	**19**	9	-

Hence, we arrive to an intransitive ordering:

$$\ldots x_1 \succ x_3 \succ x_4 \succ x_2 \succ x_1 \ldots \tag{22}$$

If we use Ranked Pairs rule, we arrive to the collective ordering [Schulze, 2016; Alvira, 2015]:

$$x_1 \succ x_3 \succ x_4 \succ x_2 \tag{23}$$

Let us solve it using Mnll:

STEP 01	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	3	There is no Condorcet Winner
v[x_1,…]	-	5	**0**	1	5	
v[x_2,…]	**0**	-	9	17	17	
v[x_3,…]	7	**0**	-	**0**	7	
v[x_4,…]	**0**	**0**	3		3	x_4 Most Preferred Option
	7	5	9	17	5	There is no Condorcet Loser

The dominant criterion is 'most preferred option', from which it follows x_4 is preferred to x_1-x_3

$$x_4 \succ \{x_1, x_2, x_3\} \tag{24}$$

We remove x_4 from the set of eligible options and review again:

STEP 02	v[…,x_1]	v[…,x_2]	v[…,x_3]	5	There is no Condorcet Winner
v[x_1,…]	-	5	**0**	5	x_1 Most Preferred Option
v[x_2,…]	**0**	-	9	9	
v[x_3,…]	7	**0**	-	7	
	7	5	9	5	There is no Condorcet Loser
		x_2			Least Preferred Option

We know x_1 is the most preferred option within the subset and x_2 is the least preferred option, so the complete ordering options is:

$$x_4 \succ x_1 \succ x_3 \succ x_2 \tag{25}$$

Now, let us position the options on the range 0-1. In the first step we selected option x_4 as most preferred option, hence its collective desirability differential regarding $x_{\neg 4}$ is:

x_4

$$\Delta d[x_4 - x_{\neg 4}] = min\big[WD[x_{\neg 4} = x_1, x_2, x_3]\big] - WD[x_4] \tag{26}$$

$$\Delta d[x_4 - x_{\neg 4}] = 5 - 3 = 2 \tag{27}$$

In Step 02, we select x_1 as most preferred option and x_2 as least preferred option. But since the cycle started at Step 01, we have to calculate their higher/lower desirability at Step 01 [yet not considering option x_4 values]:

- the most preferred option at Step 02 is x_1, and we can calculate its higher desirability in relation to the rest of eligible options [¬x_1] as:

x_1

$$\Delta d[x_1 - x_{\neg 1}] = min\big[WD[x_{\neg 1} = x_2, x_3]\big] - WD[x_1] \tag{28}$$

$$\Delta d[x_1 - x_{\neg 1}] = 7 - 5 = 2 \tag{29}$$

- the least preferred option at Step 02 is x_2, and we can calculate its lower desirability in relation to the rest of eligible options [¬x_2] as:

x_1
$$\Delta d[x_{\neg 2} - x_2] = min\left[BV[x_{\neg 2} = x_1, x_3]\right] - BV[x_2] \tag{30}$$
$$\Delta d[x_{\neg 2} - x_2] = 7 - 5 = 2 \tag{31}$$

Therefore:

x_4	$\succ$	x_1	$\succ$	x_3	$\succ$	x_2
	2,0		2,0		2,0	

Now we calculate most/least preferred options' distance to the limiting points 1 and 0:

- Distance to 1. We calculate it as the number of individuals expressing preferences about any option minus the maximum number of individuals supporting the most preferred option [x_4] in any pairwise confrontation with any other eligible option. In this case we obtain:

Distance to 1
$$\Delta d[1 - x_4] = N - max\left[n[x_4 \succ x_j]_{j=1-3}\right] = 21 - 19 = 2 \tag{32}$$

- Distance to 0. We calculate it as the maximum number of individuals who support the least preferred option [x_2] against any eligible option at the step where this option is eliminated[5]:

Distance to 0
$$\Delta d[x_2 - 0] = max\left[n[x_2 \succ x_j]_{j=1,3,4}\right] = 13 \tag{33}$$

Therefore desirability differentials are:

	x_4	$\succ$	x_1	$\succ$	x_3	$\succ$	x_2		¬X
2,0		2,0		2,0		2,0		13,0	

And we can easily position options in the 0-1 range by dividing the sum of their desirability differentials starting from ¬X [equivalent to 0] up to the option which position we want to calculate by the sum of every desirability differential [from 1 to 0]

$$x_4 = \frac{2+2+2+13}{2+2+2+2+13} = 0{,}905$$

$$x_1 = \frac{2+2+13}{2+2+2+2+13} = 0{,}810$$

$$x_3 = \frac{2+13}{2+2+2+2+13} = 0{,}714$$

$$x_2 = \frac{13}{2+2+2+2+13} = 0{,}619$$

Let us summarize above results in a table:

---

[5] Actually, option x2 is eliminated at Step 02, but it is involved in a cyclic preference relation starting at Step 01, so in fact it is necessary to review its maximum support against any other option at step 01. However in this case both values match.

	x_4	$\succ$	x_1	$\succ$	x_3	$\succ$	x_2
PREFERENCE INTENSITY/COLLECTIVE UTILITY	0,905		0,810		0,714		0,619

Let us provide some contrast to above results by calculating each option's collective preference using Borda's rule. Since every ordering comprises all the options, all scales are equal:

	3		2		1		0
8	x_1	$\succ$	x_3	$\succ$	x_4	$\succ$	x_2
2	x_2	$\succ$	x_1	$\succ$	x_4	$\succ$	x_3
4	x_3	$\succ$	x_4	$\succ$	x_2	$\succ$	x_1
4	x_4	$\succ$	x_2	$\succ$	x_1	$\succ$	x_3
3	x_4	$\succ$	x_3	$\succ$	x_2	$\succ$	x_1
21							

Then we calculate the Borda Score for each option:

BORDA SCORE		
		TOTAL
x_1	*=8*3+2*2+4*0+4*1+3*0=*	32
x_2	*=8*0+2*3+4*1+4*2+3*1=*	21
x_3	*=8*2+2*0+4*3+4*0+3*2=*	34
x_4	*=8*1+2*1+4*2+4*3+3*4=*	39

We calculate the maximum possible score any option could obtain; i.e., its score if it was declared most preferred choice by each individual:

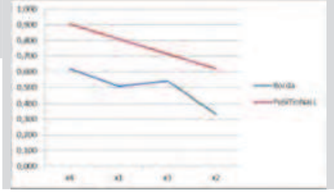
Maximum borda score $$= 21 * 3 = 63 \tag{34}$$

And we can now easily calculate each option's position in the range 0-1 by dividing its score by the maximum possible score:

BORDA RULE		
	SCORE	POSITION
x_1	*=32/63=*	0,508
x_2	*=21/63=*	0,333
x_3	*=34/63=*	0,540
x_4	*=39/63=*	0,619

Now we can compare the values obtained using Borda rule with those obtained using PsII [we arrange options according to ordering obtained using PsII]:

PREFERENCE INTENSITY / COLLECTIVE UTILITY			
	Borda	Psll	Standard Deviation
x_4	0,619	0,905	0,202
x_1	0,508	0,810	0,213
x_3	0,540	0,714	0,123
x_2	0,333	0,619	0,202



While the deviation between collective utility assignments by both rules increases [average deviation of 0.185] still the overall assignment is highly related [Pearson=0.88]. While both rules choose the same most/least preferred options, intermediate options are switched in Borda ordering, and only Psll assigns collective utility above 0,50 threshold in all cases.

EXAMPLE 02: MNLL DOES NOT MATCH CONDORCET HARE

Let us consider the following preferences [Alvira, 2016]:

19 x_1 P x_2 P x_3 P x_4

18 x_2 P x_3 P x_4 P x_1

20 x_3 P x_4 P x_1 P x_2

21 x_4 P x_1 P x_2 P x_3

78

If we review the options by pairwise comparing their relative preference we see that:

	n[…,x_1]	n[…,x_2]	n[…,x_3]	n[…,x_4]
n[x_1,…]	-	**60**	**40**	19
n[x_2,…]	18	-	**58**	37
n[x_3,…]	38	20	-	**57**
n[x_4,…]	**59**	**41**	21	-

We arrive to a cyclical relationship of relative preference between the four options:

$$\ldots x_1 P x_2 P x_3 P x_4 P x_1 \ldots. \quad (35)$$

If we use Condorcet Hare, we arrive to the collective ordering:

$$x_3 P x_4 P x_1 P x_2 \quad (36)$$

If we use Mnll:

STEP 01	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	36		There is no Condorcet Winner
v[x_1,…]	-	0	0	40	40		
v[x_2,…]	42	-	0	4	42		
v[x_3,…]	2	38	-	0	38		
v[x_4,…]	0	0	36		36	x_4	Most preferred option
	42	38	36	40	36		There is no Condorcet Loser

There is neither Condorcet Winner nor Loser, so we follow the dominant criterion [most preferred option] and choose x_4 as most preferred option. We eliminate it from the choice set and review again:

STEP 02	v[…,x_1]	v[…,x_2]	v[…,x_3]	0	There is Condorcet Winner
v[x_1,…]	-	0	0	0	x_1
v[x_2,…]	42	-	0	42	
v[x_3,…]	2	38	-	38	
	42	38	0	0	There is Condorcet Loser
			x_3		

There is a Condorcet Winner [x_1] and a Condorcet loser [x_3] in the subset, while x_2 is located in an intermediate position. Hence, the complete ordering is:

$$x_4 P x_1 P x_2 P x_3 \tag{37}$$

Let us now position the options on a 0-1 collective desirability scale. First, we review step 01:

STEP 01	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	36		There is no Condorcet Winner
v[x_1,…]	-	0	0	40	40		
v[x_2,…]	42	-	0	4	42		
v[x_3,…]	2	38	-	0	~~38~~		
v[x_4,…]	0	0	36		36	x_4	Most preferred option
	42	38	36	~~40~~	36		There is no Condorcet Loser

Since we already know the ordering of the options, for the calculations we do not use values that disagree with such ordering [x_3's row and x_4's column]. Only x_4 is chosen at this step, and since it is more preferred, we operate on the rows [defeats]. We have stated above that:

x_4
$$\Delta P[x_4 - x_{\neg 4}] = min\left[rows[x_{\neg 4} = x_1, x_2, x_3]\right] - row[x_4] \tag{38}$$

However, since x_3's row value does not match its position in the collective preference ordering, we exclude x_3 from above calculation, therefore:

x_4
$$\Delta P[x_4 - x_{\neg 4}] = min\left[rows[x_{\neg 4} = x_1, x_2]\right] - row[x_4] \tag{39}$$

$$\Delta P[x_4 - x_{\neg 4}] = 40{,}0 - 36{,}0 = 4{,}0 \tag{40}$$

In order to position options 01-03, we review the next step:

STEP 02	v[…,x_1]	v[…,x_2]	v[…,x_3]	0	There is Condorcet Winner
v[x_1,…]	-	**0**	**0**	0	x_1 Most preferred option
v[x_2,…]	42	-	**0**	42	
v[x_3,…]	2	38	-	38	
	42	38	0	0	There is no Condorcet Loser
			x_3		Least preferred option

We see the cyclic relative preference relation does no longer hold once x_4 is removed from the choice set. This means that positioning the options requires reviewing them at the previous step [Step 01], when the cycle still holds, but importantly, not taking into account options chosen at Step 01 [x_4]:

STEP 01	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	36	There is no Condorcet Winner
v[x_1,…]	-	**0**	**0**	40	40	x_1
v[x_2,…]	42	-	**0**	4	42	
v[x_3,…]	2	38	-	**0**	~~38~~	
v[x_4,…]	**0**	**0**	36		~~36~~	
	42	38	36	~~40~~	36	There is no Condorcet Loser
			x_3			

Now we can calculate the desirability difference between options:

- x_1 as most preferred option shall be calculated operating the rows:

x_1

$$\Delta P[x_1 - x_{\neg 1}] = min\big[rows[x_{\neg 1} = x_1, x_2, x_3]\big] - row[x_4] \tag{41}$$

However, since x_3's row value does not match its position in the collective preference ordering, we exclude x_3 from above calculation:

x_1

$$\Delta P[x_1 - x_{\neg 1}] = min\big[rows[x_{\neg 1} = x_1, x_2]\big] - row[x_1] \tag{42}$$

$$\Delta P[x_1 - x_{\neg 1}] = 42{,}0 - 40{,}0 = 2{,}0 \tag{43}$$

- x_3 as least preferred option shall be calculated operating the columns:

x_3

$$\Delta P[x_3 - x_{\neg 3}] = min\big[columns[x_{\neg 3} = x_1, x_2, x_3]\big] - column[x_4] \tag{44}$$

Since x_4 column is not taken into account for calculations [it does not belong to the choice set at Step 02] there are no values mismatching the preference ordering, so we calculate x_3's higher preference considering all the values of the remaining options [x_1 and x_2]:

$$\Delta P[x_3 - x_{\neg 3}] = 38{,}0 - 36{,}0 = 2{,}0 \tag{45}$$

Now we can draw a table stating the preference differences between options:

x4	≻	x1	≻	x2	≻	x3
	4,0		2,0		2,0	

Let us now calculate distance to limiting points 0-1 of the scale:

- To calculate the distance to 1, we subtract to the total number of individuals [N] the maximum number of individuals supporting option x_4 in any confrontation:

Distance to 1
$$\Delta P[1 - x_4] = N - max\left[n[x_4 \succ x_j]_{j=1-3}\right] = 78 - 59 = 19 \tag{46}$$

- To calculate the distance to 0 point, we calculate the maximum number of individuals supporting option x_3 in the pairwise matrix at step 1 [when the cyclic preference still holds]:

Distance to 0
$$\Delta P[x_3 - 0] = max\left[n[x_3 \succ x_j]_{j=1,2,4}\right] = 57 \tag{47}$$

Now, we can draw the whole scale:

	x4	≻	x1	≻	x2	≻	x3		¬X
19,0		4,0		2,0		2,0		57,0	

From above scale, we calculate each option's desirability of by adding every desirability difference starting from 0 point [every number on the right side of that option up to the ¬X], and normalize it by dividing it by the sum of all of them [including the numbers on the left side, which informs us of the distance to the 1 point]. We obtain:

$$x_4 = \frac{4+2+2+57}{19+4+2+2+57} = 0{,}774$$

$$x_1 = \frac{2+2+57}{19+4+2+2+57} = 0{,}726$$

$$x_2 = \frac{2+57}{19+4+2+2+57} = 0{,}702$$

$$x_3 = \frac{57}{19+4+2+2+57} = 0{,}679$$

We can summarize above results in a table:

	x4	P	x1	P	x2	P	x3
PREFERENCE INTENSITY/COLLECTIVE UTILITY	0,774		0,726		0,702		0,679

Again we see all options are assigned high and similar utility.

Let us provide again some contrast to above results by calculating collective preference for each option using Borda's rule:

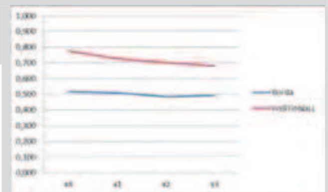
	3		2		1		0
19	x_1	P	x_2	P	x_3	P	x_4
18	x_2	P	x_3	P	x_4	P	x_1
20	x_3	P	x_4	P	x_1	P	x_2
21	x_4	P	x_1	P	x_2	P	x_3
78							

We obtain the following Borda Scores and Utility Assignments [for brevity, we skip calculations]:

	BORDA RULE	
	SCORE	POSITION
x_1	119	0,509
x_2	113	0,483
x_3	115	0,491
x_4	121	0,517

Now we can compare the values obtained using Borda rule with those obtained using PsII, we see [we arrange options according to ordering obtained using PsII]:

	PREFERENCE INTENSITY / COLLECTIVE UTILITY		
	Borda	PsII	Standard Deviation
x_4	0,517	0,774	0,154
x_1	0,509	0,726	0,155
x_2	0,483	0,702	0,132
x_3	0,491	0,679	0,182



We arrive at an average standard deviation of 0.156 and a Pearson Correlation of 0.86. While two most preferred options are equally ordered by both rules, some difference appears in relation to the two intermediate preferred options, which position is inverted by the rules.

Again, only PsII assigns in all cases collective utility above the 0.50 threshold, something consistent with the fact that any of them can become most preferred option if certain option is removed. According to PsII, choosing any of them would increase collective utility [any of them is a 'good choice'], yet Net Benefit is maximized [Opportunity Cost is minimized] by choosing the one providing the highest collective utility [x_4].

This example is quite interesting because it poses the two more difficult issues we may find when positioning options:

- Positioning options involved in a cyclic relative preference relation that is broken when we remove some option [in this case, the cycle breaks after we remove x_4]

- Calculating the desirability differentials when the ordering from *least worst defeat* to *worst defeat* and the ordering from *largest* to *smallest biggest victory* do not match the collective preference ordering.

We have seen these situations are easily solved following the explained procedure.

3.3 PSLL FOR MAKING OUR MOST IMPORTANT COLLECTIVE DECISIONS: REFERENDUMS

In most of our societies, structural issues are subject to approval via referendum. Yet referendums currently pose some flaws that Psll can help solve:

- *Most referendums are posed in a binary way;* 'yes' and 'no' are often the only accepted answers, greatly restricting the universe of eligible options. This is of the utmost importance, since it is widely acknowledge *a decision made between few eligible options may lead to an option being chosen which is not the most desired by the individuals.* However, Psll allows for decisions between as many options as desired, providing a fuzzy assignment of collective utility to every option.
- *Referendums often do not provide any means for citizens' expressing disagreement/agreement with every eligible option*. However, these should be accepted options accounted in different manner. Psll allows citizens to express them and appropriately accounting them for the overall result.
- *Referendums usually account abstention as votes preserving the same proportion yes/no than casted votes.* This implies a burden on those citizens' who prefer to preserve current course of action, and may breach a widely accepted premise for referendums; the disutility of voting should be allocated to those promoting change[6].

Let us briefly review each of them:

3.3.1 REALITY IS NOT WHITE AND BLACK AND CONSENSUS IS USUALLY IN THE GREY AREA

Most decisions do not need choosing between black and white, but accept choosing also between many types of grey. Furthermore, **it is usually in the gray area where consensus among people lies.**

However, most referendums are designed in a way citizens' have to choose between two considerably different and mutually exclusive courses of action. Why? The reason probably lies in a combination of issues:

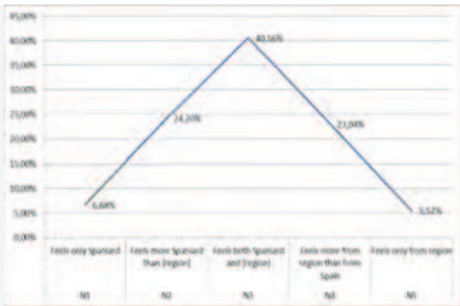
- The need to set some clear thresholds that state whether change is approved or rejected, with a clear criterion stated on how can be assessed whether options cross this thresholds or not when there are more than two eligible options
- The lack of consensus on how should Condorcet Paradox be solved if preferential vote is used
- The highly extended paradigm that Plurality Rule leads to democratic outcome, combined with the manipulability of the rule if more than two options were to be chosen using it[7].

As consequence, usual practice is to organize a binary vote support/rejection of certain proposed system's change, even if it is well know this design may easily lead to choosing as collective decision that which is actually not the collectively preferred course of action[8].

---

[6] We do not mean for every referendum abstention should be accounted as votes supporting current course of action. In some specific instances, abstention may not be taken into account at all. But as a general rule, a vote is held to remove our uncertainty regarding citizens' preferences, and abstention does not remove such uncertainty. Therefore, when the decision is made between preserving some already collectively approved course of action and modifying it, as a general rule abstention should be accounted as supporting the previous collective decision; i.e., as supporting the decision that has taken to current course of action.

[7] In other terms, if we set a usual threshold as 0.50 any election under Plurality rule would be very easy to manipulate by simply introducing many options, since no one of them would be able of reaching that threshold.



If we review Spanish citizens preferences [CIS' survey, September 2016], we see a referendum between two options [independence or not] as pro-independence parties request would be unnecessarily limiting the eligible options. While N_1,N_2, N_3 approach permanence within Spain and N_5 approach independence, N_4 requests more decentralization [locates somewhere in between]. If increasing decentralization is not an eligible option, which option would this 22.80% vote? Most likely, some of them would vote in favor of independence and some in favor of permanence. This binary approach of pro-independence parties would most likely increase the share of citizens supporting independence.

PsII allows us to solve this issue by enabling assessing as many options as desired[9], both in terms of preference ordering and in terms of collective preference on a 0-1 scale. And this cardinal valuation is crucial because thresholds can be easily set instead of referring to Single Votes, referring them to collective preference values:

- a 50% threshold equals a 0.50 value in collective preference
- a 3/5 threshold equals a 0.60 value in collective preference
- a 2/3 threshold equals a 0.666… value in collective preference
- a ¾ threshold equals a 0.75 value in collective preference
- …

Since PsII provides a fuzzy measure of collective preference for each option, we can use fuzzy thresholds and determine whether they are met by any of the options.

- If no option crosses the thresholds, then system must preserve its current course of action
- If more than one option crosses the required thresholds, then the option with higher collective preference shall be chosen.

Referendums are often understood as 'All-or-nothing' games, yet in our view that is far from their goal, which approaches more to their understanding as:

- *Means to let people choose in especially important decisions*; those which most shape their society in which they live in.
- *Means to achieve better decisions for the collective*, since a group of rational and well informed citizens will most likely support the option providing greater collective benefit.

This means that a binary referendum between two mutually exclusive options with a 51% vs 49% outcome is most likely the result of a bad political design of the referendum:

- First, it implies a binary decision on an issue on which society is mostly divided. It adopts an approach which favors polarization instead of searching for grey/consensus areas.

[8] In formal terms, this issue underlies Arrow Condition 04: "when the SWF is imposed, there is some pair of alternatives x_1 and x_2 such that the community can never express a preference for x_2 over x_1 no matter what the tastes of both individuals are, even if both individuals prefer x_2 to x_1" [Arrow, 1950: 334]. In a general sense, a decision between only two options has high probability of being imposed; i.e., has high probability there is another option which would have been more preferred by the individuals.

[9] However, it seems convenient not to exceed a reasonable number of eligible options; a number which comparison can be done in a realistic manner. As an orientate figure, Miller [1951] proposed humans cannot accurately compare more than 9 statements at the same time [in easy terms, the score in the range 1-0 assigned to each proposal is a subjective assessment of the degree of truth of the statement 'this solution is optimal for the proposed problem']. The number 5-7 should be most likely set as maximum number of eligible options.

- Second, it implies instability, since few votes change could reverse the outcome.

Psll fuzzy measures can be used as help to understand the different collective preference over the options, and when this collective preference is small, more than a 'tight victory' it should be interpreted as 'further work has to be done to find a consensual decision'. In this sense, two types of thresholds may need to be defined for important decisions:

… minimum collective preference thresholds [e.g., a change is only to be undertaken if its collective preference value is above 0.50 collective]
… minimum collective preference differential regarding current course of action [e.g., a change is only to be undertaken if it is at least more collectively preferred than current course of action]

Let us now review the other major issue: how abstention and blank votes should be accounted.

3.3.2 HOW SHOULD ABSTENTION AND BLANK VOTES BE ACCOUNTED?

Most constitutional frameworks acknowledge important decisions can [or should] be made by all citizens via referendums, which stand as tools for making/supporting societies' most important decisions. Yet, not everything is clear about these referendums.

For instance, many times these referendums are related to especially important matters, so their outcome affects the structure of the system; or in other terms; these referendums often refer to structural issues.

It is widely acknowledged structural issues modification should always be supported by qualified majorities to prevent systems' becoming unstable; i.e., deciding one day to change in one direction and the next day to change in the other direction[10]. Societies want their changes over time to be an evolution, not merely an erratic path. This would take us to setting thresholds like 2/3, 3/4 or 3/5 votes in favor for undertaking structural changes.

Additionally, these higher thresholds for undertaking structural modifications can be justified from an economic public choice perspective. A Public project/transformation should be undertaken iff it provides greater than zero Net Benefit [Stiglitz, 2000][11].

Any modification of a society usually involves a cost in collective resources [economic, material, time...] and for structural modifications this cost is usually much higher. This means that when a structural society's transformation is valued, the decision should never be linked to any absolute majority, since if the margin of victory is reduced [e.g., 51% vs 49%] the Net Benefit of the transformation will almost certainly be lower than that of preserving current status/course of action.

The small collective utility increase implied in a small margin of population's percentage in favor of the change, will not be countering the high cost and uncertainty associated to structural reforms.

---

[10] Regarding this issue, see Barberá & Jackson, 2004. More specifically, the authors state the rules for approving societies' structures modification need to be more difficult than the approval rules approval of usual societies' transformations.

[11] Usually there is always some available transformation providing Net Benefit above zero. If it is not the case [e.g., in a war every possible choice may imply a loss], then the one with the highest Net Benefit should be chosen.

Societies' structural changes imply both a high cost in collective resources [time, money...] and a high risk [big changes always allow for unexpected results].

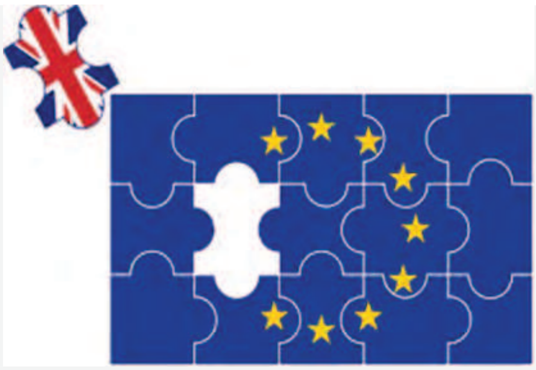
This means that for such changes being rational, the utility provided by them [if they are subject to vote, then this utility is assumed to be measured by the difference between support and opposition] must be much greater than that which is assigned to continuing current course, which risk and implementation cost is usually low [or at least much lower].

Since… $$P(structural\ change) \ll P(preserving\ current\ design) \quad (0)$$

…it follows $$eu[change] > eu[current\ design] \leftrightarrow \\ \leftrightarrow u[change] \gg u[current\ design] \quad (1)$$

Being P(...)_ the probability of achieving the assessed state, u[…] the utility such state provides and eu[…] its expected utility

Since we relate each option's collective utility to the number of individuals supporting it, *if the difference between the number of individuals supporting change and the number of individuals rejecting it is small, change cannot maximize society's expected utility; in such situation, undertaking the change is an irrational action.*



In June 2016, UK decided to vote whether remaining or exiting the EU. From the 72.2% of citizens who voted, 51.9% expressed their preference to exit the EU while 48.1% expressed their preference to stay in the EU. The huge and unpredictable [hence risky] economic cost of 'Brexiting' [both for the UK and other EU citizens] compared to the reduced and predictable [hence riskless] cost of 'Bremaining', seems unlikely to be compensated by a 3.2% of voters' preference [2.3% citizens]. In terms of expected utility, the high risk of a major change [low probability of arriving to expected results] compared to the reduced risk of continuing current course of action [high probability of arriving to a known situation] requires high difference in utility for the change being rational. Moreover, currently many politicians who advocated for the Brexit have already substantially lowered their predicted economic benefits. It seems quite likely, Brexit will produce a net collective utility loss to both UK's and EU's citizens. So… why is it made for? Instead of the consequence of the will of a nation, Brexit appears more and more to be simply the outcome of a bad politicians' decision.

Furthermore, in a world where societies' usually organizes into sub-groups [regions, …] each with its own identity, double majority criteria for approving structural reforms should be the general rule [e.g., 1/2 individuals and 1/2 contexts for less important issues; 3/5 individuals and 1/2 contexts for important issues…]

Above approach clashes with the fact that usually not 100% of citizens' vote; some of them do not vote for any eligible option and a doubt arises. How should we account abstention, null and blank votes?

It is not the same a 60% support for an options when 95% citizens' cast a ballot expressing preference for some option [it implies certainty that 57% citizens support the decision], than if only 72% citizens' cast a ballot expressing preference for some option [it only implies certainty that 43.20% - less than half- citizens support the decision].

And it has to be considered that not expressing preference for any option may be done in two different ways, which meaning can also be significantly different.

- It can be done by not casting a vote or casting a null vote.
- It can be done by blank voting.

Some authors express concerns about how these types of votes can be accounted. Let us review again our above proposal, focusing more on the issue of referendums:

How should abstention/null votes be accounted? it is *usually accepted who wants to hold a referendum should take care of the organization costs [disutility].* And voting is a disutility [cost] for citizens.

So, if we do not take into account abstention, then we are considering citizens who do not cast a vote support the measure in the same proportion than citizens who cast vote. We are then imposing the disutility of the referendum not on those citizens who want to approve a change, but also on citizens who support current course of action.

To effectively assign the disutility of holding the referendum to those promoting change, abstention needs to be accounted as votes in support of current course of action.

How should blank votes be accounted? There is some debate on whether a blank vote means equal support or equal rejection of every eligible option.

Since Psll uses preferential voting, it is possible for each citizen to clearly state whether he supports every possible option [he is indifferent between current course and change] or he rejects them all [he dislikes current situation but he does not like the proposed change[s]].

- For the first, citizen only needs to assign a first place preference for every eligible option
- For the second, citizen's only needs to cast an unmarked ballot [blank vote].

Since Psll provides a measure in the 0-1 range, abstention and null votes can be accounted as support for current course of action, expressed indifference as support for every eligible option, and blank votes as rejection for any eligible option.

Namely, the resulting collective preference values for each option can effectively account all these issues, since instead of requesting a 3/5 of the votes to accept a structural change we can set a collective preference threshold, allowing all types of votes to be taken properly into account.

- a high abstention will imply change is only to be approved if high percentage of voters' support change.
- a high percentage of blank votes will make less probable any eligible change is undertaken, but would make evident the need to search for some type of change [citizens are stating they dislike both change and current course of action].

This approach enables differentiating two types of decisions:

- In structural decisions [or when a government decisions need to be supported/rejected], thresholds can be unified at 0.50, while abstention should be accounted as supporting maintaining current course of action [or government decision][12].
- In non-structural decisions [or when no governments decision has been yet made], accounting abstention as rejecting change may lead to immobilism, therefore it seems better not accounting abstention.

Also accounting abstention may be an issue in countries where casting a ballot implies a risk for personal safety. In these places, it may be better not to account abstention, and simply set thresholds fitted to the importance of the voted change[13].

For a better understanding, let us assess two recent/currently ongoing situations using PsII.

[12] An exception is when no current course of action exists and a first decision needs to be made, since in these cases abstention cannot be accounted in any sense. However, in order to avoid immobilism, another option is that if any proposal receives the highest support, but still does not reach the 0.50 threshold due to abstention, a second vote might be allowed without accounting abstention.

[13] In the short term future, electronic democracy could also facilitate voting without a risk.

3.3.3 EXAMPLE: A CONSULTATION FOR THE TERRITORIAL MODEL IN SPAIN

During last years, some Catalonian politicians are claiming Catalonian citizens should have the right to 'self-determination' stating 'it is a basic right in any democracy'.

Building on this paradigm, even a vote has been recently held in Catalonia regional parliament taking into account only Catalonian politicians, and a majority of pro-independent politicians has supported to hold a referendum in a year to decide whether to become independent or not[14].

While the possibility of holding a referendum in relation to almost any issue [excluding of course issues colliding with fundamental rights] could be argued to be democratic, some Catalonian pro-independence politicians [not all of them] propose a design of a referendum which restricts the people that can vote so non-Catalonians are excluded from this right[15].

We have thoroughly explained in a previous text why *the right to self-determination is not per se a democratic right [Alvira, 2017]*, but we have also thoroughly stated in previous texts that governments' actions should be much more linked to citizens' preferences, and referendums should become a citizens' right, not a politicians' prerogative [Alvira, 2015 & 2017]. So … How can we review this issue in relation to all Spanish citizens' preferences [i.e., without undemocratically restricting the universe of voters to Catalonians]?

The fact is that in order to know which Spanish citizens' preferences are, we do not need to hold a referendum. Spanish Center for Sociological Research [Centro de Investigaciones Sociológicas, CIS] has made at least two surveys along this year [June and September 2016] which among other issues have requested citizens' opinion in relation to two related yet slightly different questions:

- preferences in relation to *Territorial Model*
- feelings in relation to *sense of belonging* to country/region

Somehow the first refers to citizens' preferred government course of action for the following years, while the second refers to each citizen's expected vote if a referendum was held now. Let us review both of them.

SPANISH CITIZENS' PREFERENCES IN RELATION TO TERRITORIAL MODEL

According to CIS' survey [September 2016, Question 33], Spaniards have the following preferences regarding Territorial Model:

SPANISH CITIZENS PREFERENCES REGARDING TERRITORIAL MODEL [%]

M_1	A state with one central government without autonomies	16,6 %
M_2	A state in which the autonomous communities have less autonomy than at present	10,3 %
M_3	A State with autonomous communities currently	35,7 %
M_4	A state in which the autonomous communities have greater autonomy than at present	13,4 %
M_5	A state in which the autonomous communities recognized the possibility of becoming independent states	9,8 %

[14] This vote has been held against the criterion stated by Spain's' Constitutional Court, since Catalonia Parliament has not the right to vote this type of issues, for which Spanish Constitution requires the vote of every Spanish citizen is accounted.

[15] In the view of these politicians, only Catalonians should be allowed to express on this matter, or in other terms, Spanish citizens not residing in Catalonia should lose their Constitutional right to vote in such decision

Does not know	10,5 %
Does not answer	3,7 %

Universe: 2483 interviews throughout all Spain.

Above preferences refer to Government's preferred course of action. So... which would be the democratic collective decision [the collectively preferred government's course of action] emerging from these individuals' preferences?

If we assume individual rationality [otherwise requesting individuals' preferences would be irrational] we can assume preferences are single-peaked.

Also, as stated before, we account answers not expressing preferences in the following manner:

- We account 'does not answer' as abstention in a real election [i.e., citizens supporting current course of action M_3][16]
- We split 'does not know' into 50% citizens expressing complete and equal support for every eligible option [we divide it into the five eligible options] and 50% citizens expressing complete and equal rejection for every eligible option[17].

This assumption takes us to the preferences[18]:

Express support		Abstention [does not answer]		Blank Votes [does not know]		Total assumed support	
16,6 %	+		+	1,05 %	=	17,65 %	M_1 P M_2 P M_3 P M_4 P M_5
10,3 %	+		+	1,05 %	=	11,35 %	M_2 P M_1 I M_3 P M_4 P M_5
35,7 %	+	3,75%	+	1,05 %	=	40,45 %	M_3 P M_2 I M_4 P M_1 I M_5
13,4 %	+		+	1,05 %	=	14,45 %	M_4 P M_3 I M_5 P M_2 P M_1
9,8 %	+		+	1,05 %	=	10,85 %	M_5 P M_4 P M_3 P M_2 P M_1
		Total individuals				100 %	

From above preferences, we can build a PC matrix:

	M_1	M_2	M_3	M_4	M_5
M_1	-	18	29	29	**29**
M_2	**77**	-	29	**69**	**69**
M_3	**77**	**66**	-	**69**	**84**
M_4	**66**	66	25	-	**84**

[16] We actually do not think citizens' not providing an answer equal expected abstention in a real election, but this gives us the opportunity to review how abstention can be accounted in real world referendums. Also, since the percentage of citizens' not providing an answer is small, results barely change.

[17] We actually do neither think citizens who 'do not know' can be expressing equal and complete support of all options, since options imply incompatible courses of action [in other terms, in this case a citizen supporting all possible courses of action would be irrational]. However, again it allows us to show how to model these types of votes in real world elections [and also, the number of individuals stating 'does not know' is also reduced, so results are neither substantially modified].

[18] In this case, the number of votes involved is reduced [14.2%] and the result is barely affected. In real elections with a high percentage of abstention/blank votes, result can be noticeable different.

M_5	25	25	25	11	-

Above matrix implies a linear ordering of the options according their collective preference:

$$M_3 \ P \ M_2 \ P \ M_4 \ P \ M_1 \ P \ M_5$$

In other terms; the most preferred option by Spanish citizens is that Government continues with a more or less similar to actual course of action [M_3], while the least preferred option is that government allows regions to hold self-determination referendums [M_5].

We can also assess above preference under Borda Count. First we define the measurement scale:

	4		3		2		1		0
17,65	M_1	P	M_2	P	M_3	P	M_4	P	M_5
11,35	M_2	P	M_1 I M_3	P	M_4	P			M_5
40,45	M_3	P	M_2 I M_4	P					M_1 I M_5
14,45	M_4	P	M_3 I M_5	P	M_2	P			M_1
10,85	M_5	P	M_4	P	M_3	P	M_2	P	M_1

From above preferences/scale, we obtain the following Borda Scores:

	BORDA COUNT	
OPTION	BORDA SCORE	NORMALIZED BORDA SCORE
M_1	105	0,26
M_2	259	0,65
M_3	296	0,74
M_4	252	0,63
M_5	87	0,22
MAXIMUM POSSIBLE BORDA SCORE	400	

Therefore, according Borda Count, collective preference in relation Government course of action is:

$$M_3 \ P \ M_2 \ P \ M_4 \ P \ M_1 \ P \ M_5$$

We see these two rules lead to a coincident ordering, let us now go a bit deeper in our analysis using Psll and Prll , which will allow us to observe some interesting issues.

First we use Psll to calculate the collective utility provided by each option:

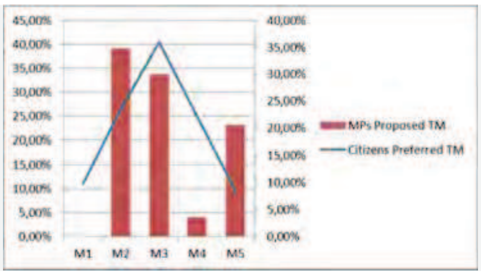
		M_3		**M_2**		**M_4**		**M_1**		**M_5**		
	16		37		4		37		4		25	122
Psll		0,868		0,568		0,538		0,237		0,207		2,418

Since Psll is Condorcet Consistent, we see the Condorcet ordering is preserved, and noteworthy, we see the high collective preference assigned to option M_3 [maintaining a State model more or less similar to actual] compared to any other option, and especially to option M_5, which obtains a quite low value.

From above intensities of preferences/collective utility assignment, we can calculate which would be a representative allocation of seats in a chamber using PrII :

	M_3	M_2	M_4	M_1	M_5	
16	37	4	37	4	25	122
PsII	0,868	0,568	0,538	0,237	0,207	2,418
PrII	35,91%	23,49%	22,24%	9,81%	8,56%	
	125,70	82,20	77,82	34,33	29,95	350
Seats [Pure Proportionality]	**126**	**82**	**78**	**34**	**30**	350
Seats [d'Hondt]	**126**	**82**	**78**	**34**	**30**	350

It is interesting comparing the allocation of seats emerging from citizens' preferences with the actual allocation of seats in Spanish parliament after past elections:



While citizens' preferences show a perfect single peaked structure, current distribution of seats between parties shows a partly reversed shape. This is a strong empirical proof of two issues:

- *current PR rule leads to allocating seats non-consistently with citizens' preferences*
- *Part of the importance that the claim for pro-independence referendums has gained over past year is due to this misallocation of seats, which gives much more relevancy to parties requesting referendums than citizens would like to assign them.*

We observe a noticeable difference between citizens' preferences and MPs' preferences. While MPs located around M_2 approximately represent citizens' preferences regarding M_1 and M_2, MPs located at M_5 are almost three times the figure that would actually represent citizens [23.14% against 8.5%].

Current importance of the pro-independence issue in Spanish political agenda seems to be highly magnified due to incorrect allocation of seats by current RER, which has assigned pro-independence parties much more seats than those which would represent society[19], linking Spanish overall governability to a debate which does not represent actual citizens' preferences.

SPANISH CITIZENS' SENSE OF BELONGING

This second question [CIS' survey, September 2016, Question 34] allows us to anticipate the expected result of a pro-independence referendum.

SPANISH CITIZENS 'SENSE OF BELONGING'- NATIONALIST FEELING [%]

N_1	Feels only Spaniard	15,5 %
N_2	Feels more Spaniard than [region]	6,8 %
N_3	Feels both Spaniard and [region]	52,8 %

[19] In other terms, some MPs are not requesting their voters' preferred TM, but their party preferred TM. The reason for this striking divergence seems to be highly linked to electoral alliances, an issue we prefer not to review here.

N_4	Feels more from region than from Spain	12,5 %
N_5	Feels only from region	5,9 %
	None of above	5,0 %
	Does not know	0,4 %
	Does not answer	1,2 %
	Universe: 2483 interviews throughout all Spain.	100 %

Again individual rationality allows us to assume single-peaked preferences. And this time we account vote in the following sense[20]:

- Citizens answering 'None of the above' are accounted as rejection of all eligible options.
- Citizens answering 'does not know' are accounted 50% as total support for all options and equal indifference between them; and 50% as null support for any option and total indifference among them.
- Citizens not providing an answer are accounted as abstention, therefore supporting current course of action.

Above assumptions take us to the preferences:

15,6	N_1	P	N_2	P	N_3	P	N_4	P	N_5
6,9	N_2	P	N_1	I	N_3	P	N_4	P	N_5
54,1	N_3	P	N_2	I	N_4	P	N_1	I	N_5
12,6	N_4	P	N_3	I	N_5	P	N_2	P	N_1
6,0	N_5	P	N_4	P	N_3	P	N_2	P	N_1

From above preferences, we can build a PC matrix:

	N_1	N_2	N_3	N_4	N_5
N_1	-	16	23	23	**23**
N_2	**80**	-	23	**77**	**77**
N_3	**80**	**73**	-	**77**	**89**
N_4	**73**	73	19	-	**89**
N_5	19	19	19	6	-

Above matrix implies a linear Condorcet Ordering of the options according to their collective preference:

$$N_3 \ P \ N_2 \ P \ N_4 \ P \ N_1 \ P \ N_5$$

In other terms; the collectively most preferred option if a referendum was held today would be N_3; i.e., that no region becomes independent and the least collectively preferred option would be N_5, i.e., that some region becomes independent.

Let us again check the results we obtain under Borda rule. First we define the measurement scale:

[20] Again, the impact of these assumptions is in this case reduced, since figures are small [the total votes to be reallocated add up to 6.6%].

	4		3			2		1		0
15,6	N_1	≻	N_2		≻	N_3	≻	N_4	≻	N_5
6,9	N_2	≻	N_1	N_3	≻	N_4	≻			N_5
54,1	N_3	≻	N_2	N_4	≻					N_1 N_5
12,6	N_4	≻	N_3	N_5	≻	N_2	≻			N_1
6,0	N_5	≻	N_4		≻	N_3	≻	N_2	≻	N_1

From above preferences/scale, we obtain the following Borda Scores:

BORDA COUNT		
OPTION	BORDA SCORE	NORMALIZED BORDA SCORE
N_1	83	0,21
N_2	268	0,67
N_3	318	0,80
N_4	260	0,65
N_5	62	0,15
MAXIMUM POSSIBLE BORDA SCORE	400	

Therefore, we also arrive to a coincident ordering:

$$N_3 \;\; P \;\; N_2 \;\; P \;\; N_4 \;\; P \;\; N_1 \;\; P \;\; N_5$$

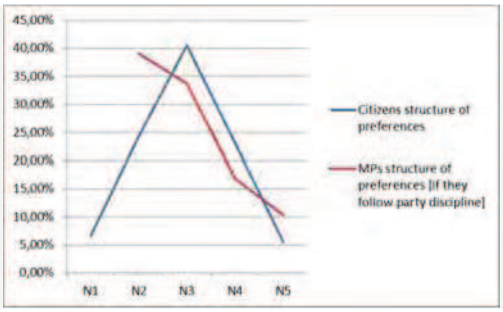
Let us again go a bit deeper in our analysis using the two proposed rules: Psll and Prll , which again will allow us to observe some interesting issues.

First we use Psll to calculate the collective utility provided by each option:

		N_3	P	**N_2**	P	**N_4**	P	**N_1**	P	**N_5**		
	11		50		4		50		4		19	138
Psll		0,922		0,557		0,529		0,164		0,135		2,307

Since Psll is Condorcet Consistent, again the Condorcet ordering is preserved. Noteworthy, we see the high preference for option N_3 compared to any other option, and especially to option N_5. If we consider citizens 'feeling from region and not from Spain' to be the ones preferring independence over coexistence, then the independence option is by large collectively rejected. Only 5.6% of Spanish citizens would actually prefer some regions' independence.

Again, it is interesting to review which would be an allocation of seats representative of above intensities of preferences/collective utility assignment, and compare it with current allocation of seats as per last elections. We see:



Again we see a divergence between citizens' and MPs' preferences. If MPs follow party discipline, they almost double citizens' preferences for independence processes [10.29% vs 5.52%]. Furthermore, it is not so clear which would be UP's deputies vote [45/12.86% MPs] and whether UP would impose party discipline for its MPs [e.g., UP's deputies in Basque Country would most likely prefer to vote in favor of independence]. If this would be the case, then MPs preferences would be further departing actual citizens' preferences.

Noteworthy, if we use Prll as RER, allocation of seats greatly increases its relation with citizens' preferences, or in other terms; the resulting MPs' preferences would more closely match those of allegedly represented citizens.

STRUCTURE OF PREFERENCES REGARDING GOVERNMENT'S COURSE OF ACTION

			M_1	M_2	M_3	M_4	M_5	Pearson (1)
Citizens preferences		%	10,86%	22,91%	34,97%	21,66%	9,61%	-
		Seats	38	80	122	76	34	
MPs	Current PR rule	%	-	39,14%	33,71%	4,00%	23,14%	0,324
		Seats	-	137	118	14	81	
	Prll	%	-	20,57%	52,86%	3,14%	23,43%	0,599
		Seats	-	72	185	11	82	
	Prll +d'Hondt	%	-	19,71%	56,86%	3,14%	20,29%	0,672
		Seats	-	69	199	11	71	

(1) MPs' preferences compared to citizens preferences

We see the relation between MPs' preferences and citizens' preferences more than doubles when allocation of seats is done using Prll + d'Hondt.

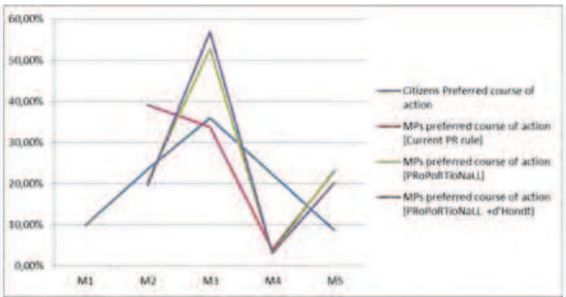
EXPECTED VOTE

			N_1	N_2	N_3	N_4	N_5	Pearson (1)
Citizens expected vote		%	7,10%	24,15%	39,96%	22,92%	5,87%	-
		Seats	25	84	140	80	21	
MPs expected vote (2)	Current PR rule	%	0,00%	39,14%	33,71%	16,86%	10,29%	0,808
		Seats	-	137	118	59	36	
	Prll	%	-	20,57%	52,86%	16,86%	9,71%	0,914
		Seats	-	72	185	59	34	
	Prll + d'Hondt	%	-	19,71%	56,86%	14,29%	9,14%	0,891
		Seats	-	69	199	50	32	

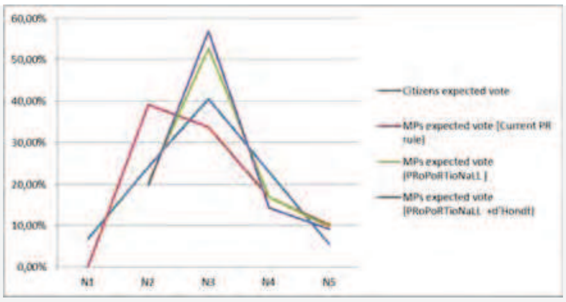
(1) MPs' expected vote compared to citizens' expected vote
(2) Regarding this point, it is not totally clear if UP would impose party discipline over its MPs. If the case was every UP deputy was allowed to vote according his ideology, then non-relatedness to citizens' preferences would most likely increase.

If we compare MPs' expect vote if they abide by party discipline, we see MPs' preferences approach more citizens' preferences.

Still, this makes evident great part of the current importance being given to this issue in Spain is due to RER currently used in Spain, which favors allocation of seats in a way which does not fit citizens' preferences. A RER providing higher fitness of allocation of seats to actual citizens' preferences could greatly help to take the debate towards actual citizens' preferences[21].



Preferred governments course of action



Expected vote in case a referendum is hold today

Noteworthy, **above data shows the great paradox that while only 5.6% of Spanish citizens would be expected to vote in favor of independence of any region, there is actually a 23.14% of Spanish MPs requesting such referendum is hold**, and it has already been argued as the main reason for not arriving to pacts for forming a government.

OVERCOMING TM'S PARADOX: WHICH SHOULD EACH PARTY MOST PREFERRED COURSE OF ACTION?

All parties strongly assert their course of action follows their voters' preferences. However, comparison of Spanish preferences according to CIS' survey and MP's preferences show an important mismatch. If parties intended course of action follows their voters' preferences, why is that mismatch happening?

Which party or parties are inadvertently [or advertently] proposing a course of action which actually does not emerge from their voters preferences? [i.e.; an undemocratic course of action in relation to its voters' preferences]

In order to shed some light on this puzzle, let us assess each party's voters' preferences and compare them with party's policy.

CIS' survey details each party's voters preferences related to TM, so we can calculated both each party's voters ordering of the different possible courses of action as well as their assessment in term of preference intensities.

CONSISTENCY BETWEEN VOTERS AND PARTY PREFERENCES

	Voters Preference Ordering	Voters Preferred Course Of Action	Parties Declared Course Of Action
PP	$M_2 \succ M_3 \succ M_1 \succ M_4 \succ M_5$	M_2	M_2
C's	$M_2 \succ M_3 \succ M_1 \succ M_4 \succ M_5$	M_2	M_3
PSOE	$M_3 \succ M_2 \succ M_4 \succ M_1 \succ M_5$	M_3	M_3
UP	$M_3 \succ M_4 \succ M_2 \succ M_5 \succ M_1$	M_3	M_5
EM	$M_3 \succ M_4 \succ M_2 \succ M_5 \succ M_1$	M_3	M_5

[21] Noteworthy, the average correlation for both above issues [preferred government's course of action and expected vote] is 0.560 for Current RER; 0.744 for PRoPoRTioNaLL and 0.770 for PRoPoRTioNaLL + d'Hondt. In other terms, PRoPoRTioNaLL highly increases MPs' preference relation to citizens' preferences [132,96% and 137,56% if d'Hondt is used]

CP	M_4	≻	M_3	≻	M_5	≻	M_2	≻	M_1	M_4	M_4
ECP	M_4	≻	M_5	≻	M3	≻	M_2	≻	M_1	M_4	M_5
EAJ-PNV	M_4	≻	M_5	≻	M_3	≻	M_2	≻	M_1	M_4	M_4
ERC	M_5	≻	M_4	≻	M_3	≻	M_2	≻	M_1	M_5	M_5
CDC	M_5	≻	M_4	≻	M_3	≻	M_2	≻	M_1	M_5	M_5

Above Table shows us there are some parties which declared course of action does not fit their voters' preferences. Let us further review these inconsistencies, since they have different meaning:

- We see one party [C's] declares an intended course of action which `departs from its voters' most preferred course of action approaching more the consensual decision of the whole society [M_3 vs M_2]. In principle this stands as a reasonable modification, since the search for consensus is the more stable and utility maximization course of action[22].
- We see three parties [UP, EM and ECP] which declared courses of action depart their voters' most preferred course of action towards more extreme [i.e., less consensual] positions. Furthermore, in the case of UP and EM, this declared course of action greatly departs their voters' most preferred course of action [M_5 vs M_3]. Strikingly, while these last two parties' voters' locate in the consensual course of action of the whole society, these parties locate in the most extreme [and likely irreversible] possible course of action.

While C's position has enabled government [C's has both been able to arrive to pact with PP –which mostly locates at M_2- and PSOE –which most likely locates at M_3, since it has been the governing party for most last Spain's parliamentary period], UP-EM-ECP position has already prevented government, since they have stated M_5 as a prerequisite for any candidate being supported by them.

Why?

In the case of EM and ECP it can be understood as a negotiation strategy; history has shown us many times regional parties ask for independence and in return obtain a privileged agreement for their region[23]. In the case of UP, we have no explanation at all which can justify this extreme positioning. Furthermore, we find difficultly understanding why is a left party advocator of the State and the common good, fighting for holding independence referendums against the actual preferences of its voters and society as a whole?[24]

---

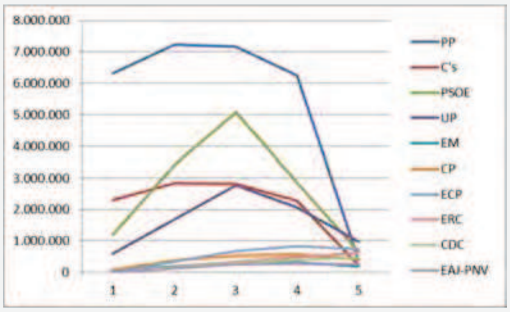
[22] Of course, we are trusting C's declared course of action [somehow similar to M3] would be their actual course of action in case they govern, but we have no actual evidence whether they would undertake a different course of action.

[23] However, this strategy searches for unequal treatment of different regions, hence we must reject it as an undemocratic course of action. Also, it can easily lead to polarization. Though some parties state it is their only course of action against too centralist governments, in Alvira 2015 we have proposed 'Majority Veto' as tool which allows preventing undesired governments action without resorting to create territorial inequality and citizens polarization.

[24] Actually UP leaders state a pro independence referendum is a way to achieve piece in these regions, since 'No' will win, and people will afterwards consciously accept permanence within Spain as the democratically preferred solution. However, history cannot provide a single example of this, but many of the opposed effects. So... why is UP both fighting against History, the will of its voters and the consensual choice of all Spanish citizens?

VOTERS COLLECTIVE PREFERENCE

	VOTERS	M_1	M_2	M_3	M_4	M_5
PP	7.906.185	0,798	0,915	0,907	0,790	0,051
C's	3.092.817	0,742	0,915	0,909	0,737	0,072
PSOE	5.411.106	0,220	0,629	0,938	0,529	0,120
UP	3.201.170	0,185	0,525	0,865	0,646	0,306
EM	344.143	0,171	0,537	0,903	0,890	0,524
CP	655.895	0,125	0,595	0,807	0,867	0,655
ECP	848.526	0,018	0,407	0,794	0,971	0,888
EAJ-PNV	286.215	0,003	0,428	0,849	0,991	0,908
ERC	629.294	0,005	0,273	0,532	0,781	0,990
CDC	481.839	0,003	0,305	0,606	0,851	0,964



Noteworthy, a state of opinion can be generated. If a party consistently states 'we are not free and only independence will allow us to develop as we deserve' its voters will sooner than later get convinced of it, be it true or not [usually, it is not]. Since not all citizens support independence, such course of action increases polarization. For instance, we currently see increasing polarization in Catalonia due of half politicians claiming independence while other half politicians claim permanence.

As we have already stated before, self-determination is not a basic democratic right. It is not stated in any currently existing society's Constitution neither is advocated by pro-independence parties in their proposed statutes/constitutions.

If what these parties want to achieve is a *more democratic system*, then it is greatly questionable their goal is different than that of the rest of Spanish citizens, i.e., it is greatly questionable it can only be achieved by means of independence.

Our political system has many flaws? True, but independence is not necessary to improve it neither it ensures such or any improvement at all. There are other much more efficient and peaceful ways to improve our currently low quality political system….

3.3.4 ANOTHER EXAMPLE: BR-IN OR BR-OUT: A [DIS]UNITED KINGDOM?

Past June, a referendum designated as Brexit was held in Great Britain and Northern Ireland, in order to decide whether to remain inside the EU or start an independent path outside the EU. After the referendum provided a 51.9% support for exit against a 48.1% support for permanence within the EU, the British government has stated it is clear that the will of British citizens is to leave the EU, so the UK is currently heading in such direction. However, the way this referendum was organized challenge some well accepted perspectives in democracy:

- The principle that both *nations* and *citizens* are to be treated as most equally as possible.
- The principle that structural changes shall only be undertaken if supported by sufficient [qualified] majority.

Let us review it, since from these principles it is not so clear which is the will of British individuals [or which is the rational course of action for UK]

A TIE BETWEEN NATIONS IS NOT A VOTE SUPPORTING CHANGE

While the very designation itself of United Kingdom refers to the union of four nations [England, Northern Ireland, Scotland, Wales] the referendum does not use the usual procedure for collective decision making when both citizens and nations should have a say, which is to establish double majority requirements.

There is wide agreement that not only citizens are to be represented in a Nation's collective decisions. Also its different identities should be represented[25]. Problem arises when different identities comprise different percentages of population and balance needs to be set in order to preserve balance between both individuals and contexts. For instance:

- Switzerland: Approval of referendums is subject to support by more than half citizens and cantons[26].
- European Union Constitution draft [2003]: Art 58.6. Approval is subject to affirmative vote of more than 2/3 citizens and a majority 1/2 of states.

Double majorities are usual in many collective decision processes where balance between two scales of agents needs to be achieved. And if we apply a double majority criterion to British citizens' votes:

DOUBLE MAJORITY REQUIREMENT

CITIZENS	LEAVE	17.410.742	51,89%
	REMAIN	16.141.241	48,11%
	TOTAL	33.551.983	
NATIONS	LEAVE	2	50,00%
	REMAIN	2	50,00%
	TOTAL	4	

[25] In this sense, Swiss Constitution [Art. 01] states that "The People and the Cantons of … form the Swiss Confederation".

[26] Noteworthy, great difference may exist between cantons' population. E.g., Zurich is the most populated with 1,463,459 inhabitants, while Uri has only 35,973 inhabitants. The relation [40:1] is almost 33% higher than the relation between the population of Northern Ireland and England [31:1] However, in Switzerland both cantons are assigned the same weight [one vote] in referendums.

Hence, if we apply the lowest double majority requirements [over 50 citizens + over 50 nations] Brexit votes do not lead to Br-out but to Br-in, since one of the two requirements is not met [there is a tie between nations]. If we were to apply a stricter double majority criterion [e.g.; 66% citizens/ 50%nations as planned in Art 58.99 of the EU constitution] then none of the requirements would be met by Brexit.

On the contrary, an almost-tie between citizens [51.9% vs 48.1%] and a tie between nations is being accounted by the British government as support for Br-out [i.e., as support for change], breaching Anonymity between regions [those more populated regions acquire more importance for defining overall preferences]

Furthermore, **since more than 80% of total UK electorate locates in England, a simple majority criterion as used in Brexit implies any structural change can be undertaken with the only support of England.** In social choice terms, **current UK`s voting system assigns England dictatorial power over Wales, Scotland and North Ireland.**

A SMALL MAJORITY IMPLYING A MINORITY OF CITIZENS

Though British politicians state the Brexit referendum left no doubt on citizens' preferences, the fact is that the percentage of citizens supporting Br-exiting is only slightly over the percentage of citizens supporting Br-emaining.

BREXIT RESULTS

		% VOTERS
BR-ESCAPE	17.410.742	51,89%
BR-EMAIN	16.141.241	48,11%
TOTAL VALID VOTES	33.551.983	

This breaches again two crucial issues in social choice; structural changes shall only be undertaken when citizens' support is much larger than its rejection. However, the fact a qualified majority has not been requested allows a huge change for UK building on an almost tie among citizens.

Furthermore, if we account for abstention and null votes[27], we see Brexit results do not ensure that even a simple majority of UK citizens support Br-out.

BREXIT RESULTS ACCOUNTING ABSTENTION AND NULL VOTES

		% CITIZENS
BR-OUT	17.410.742	37,44%
BR-IN	16.141.241	34,71%
Null Votes	25.359	0,05%
Abstention	12.927.345	27,79%
Electorate	46.501.241	100,00%

Data above shows we only know for certain 37.40% of British citizens support Br-out; Brexit does not remove uncertainty regarding British preferences but even leaves much doubt regarding them.

[27] Noteworthy also, UK government stated before Brexit its result would not be mandatory, so, citizens could not actually assign it the importance it is more likely it will have.

WHAT WOULD PSLL SAY

We have already seen two interesting features of *Psll* for referendums. Though the rules for accounting abstention and blank votes should be clearly stated always before the vote is held, let us review data above using the rules we have proposed for the general case:

With data above, and considering abstention as vote supporting status quo[28], we arrive to the following preferences/PC matrix:

37,44%		37,44%	BR-OUT	P	BR-IN
34,71%	27,85%	62,56%	BR-IN	P	BR-OUT
		100,00%			

And from above preferences, we arrive to the PC matrix:

	BR-OUT	BR-IN
BR-OUT	-	0,37
BR-IN	**0,63**	-

Above PC matrix states a collective preference for Br-in vs Br-out, and if we apply Psll to assign collective utility, we see:

	BR-IN		BR-OUT		
	0,37	0,63		0,37	1,37
Psll	0,728		0,272		

If we account abstention as supporting current course of action, the collective preference for [Br]emaining is more than 2.5 times that for [Br]escaping.

CONCLUSIONS

The way Brexit was organized poses some important flaws from Social Choice point of view of:

- It did not set any double majority requirement, assigning *de facto* dictatorial power to England [83% UK population].
- It assigned citizens who did not cast a vote the same preferences than those casting a vote [i.e., it considered non-voting citizens as 51.9% supporting Br-out and 48.1% supporting Br-in], yet this is almost certainly not true. If we are to follow usual approach that the effort/disutility of approving structural change should lay on those proposing change, abstention should be accounted as vote in support of current status [Br-in] and not as a mirror of those votes casted by citizens.
- It does not provide a way to express rejection for all eligible options. Some voters have stated after *Brexit* their vote supporting Br-exiting wanted to express dissatisfaction with current British political system, but they did not actually want UK to leave the EU. While we cannot know for sure if this is true, *Brexit* did not provide a means for these voters expressing this view, which is also a peaceful and democratic political view.

---

[28] This approach can be further sustained by the fact that Br-in was already democratically elected in 1975 by a qualified majority of British citizens and unanimity of Uk's nations.

- It only allowed for a binary choice on an issue that could admit infinite solutions. UK cannot necessarily choose between Br-emaining and Br-exiting. Many intermediate situations can be proposed and, most likely, successfully negotiated between the UK and the EU[29].

All this issues challenge the idea that Brexit implies the democratic decision of UK's citizens is UK leaves the EU[30], and supports the idea that if democracy is the underlying reason for holding Brexit past June, then another referendum should be hold/ allowed if requested by citizens [as it seems many British citizens are in fact requesting].

If citizens'/nations' are to be given the right to participate in important public decisions, then it is not a one-time-use-right. Of course some rules need to be implemented [e.g., a minimum time between two votes for the same issue; a minimum citizens/in a minimum of nations request may be required for this vote is held,…], but if such requirements are fulfilled, then **citizens should not be blamed for wanting to be sure they are actually undertaking their preferred course of action, which consequences may last for decades.**

Otherwise, democracy is not be the actual reason for Brexit, and then... in what sense would Brexit results be relevant?

In our view, if a 'To Br-out or not to Br-out' referendum is sufficiently requested by citizens; it should be hold in some months, since it is simply citizens' right to express their preferences. In order for this referendum to be more democratic, we believe the ideas and rules herein explained may be useful.

A BIT OF HISTORY: THE 1975 BRIN OR BROUT REFERENDUM

Brexit was not the first time British citizens were called to cast a vote supporting UK membership to the EU or not. In 1975, only two years after UK joined the EU a first referendum was held. Since it was almost at the beginning, we could somehow understand it was a referendum to corroborate the will of joining the EU.

Contrary to Brexit, this referendum result complied with almost every accepted premise for considering a collective choice as positive:

5 JUNE 1975 REFERENDUM					
		CITIZENS		NATIONS	
		% VOTERS	% CITIZENS		
LEAVE	8.470.073	32,70%	20,94%	0	0%
REMAIN	17.378.581	67,09%	42,96%	4	100%
NULL VOTES	54.540	0,21%	0,13%		
TOTAL VOTES	25.903.194		64,03%		
ELECTORATE	40.456.877		100,00%		

[29] Noteworthy, to know which intermediate position is preferred by British citizens and try to negotiate with the EU, British politicians only needed to make a survey; they did not need to hold a referendum !!.

[30] In other terms; above review proves we do not know whether leaving the EU is actually the most preferred course of action of British citizens, yet we know for sure it is not for most British Nations.

- Usual thresholds for structural decisions were fulfilled since a qualified majority of citizens supported ‘Br-in’ against those who supported ‘Br-out’[31].
- Usual double majority thresholds were fulfilled since in the four nations the most preferred option was ‘Br-in’ [therefore, both 50/50 and 66/50 thresholds were reached].

Noteworthy, the 1975 referendum also complied with the majority criterion even if we account abstention as support for current course of action, since UK was already in the EU when the 1975 referendum was hold; i.e., maintaining current course of action meant Br-in, not Br-out. This can be checked using *Psll*:

First, we draw our PCm:

				BR-ESCAPE	BR-EMAIN	
20,94%		20,94%	BR-ESCAPE	-	0,21	0,21
42,96%	36,11%	79,06%	BR-EMAIN	0,79	-	0,79
		100,00%				

From above PCm we obtain preference intensities:

		BREMAIN		BRESCAPE		
	0,21		0,79		0,21	1,21
Psll		0,827		0,173		

We see the collective preference for Br-in/Br-emain was above four times the collective preference for Br-out/Br-escape, leaving no doubt it was the democratic decision of UK’s citizens.

In summary, the 1975 referendum provided a start point which sets a legitimate course of action before Brexit. The fact this has been decided to be changed after Brexit, even if Brexit did not fulfill most usual requirements for these type of decisions, casts great doubts on whether Brexit results should be considered the democratic UK’ citizens choice or simply UK’s government choice.

While after 1975 referendum, the Home Secretary Roy Jenkins stated: “it puts the uncertainty behind us” [BBC news], it seems Brexit more than anything else has brought a lot of uncertainty to Britain in more than one sense.

However, even if another referendum would be advisable, a fact clearly needs to be taken into account by UK and EU politicians, the great change in UK’s citizens’ view regarding the EU [which has other parallels in other EU’s countries] should be further analyzed to find out why citizens are lowering their support for the European project.

[31] Noteworthy, also the total percentage of UK’s citizens in favor of Br-in in 1975 was higher than in favor of Br-out in 2016 [42,96% vs 37,44%]

4 PrII: A RULE FOR ELECTING REPRESENTATIVES (RER)

4.2 PRLL AS A RULE FOR ELECTING REPRESENTATIVE [DECISION MAKING] BODIES

PrII is a RER which works as a 'plug-in' to be used after PsII. It allows us to obtain an allocation of seats according [PrII y] to the collective preference for each eligible option [obtained using PsII].

That is, PrII is *a PR rule that assigns each party a number of representatives according its collective desirability.* This sets some parallel to other PR rules, but also some differences:

- Since the input variable is each party/option collective preference, PrII is a PR that can take to allocation of seats quite different to other PR rules.
- Since it is a continuous variable, mechanisms used for achieving seats concentration with PR rules referring to FPP can be used;
    - Allocation of seats can be done preserving pure proportionality to the input variable, but also using any non-proportional rule [Largest Remainders or Highest Averages].
    - District sizes and more than one tier systems can be used
    - Appropriate thresholds can be set.
- Since most choices preferences are single peaked, centered parties usually receive higher collective preference values, receiving more seats as consequence. In other words, centered parties usually receive more seats than extreme parties.

As consequence, it provides two interesting and relevant differentiating features:

First, it ensures **the number of representatives obtained by each party is related to the degree such party is preferred/non-preferred by citizens.** If political parties are ordered from highest to lowest number of representatives, this ordering matches their ordering from highest to lowest collective desirability [and it matches their Condorcet Ordering].

As consequence, it ensures parliament's overall preferences closely resemble citizens' preferences. This implies any binary decision made by the parliament would closely resemble society's consensus choice, if all citizens were to make the decision altogether. **It provides high representativeness.**

Second, since centered parties receive more seats, polarization is reduced, while pacts should be easier. **It provides increased governability.**

We believe these two features to be the more fundamental goals of RERs, and to achieve them PrII stands as an optimum RER. Let us now review the computing procedure.

4.2.1 COMPUTING PROCEDURE

PrII is an algorithm to be applied after MnII + PsII have been applied; i.e., after the ordering of the options has been defined [MnII] and their collective desirability calculated [PsII].

Since we have already reviewed many examples in the above chapter of how both rules are to be used, let us now use some of those examples so we already know each option's collective desirability values. We will be explaining PrII operating procedure in parallel to its use over some examples previously reviewed.

4.2.1.1 A CLASSICAL EXAMPLE

Let us consider the following set of individuals' preferences [Dodgson, 1884: 31]:

21.840	x_1	P	x_2	P	x_4
10.160	x_1	P	x_3	P	x_2
7.999	x_5				

The procedure for proportional allocation of seats, is applied once the position of all eligible options in the 0-1 range [their collective utility] has been calculated, i.e., after Psll rule has been used [this was explained before so we do not repeat it here; we only list the values obtained before].

	x_1	P	x_2	P	x_4	P	x_3	P	x_5	TOTAL
COLLECTIVE UTILITY/DESIRABILITY	0.878		0.512		0.333		0.155		0.122	**2.00**

We add each option's collective desirability [in the example, we obtain 2.00 –see right side of the table-]. This is the total collective utility assigned to the set of all parties, so we assign an equivalent value to total number/percentage of seats:

$$100\% = \sum_{j=1}^{|X|} P[x_j] = \sum_{j=1}^{|5|} P[x_j] = 2.0 \tag{48}$$

Then to calculate the percentage of seats to be allocated to each political party/option, we divide its relative collective desirability [its position on the 0-1 range] by above sum.

$$Seats[x_i]_\% = \frac{P[x_i\,]}{\sum_{j=1}^{|X|} P[x_j]} \tag{49}$$

In the example, if we consider there are 20 seats to be allocated, we obtain:

Prll	x_1	P	x_2	P	x_4	P	x_3	P	x_5	TOTAL
PREFERENCE INTENSITY [Psll]	0,878		0,512		0,333		0,155		0,122	2,000
ALLOCATION OF SEATS [Prll]	43,90%		25,58%		16,67%		7,75%		6,10%	100,00%
SEATS [PURE PROPORTIONALITY]	8,78		5,12		3,33		1,55		1,22	**20**
SEATS [ROUNDED NUMBERS]	9		5		3		2		1	**20**

Therefore, proportional allocation of seats is easily and quick done. Additionally, instead of pure proportionality, we can use any Largest Remainders or Highest Averages method rule to increase concentration of seats. For instance, we can use d'Hondt:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
x_1	0,878	0,4390	0,2926	0,2195	0,1756	0,1463	0,1254	0,1097	0,0975	0,0878	0,0798	0,0732
x_2	0,512	0,2558	0,1705	0,1279	0,1023	0,0853	0,0731	0,0639	0,0568	0,0512	0,0465	0,0426
x_4	0,333	0,1667	0,1111	0,0833	0,0667	0,0556	0,0476	0,0417	0,0370	0,0333	0,0303	0,0278
x_3	0,155	0,0775	0,0517	0,0388	0,0310	0,0258	0,0222	0,0194	0,0172	0,0155	0,0141	0,0129
x_5	0,122	0,0610	0,0407	0,0305	0,0244	0,0203	0,0174	0,0153	0,0136	0,0122	0,0111	0,0102

Since there is a high number of seats [20] available, allocation of seats is not modified [however, if we reduced the number of available seats to 15, x_5 would not be receiving any seats].

To provide some contrast to these values, let us also calculate allocation of seats using Borda [we use the already calculated positioning of the options using Borda rule]. We do this by following an equivalent procedure:

BORDA	x_1	P	x_2	P	x_4	P	x_3	P	x_5	TOTAL
PREFERENCE INTENSITY [Borda]	0,800		0,449		0,182		0,169		0,200	1,800
ALLOCATION OF SEATS	44,45%		24,93%		10,11%		9,41%		11,11%	100,00%
SEATS [PURE PROPORTIONALITY]	8,89		4,99		2,02		1,88		2,22	**20**
SEATS [ROUNDED NUMBERS]	9		5		2		2		2	**20**

The resemblance between both allocations of seats is evident, and quite different to that we obtain if using First Preferred Party [FPP]/Single Vote as input variable:

SINGLE VOTE/FPP AS INPUT	x_1	P	x_2	P	x_4	P	x_3	P	x_5	TOTAL
PREFERENCE INTENSITY	0,800		-		-		-		0,200	1,000
ALLOCATION OF SEATS	80,00%		-		-		-		20,00%	100,00%
SEATS [PURE PROPORTIONALITY]	16,00		-		-		-		4,00	**20**
SEATS [ROUNDED NUMBERS]	16		-		-		-		4	**20**

Above allocation of seats shows us that both preferential rules [Prll and Borda] provide a much more accurate representation of citizens' preferences than current rules using single votes as input [most majoritarian and PR rules].

4.2.2.2 AN EXAMPLE OF THREE PARADOXES OF MOST ELECTORAL RULES

Let us review an example that shows us that any electoral rule taking as input Single Votes can lead to allocation of seats which does not represent actual citizens' preferences, showing three paradoxical issues [Van Deemen & Vergunst, 1999]:

- A party having a majority over other party receives less seats
- A Condorcet Winner does not receive the largest number of seats
- The majority relation may be the reverse of the ranking of parties in terms of seats.

To show it, we review a famous set of individuals' preferences; that which allowed J.C. de Borda to prove Plurality rule may choose as winner the least collectively preferred option.

	2		1		0
1	x_1	P	x_2	P	x_3
7	x_1	P	x_3	P	x_2
7	x_2	P	x_3	P	x_1
6	x_3	P	x_2	P	x_1
21					

To add contrast, we calculate the input variable for the allocation of seats using Single Vote, Borda and Prll .

Using Single Vote [SV] equates as taking into account only individuals' first choice, dividing each option's votes by the total number of votes

SINGLE VOTE / FIRST PREFERRED PARTY			
	Votes as First Choice	Preference Intensity / Collective Utility	Input Variable for Allocation of Seats
x_1	8	0,3810	38,10%
x_2	7	0,3333	33,33%
x_3	6	0,2857	28,57%
	21	1,0000	100,00%

If we use Borda, we calculate each option's score. Then we calculate each option's Preference intensity/collective utility[32] and divide each option's collective utility by the sum of all them:

BORDA			
	Borda Score	Preference Intensity / Collective Utility	Input Variable for Allocation of Seats
x_1	16	0,381	25,40%
x_2	21	0,500	33,33%
x_3	26	0,619	41,27%
		1,500	100,00%

If we use Prll , first we draw the PC matrix:

	n[…,x_1]	n[…,x_2]	n[…,x_3]
n[x_1,…]	-	8	8
n[x_2,…]	**13**	-	8
n[x_3,…]	**13**	**13**	-

From above matrix, we draw the TM matrix:

	v[…,x_1]	v[…,x_2]	v[…,x_3]	**0**	There is Condorcet Winner
v[x_1,…]	-	5	5	5	
v[x_2,…]	**0**	-	5	5	
v[x_3,…]	**0**	**0**	-	0	x_3
	0	5	5	0	There is Condorcet Loser
	x_1				

[32] We could actually directly calculate each option's allocation of seats as its Borda score divided by the sum of the three options' scores. However, we will use Preference Intensity values to make it clearer.

Then we assign collective utility to every option using PsII:

- the most preferred option is x_3, and we can calculate its higher desirability in relation to the rest of eligible options [$\neg x_3$] as:

x_3
$$\Delta d[x_3 - x_{\neg 3}] = min\big[rows[x_{\neg 3} = x_1, x_2]\big] - row[x_3] \quad (50)$$

$$\Delta d[x_3 - x_{\neg 3}] = 5 - 0 = 5 \quad (51)$$

- the least preferred option is x_1, and we can calculate its lower desirability in relation to the rest of eligible options [$\neg x_1$] as:

x_1
$$\Delta d[x_{\neg 1} - x_1] = min\big[columns[x_{\neg 1} = x_2, x_3]\big] - column[x_1] \quad (52)$$

$$\Delta d[x_{\neg 1} - x_1] = 5 - 0 = 5 \quad (53)$$

Therefore:

x_3	P	x_2	P	x_1
	5,0		5,0	

Now we need to calculate the distance to the limiting points 1 and 0, which we do in relation to most/least preferred options:

- *Distance to 1.* We calculate it as the number of individuals expressing preferences about any option [including blank votes] minus the maximum number of individuals supporting the most preferred option [x_3] in any pairwise confrontation with any other eligible option. In this case we obtain

Distance to 1
$$\Delta d[1 - x_3] = N - max\left[n[x_3 P x_j]_{j=1,2}\right] = 21 - 13 = 8 \quad (54)$$

- *Distance to 0.* We calculate it as the maximum number of individuals who support the least preferred option [x_1] against any eligible option [including those individuals who explicit indifference among all eligible options]:

Distance to 0
$$\Delta d[x_1 - 0] = max\left[n[x_1 P x_j]_{j=2,3}\right] = 8 \quad (55)$$

Therefore desirability differentials are:

	x_3	P	x_2	P	x_1		¬X
8,0		5,0		5,0		8,0	

And we can position options in the 0-1 range by simply dividing the sum of their desirability differentials starting from ¬X [equivalent to 0] up to the option which position we want to calculate by the sum of every desirability differential [from 1 to 0]:

$$x_3 = \frac{8+5+5}{8+5+5+8} = 0{,}692$$

$$x_2 = \frac{8+5}{8+5+5+8} = 0{,}500$$

$$x_1 = \frac{8}{8+5+5+8} = 0{,}308$$

We can summarize above preferences' intensities in a table:

	x_3	P	x_2	P	x_1
PREFERENCE INTENSITY/COLLECTIVE UTILITY	0,692		0,500		0,308

The review of this set of individuals' preferences, which gave rise to Borda's Paradox, allows us to see SV/FPP as input variable can lead to choosing as most preferred option an option which is clearly less preferred than non-preferred [0.308]; an option [x_1] which is less than half [44%] preferred than the actual most preferred option [x_3].

This is a quite strong statement supporting electoral systems' changing to Condorcet methods.

Still, let us calculate allocation of seats to each option according to Prll . We add each option's collective desirability values [in the example, we obtain 1.5].

Total percentage of seats

$$100\% = \sum_{j=1}^{|X|} P[x_j] = \sum_{j=1}^{|5|} P[x_j] = 1.5 \tag{56}$$

Then to calculate the percentage of seats to be allocated to each political party, we divide its relative collective desirability [its position on the 0-1 scale] by above sum.

x_i

$$Seats[x_i]_{\%} = \frac{P[x_i\,]}{\sum_{j=1}^{|X|} P[x_j]} = \frac{P[x_i\,]}{1.5} \tag{57}$$

We obtain the following allocation of seats:

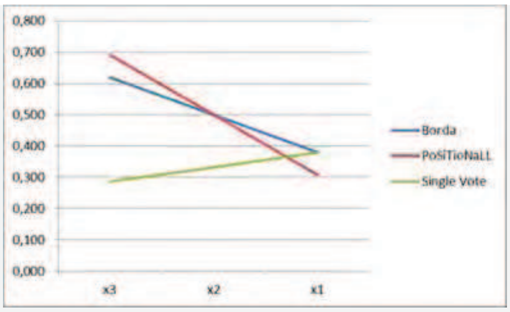
	x_3	P	x_2	P	x_1	TOTAL
PREFERENCE INTENSITY [Psll]	0,692		0,500		0,308	**1,50**
INPUT VARIABLE FOR ALLOCATION OF SEATS [Prll]	46,15%		33,33%		20,51%	**100,00%**

Let us now compare the three obtained input variables for allocation of seats [we arrange options according Mnll ordering, coincident with Borda in this example]:

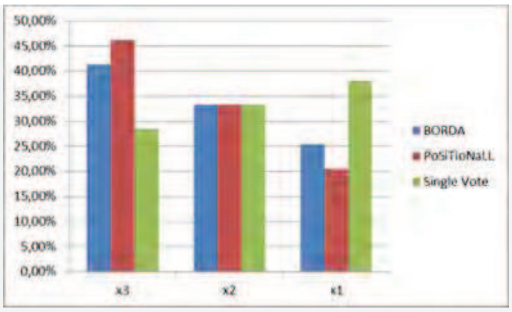
COMPARISON OF INPUT VARIABLE USED BY DIFFERENT ALLOCATION RULES									
	Input variable for Allocation Of Seats			Standard Deviation			Pearson Correlation		
Option	BORDA	PsII	Single Vote	Borda-PoS	PoS-SV	Borda-SV	Borda-PoS	PoS-SV	Borda-SV
x_3	41,27%	46,15%	28,57%	0,024	0,088	0,063			
x_2	33,33%	33,33%	33,33%	0,000	0,000	0,000	1,000	-1,000	-1,000
x_1	25,40%	20,51%	38,10%	0,024	0,088	0,063			

Data allows us to clearly appreciate the high resemblance of Borda and PrII , which provide a matching ordering of the options and similar values of input variables. On the contrary, if we were to use SV as input variable, we arrive to a reversed ordering of the parties.

This can be more clearly appreciated in the graphical representation:



INTENSITY OF PREFERENCES



ALLOCATION OF SEATS

We see the three paradoxes stated by Van Deemen & Vergunst [1999] appear for this set of individuals' preferences.

4.2.2.3 A MEANINGFUL CASE: CURRENT PR RULES FAVOR EXTREME PARTIES AGAINST CENTERED OPTIONS

Let us review a highly unlikely case but useful to make evident most currently used electoral laws favor extreme options against centered options. Let us consider four political parties present candidatures for a six seats representative body. Three parties can be located in a right-left ordered as x_1 [right wing]; x_2 [center] and x_3 [left wing]. The fourth political party cannot clearly be located in L-R terms, since it is oriented on a different ideology [e.g., a party in favor of animals' rights]. A group of 5,000 citizens vote expressing the following preferences:

$$2500 \quad x_1 \; P \; x_2 \; P \; x_3 \; P \; x_4$$

$$2500 \quad x_3 \; P \; x_2 \; P \; x_1 \; P \; x_4$$

$$5000$$

If we use Single Vote, we arrive to:

SINGLE VOTE			
	Votes As Most Preferred Option	Input Variable for Allocation Of Seats	Allocated Seats
x_1	2,500	50.00%	3

x_2	0	0.00%	-
x_3	2,500	50.00%	3
x_4	0	0.00%	-
	5,000	100.00%	6

If we use Borda Rule, we consider the following scoring scale:

	3		2		1		0
2,500	x_1	P	x_2	P	x_3	P	x_4
2,500	x_3	P	x_2	P	x_1	P	x_4
5,000							

Therefore, we obtain the following scores and allocation of seats:

BORDA RULE			
	Borda Score	Input Variable for Allocation Of Seats	Allocated Seats
x_1	10,000	33,33%	2
x_2	10,000	33,33%	2
x_3	10,000	33,33%	2
x_4	0	0,00%	-
	30,000	100,00%	6

Let us first see whether there is a Condorcet Winner. First we draw the PC matrix:

	d[…,x_1]	d[…,x_2]	d[…,x_3]	d[…,x_4]
d[x_1,…]	-	2500	2500	**5000**
d[x_2,…]	2500	-	2500	**5000**
d[x_3,…]	2500	2500	-	**5000**
d[x_4,…]	0	0	0	-

From above matrix, Condorcet Winner Criterion states a tie between x_1, x_2 and x_3, being the three options more preferred than x_4.

If we use Prll , from above pairwise comparison matrix, we calculate the TM matrix:

	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	0	
v[x_1,…]	-	0	0	**0**	0	x_1
v[x_2,…]	0	-	0	**0**	0	x_2
v[x_3,…]	0	0	-	**0**	0	x_3
v[x_4,…]	5,000	5,000	5,000		50	
	5,000	5,000	5,000	0	**0**	
				x_4		

Then we assign collective utility to every option using PsII:

- the most preferred options are x_1,x_2 and x_3, and we can calculate their higher desirability in relation to the rest of eligible options [$\neg x_3$=x_4] as:

x_1, x_2 & x_3

$$\begin{aligned}\Delta d[\{x_1,x_2,x_3\} - \neg\{x_1,x_2,x_3\}] \\ = min\big[rows[\neg\{x_1,x_2,x_3\} = x_4]\big] \\ - row[\{x_1,x_2,x_3\}]\end{aligned} \tag{58}$$

$$\Delta d[\{x_1,x_2,x_3\} - \neg\{x_1,x_2,x_3\}] = 5000 - 0 = 5000 \tag{59}$$

- the least preferred option is x_4, and we can calculate its lower desirability in relation to the rest of eligible options [$\neg x_4$] as:

x_4

$$\Delta d[x_{\neg 4} - x_4] = min\big[columns[x_{\neg 4} = x_1,x_2,x_3]\big] - column[x_4] \tag{60}$$

$$\Delta d[x_{\neg 4} - x_4] = 5000 - 0 = 5000 \tag{61}$$

Therefore:

x_1	I	x_2	I	x_3	P	x_4
	0,0		0,0		5000,0	

Now we need to calculate the distance to the limiting points 1 and 0, which we do in relation to most/least preferred options:

- Distance to 1. We calculate it as the number of individuals expressing preferences about any option [including blank votes] minus the maximum number of individuals supporting the most preferred options [x_1, x_2, x_3] in any pairwise confrontation with any other eligible option. In this case we obtain

Distance to 1

$$\Delta d[1 - \{x_1,x_2,x_3\}] = 5000 - 5000 = 0 \tag{62}$$

- Distance to 0. We calculate it as the maximum number of individuals who support the least preferred option [x_4] against any eligible option [including those individuals who explicit indifference among all eligible options]:

Distance to 0

$$\Delta d[x_4 - 0] = 0 \tag{63}$$

Therefore desirability differentials are:

	x_1	I	x_2	I	x_3	P	x_4	P	¬X
0,0		0,0		0,0		5000,0		0,0	

And we can position options in the 0-1 range by simply dividing the sum of their desirability differentials starting from ¬X [equivalent to 0] up to the option which position we want to calculate by the sum of every desirability differential [from 1 to 0]:

$$X_1 = \frac{0+5000+0+0}{0+5000+0+0+0} = 1{,}000$$

$$x_2 = \frac{0+5000+0}{0+5000+0+0+0} = 1{,}000$$

$$x_3 = \frac{0+5000}{0+5000+0+0+0} = 1{,}000$$

$$x_4 = \frac{0}{0+5000+0+0+0} = 0{,}000$$

We can summarize above preferences' intensities in a table:

	x_1	I	x_2	I	x_3	P	x_4
PREFERENCE INTENSITY [Psll]	1,000		1,000		1,000		0,000

From above table we can easily calculate the allocation of seats to each option. We add the positions/collective desirability values of each option [in the example, we obtain 3.0].

Total percentage of seats

$$100\% = \sum_{j=1}^{|X|} P[x_j] = \sum_{j=1}^{|5|} P[x_j] = 3.0 \tag{64}$$

Then to calculate the percentage of seats to be allocated to each political party, we divide its relative collective desirability [its position on the 0-1 scale] by the above sum.

x_i

$$S[x_i] = \frac{P[x_i\,]}{\sum_{j=1}^{|X|} P[x_j]} = \frac{P[x_i\,]}{3.0} \tag{65}$$

We obtain the following allocation of seats:

	x_1	I	x_2	I	x_3	P	x_4
Preference Intensity [Psll]	1,000		1,000		1,000		0,000
Input variable for Allocation Of Seats [Prll]	33,33%		33,33%		33,33%		0,00%
Seats allocated	2		2		2		0

Let us now compare the three obtained allocation of seats [we arrange options according Mnll ordering]:

	COMPARISON OF ALLOCATION RULES		
	Borda	Prll	First Past The Post/STV
x_1	33,33%	33,33%	50,00%

x_2	33,33%	33,33%	0,00%
x_3	33,33%	33,33%	50,00%
x_4	0,00%	0,00%	0,00%

We see most electoral rules based on SV/FPP eliminate the centered party, assigning its quota to both extreme parties. Or more graphically, the resulting chamber eliminates moderated parties in favor of extreme parties.

While Borda and PrII equally divide the seats among the three more preferred parties, electoral rules based on SV and STV assign no seats to the centered party, generating a polarized chamber which decisions will difficultly resemble individuals' preferences, and may lead to polarization and instability[33].

[33] Noteworthy, while this example reflects a very unlikely set of individuals' preferences, similar 'polarization' effects can be observed when applying FPTP/PR/STV to not so unlikely sets, showing this is a conceptual flaw underlying such rules. For a practical case, see next chapter: **Error! Reference source not found.**

5 CONCLUSIONS

We have explained a family of three rules [or maybe one rule with three different parts], whose main feature is that they are consistent with the recognition, explicit in most countries' Constitutions that the views of all individuals should have the same value.

The first rule, which we designate as Mnll, provides an ordering of the options according their collective preference that proves to be consistent with most non-refutable Social Choice Criteria.

The second rule, Psll allows us to assign each the eligible option a collective utility/ desirability value in the range 0-1, providing some very interesting features not necessarily appreciable in options' linear orderings. The high correlation obtained with Borda Count combined with the fact Psll is Condorcet consistent [and complies with the Majority Principle] allows us to state Psll can be used for many collective decision making processes.

One of its most interesting applications is for holding referendums, since it allows citizens to express different views on the issues discussed [e.g., rejection or support for one option, but also for every eligible option], as well as accounting for abstention in a more consistent manner.

And the third rule, Prll allows us to overcome paradoxes of current rules for allocation of seats. In most societies, who forms the government has considerable power because is who redacts and approves [or rejects] laws and regulations. And the fact that current rules for Proportional Representation/allocation of seats do not meet the Condorcet criterion can lead to paradoxical situations, i.e., *it can even lead to the government being occupied by the least preferred political party.*

Nowadays, our societies are largely regulated by rules often drafted and approved by political parties whose ideology does not match [and can even be very opposed to] that of most citizens.

This is of the utmost importance. We started the text saying that the three rules herein proposed build on the paradigm of considering that all individuals should have equal rights[34]. And yet, in most Constitutions equality is limited to 'equality before the law'. As consequence, the desirable equality in our societies is usually defined by laws, which acquire fundamental importance.

Most of the regulatory framework that shapes life in our societies is composed by laws often enacted and passed by governments which are not the most preferred [and can even be highly rejected] by most citizens. Current allocation rules allow political parties to become the governing party if supported by a clear minority of the population. Hence attending to such minority's ideology becomes the goal of such governments[35].

That's one of the paradoxes Prll help to solve. Our belief is that a Government should always be concerned with every citizen [or at least with most of them], and never only with a minority of them. **To achieve that, we need to use rules for electing representatives which are Condorcet Consistent.**

---

[34] We state individuals meaning human beings but also different cultural identities. Balance between different types of individuals may require double majorities' requirements as explained in the text.

[35] For instance, First Past The Post can take to being the most preferred option a party that only 30,1% of citizens locate as first option [Norris, 1997]; i.e., even if almost 70% of citizens' disagree with said party's ideology

Besides, our review of Spanish Elections shows PR and majoritarian rules over represent extreme parties, leading to more polarized than actual representative parliaments. On the contrary, the use of Condorcet Methods may help pacific coexistence in highly divided environments where the use of plurality rules may lead to increasing confrontation.

ANNEX II: AN EXAMPLE WHERE PSLL AND BORDA GREATLY DIFFER

Let us consider the following individuals preferences [Tideman, 1987:198. Example 5]

7	x_1	P	x_2	P	x_3	P	x_4	P	x_5
3	x_5	P	x_4	P	x_1	P	x_2	P	x_3
6	x_4	P	x_5	P	x_2	P	x_3	P	x_1
3	x_2	P	x_3	P	x_1	P	x_5	P	x_4
5	x_5	P	x_3	P	x_1	P	x_2	P	x_4
3	x_4	P	x_3	P	x_1	P	x_2	P	x_5
27									

If we review the options by pairwise comparing their relative preference we see that:

	d[…,x_1]	d[…,x_2]	d[…,x_3]	d[…,x_4]	d[…,x_5]
d[x_1,…]	-	**18**	10	**15**	13
d[x_2,…]	9	-	**19**	**15**	13
d[x_3,…]	**17**	8	-	**15**	13
d[x_4,…]	12	12	12	-	**16**
d[x_5,…]	**14**	**14**	**14**	11	-

We arrive to a cyclical relationship of relative preference between the five options:

$$\ldots x_1 P x_2 P x_3 P x_4 P x_5 P x_1 \ldots. \tag{66}$$

If we use Mnll :

STEP 01	v[…,x_1]	v[…,x_2]	v[…,x_3]	v[…,x_4]	v[…,x_5]	3
v[x_1,…]	-	**0**	7	**0**	1	7
v[x_2,…]	9	-	**0**	**0**	1	9
v[x_3,…]	**0**	11	-	**0**	1	11
v[x_4,…]	3	3	3		**0**	3
v[x_5,…]	**0**	**0**	**0**	5		5
	9	11	7	5	1	**1**
					x_5	

There is neither Condorcet Winner nor Loser. We see the dominant criterion is Least Preferred Option, so x_5 is less preferred than x_1-x_4.

$$\{x_1, x_2, x_3, x_4\} P x_5 \tag{67}$$

We remove it from the choice set and review again:

STEP 02	v[...,x_1]	v[...,x_2]	v[...,x_3]	v[...,x_4]	3	There is no Condorcet Winner
v[x_1,...]	-	**0**	7	**0**	7	
v[x_2,...]	9	-	**0**	**0**	9	
v[x_3,...]	**0**	11	-	**0**	11	
v[x_4,...]	3	3	3		3	
	9	11	7	0	**0**	
				x_4		There is Condorcet Loser

There is no Condorcet Winner, nor Condorcet Loser. We see the dominant criterion is Least Preferred Option, so x_4 is less preferred than x_1-x_3.

$$\{x_1, x_2, x_3\}Px_4Px_5 \tag{68}$$

We remove it from the choice set and review again:

STEP 03	v[...,x_1]	v[...,x_2]	v[...,x_3]	**7**	There is no Condorcet Winner
v[x_1,...]	-	**0**	7	7	x_1 Most Preferred Option
v[x_2,...]	9	-	**0**	9	
v[x_3,...]	**0**	11	-	11	
	9	11	7	7	There is Condorcet Loser
			x_3	Least Preferred Option	

Still there is no Condorcet Winner, nor Condorcet Loser. Since there are only three remaining eligible options, both criteria necessarily match so we can order the three options. Hence, the complete ordering of the options is:

$$x_1Px_2Px_3Px_4Px_5 \tag{69}$$

Now let us position the options in the range 1-0:

Difference in preference between option x_5 and option x_4 must be calculated operating on the columns at Step 01 matrix:

STEP 01	v[...,x_1]	v[...,x_2]	v[...,x_3]	v[...,x_4]	v[...,x_5]	3
v[x_1,...]	-	**0**	7	**0**	1	7
v[x_2,...]	9	-	**0**	**0**	1	9
v[x_3,...]	**0**	11	-	**0**	1	11
v[x_4,...]	3	3	3		**0**	3
v[x_5,...]	**0**	**0**	**0**	5		5
	9	11	7	5	1	**1**
					x_5	

Since x_5 is less preferred to every other option, the formula is:

x_5

$$\Delta P[x_5 - x_{\neg 5}] = min\left[columns[x_{\neg 5} = x_1, x_2, x_3, x_4]\right] - column[x_5] \tag{70}$$

$$\Delta P[x_5 - x_{\neg 5}] = 5 - 1 = 4 \tag{71}$$

Difference between options x_3 and x_4 should be calculated at the step where x_4 is removed from the choice set [Step 02], but at this step we observe the cyclic relation has been broken, so we need to calculate it at the step before [Step 01], not taking into account options already chosen when arriving to Step 02 [i.e., x_5]:

x_4

$$\Delta P[x_4 - x_{\neg 4}] = min\left[columns[x_{\neg 4} = x_1, x_2, x_3]\right] - column[x_4] \tag{72}$$

$$\Delta P[x_4 - x_{\neg 4}] = 7 - 5 = 2 \tag{73}$$

Difference between options x_1-x_3 should be calculated at step where they are removed from the choice set [Step 03], but at this step we observe the cyclic relation has been broken, so we need to calculate it at a previous step [Step 01], not taking into account options already chosen when arriving to Step 03 [i.e., x_5 and x_4]:

- for x_1, we operate on the rows [defeats]:

x_1

$$\Delta P[x_1 - x_{\neg 1}] = min\left[rows[x_{\neg 1} = x_2, x_3]\right] - row[x_1] \tag{74}$$

$$\Delta P[x_1 - x_{\neg 1}] = 9 - 7 = 2 \tag{75}$$

- for x_3 we operate on the columns [victories]:

x_4

$$\Delta P[x_3 - x_{\neg 3}] = min\left[columns[x_{\neg 3} = x_1, x_2]\right] - column[x_3] \tag{76}$$

$$\Delta P[x_3 - x_{\neg 3}] = 9 - 7 = 2 \tag{77}$$

Now we can draw a table stating the preference differences between options:

x_1	P	x_2	P	x_3	P	x_4	P	x_5
	2,0		2,0		2,0		4,0	

Let us now calculate distance to limiting points 0-1 of the scale:

- Distance to 1, we subtract to the total number of individuals [N] the maximum number of individuals supporting the most preferred option [x_1] in any confrontation:

Distance to 1

$$\Delta P[1 - x_1] = N - max\left[n[x_1 P x_j]_{j=2-5}\right] = 27 - 18 = 9 \tag{78}$$

- Distance to 0 point, we calculate the maximum number of individuals supporting the least preferred option [x_5] in the pairwise matrix at Step 01 [when x_5 is removed from the choice set and cyclic preference relation still holds]:

Distance to 0 $$\Delta P[x_5 - 0] = max\left[n[x_5 P x_j]_{j=1-4}\right] = 14 \quad (79)$$

Now, we can draw the whole scale:

	x_1	P	x_2	P	x_3	P	x_4	P	x_5	P	¬X
9,0		2,0		2,0		2,0		4,0		14,0	

From above scale, we calculate the desirability of each option by adding every desirability difference regarding the 0 point [every number on the right side of that option up to the ¬X], and normalize it by dividing it between the sum of all of them [including the numbers on the left side, which informs us of the distance to the 1 point]. We obtain:

$$x_1 = \frac{14+4+2+2+2}{14+4+2+2+2+9} = 0{,}727$$

$$x_2 = \frac{14+4+2+2}{14+4+2+2+2+9} = 0{,}667$$

$$x_3 = \frac{14+4+2}{14+4+2+2+2+9} = 0{,}606$$

$$x_4 = \frac{14+4}{14+4+2+2+2+9} = 0{,}545$$

$$x_5 \quad \frac{14}{14+4+2+2+2+9} = 0{,}424$$

We can summarize it in a table:

	x_1	P	x_2	P	x_3	P	x_4	P	x_5
PREFERENCE INTENSITY/COLLECTIVE UTILITY	0,727		0,667		0,606		0,545		0,424

Let us now calculate each option's collective utility using Borda. Since every ordering comprises all the options, all scales are equal:

	4		3		2		1		0
7	x_1	P	x_2	P	x_3	P	x_4	P	x_5
3	x_5	P	x_4	P	x_1	P	x_2	P	x_3
6	x_4	P	x_5	P	x_2	P	x_3	P	x_1
3	x_2	P	x_3	P	x_1	P	x_5	P	x_4
5	x_5	P	x_3	P	x_1	P	x_2	P	x_4
3	x_4	P	x_3	P	x_1	P	x_2	P	x_5
27									

Then we calculate the Borda Score for each option:

	BORDA SCORE	
	POINTS	TOTAL
x_1	*=7*4+3*2+6*0+3*2+5*2+3*2=*	56
x_2	*=7*3+3*1+6*2+3*4+5*1+3*1=*	56
x_3	*=7*2+3*0+6*1+3*3+5*3+3*3=*	53
x_4	*=7*1+3*3+6*4+3*0+5*0+3*4=*	52
x_5	*=7*0+3*4+6*3+3*1+5*4+3*0=*	53

We calculate the maximum possible score any option could obtain; i.e., its score if it was declared most preferred choice by each individual:

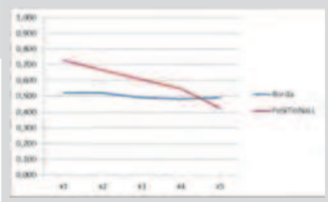
Maximum Borda score
$$= 27 * 4 = 108 \tag{80}$$

And we can now easily calculate each option's position in the range 0-1 by dividing its score by the maximum possible score:

	BORDA RULE	
	SCORE	POSITION
x_1	*=56/108=*	0,519
x_2	*=56/108=*	0,519
x_3	*=53/108=*	0,491
x_4	*=52/108=*	0,481
x_5	*=53/108*	0,491

Now we can compare the values obtained using Borda rule with those obtained using Psll [we arrange options according to ordering obtained using Psll]:

	PREFERENCE INTENSITY / COLLECTIVE UTILITY		
	Borda	Psll	Standard Deviation
x_1	0,519	0,727	0,022
x_2	0,519	0,667	0,011
x_3	0,491	0,606	0,007
x_4	0,481	0,545	0,002
x_5	0,491	0,424	0,002



We arrive at an average standard deviation of 0.009 and a Pearson Correlation of 0.764. Yet, while numerical data apparently states both rules lead to similar orderings, the graphical analysis shows they actually do not.

This example becomes interesting because preferences include a set of clones $\{x_1, x_2, x_3\}$. Let us calculate each option's collective preference according to both Borda and Psll when we consecutively eliminate each possible option inside the set of clones.

COLLECTIVE UTILITY ASSIGNED TO EACH OPTION FOR DIFFERENT CHOICE SETS [REMOVING DIFFERENT CLONES]										
CHOICE SET	BORDA					PSLL				
	UTILITY ASSIGNED TO EACH OPTION					UTILITY ASSIGNED TO EACH OPTION				
	x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5
X	0,615	0,615	0,593	0,585	0,593	0,727	0,667	0,606	0,545	0,424
X-{x_1}	-	0,685	0,583	0,620	0,611	-	0,714	0,643	0,571	0,500
X-{x_2}	0,602	-	0,667	0,620	0,611	0,600	-	0,667	0,533	0,467
X-{x_3}	0,676	0,593	-	0,620	0,611	0,690	0,621	-	0,552	0,483
X-{x_2,x_3} (1)	0,679	-	-	0,679	0,642	0,600	-	-	0,533	0,467
Deviation (2)	0,005	0,005	0,004	0,005	0,001	0,012	0,004	0,002	0,001	0,003
Average Deviation			0,004					0,005		

NOTES:
Removing x_2&x_3 necessarily leads to same results than removing x_1&x_3 or x_1&x_2
Calculated for each option collective utility in all situations where the option belongs to the choice set X

We see both rules provide apparently similarly consistent results, since when we review the variation in the collective utility assigned to each option, we see reduced variations [average standard deviation of 0,004 for Borda and 0,005 for Psll].

However, this apparent consistency of both rules is broken when we review the Pearson correlation of different orderings. Since the transformation we have undertaken have in all cases limited to inside the set of clones, it should be that the overall ordering of the clones relating options outside the set of clones and the options outside the set of clones among themselves does not change [in statistical terms, it implies we should find a positive and preferably high correlation between the series of options' collective utility values]. However, if we check the above values, we obtained the following correlations:

CORRELATION BETWEEN UTILITY VALUES OBTAINED FOR EACH CHOICE SET										
	BORDA					PSLL (1)				
	X	X-{x_1}	X-{x_2}	X-{x_3}	X-{x_2,x_3}	X	X-{x_1}	X-{x_2}	X-{x_3}	X-{x_2,x_3}
X	-	-	-	-	-	-	-	-	-	-
X-{x_1}	0,827	-	-	-	-	0,983	-	-	-	-
X-{x_2}	-0,433	-0,922	-	-	-	0,744	0,982	-	-	-
X-{x_3}	0,271	-0,901	-0,792	-	-	0,990	0,982	0,982	-	-
X-{x_2,x_3}	0,277	1,000	0,000	0,610	-	0,993	1,000	1,000	0,982	-
Average	0,000					0,964				

(1) Noteworthy, for all utility assignments according Psll we obtain correlations above 0.90, implying high consistency. We only obtain a lower value when we compare X-{x_2} against X [i.e., when x_2 is removed] because the cycle is broken in a way x_3 becomes more preferred than x_1 [see above table detailing each option's collective utility]. This is something totally consistent with Condorcet Criterion and the nature of cycles. Still, the change in the ordering only involves options x_1 and x_3, and the collective utility assigned to the options is still quite similar. This is quite meaningful.

Utility assignments according to Psll show an almost prefect correlation [R=0,964] implying high consistency between orderings/assignments, while utility assignments according to Borda show huge variations in consistency, and an overall lack of correlation [R =0.000]. This implies that Borda rule may be used to provide some contrast for collective utility assignments, but we should not in general use it as the rule for computing each option's collective utility.