A mystery circle arising from Laurent expansion

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Abstract. Let α and β be externally touching circles of radii a and b. For a parametric equation $g_z(x, y) = 0$ of circles touching α and β, we consider the following Laurent expansion of $g_z(x, y)$ around $z = d$, where $d = \sqrt{ab}/(a + b)$.

$$g_z = \cdots + C_{-1}(z-d)^{-1} + C_0 + C_1(z-d) + C_2(z-d)^2 + \cdots .$$

Then $C_{-1} = 0$ is an equation of one of the external common tangent of α and β. Also $C_i = 0$ $(i = 1, 2, 3, \cdots)$ is an equation of the other external common tangent of α and β. The equation $C_0 = 0$ represents a notable circle passing through the points where the line expressed by $C_{-1} = 0$ touches α and β.

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In this note we consider an equation $g_z(x, y) = 0$ with a parameter z for a real number z, which represents the circles touching two given externally touching circles. There are two singular cases $z = \pm d$ $(d > 0)$, in which the denominator of $g_z(x, y)$ equals zero. The cases are supposed to yield the two external common tangents of the two given circles, but the equation $g_z(x, y) = 0$ does not make any sense in this event. On the other hand we can show that considering the Laurent expansion of $g_z(x, y)$ around d, we can get equations of the two tangents beside an equation of one notable circle related to the two tangents. The purpose of this note is to describe this unexpected phenomenon and call attention to the fact.

For a point C on the segment AB such that $|BC| = 2a$ and $|CA| = 2b$ in the plane, let α and β be the circles of diameters BC and CA, respectively (see Figure 1). We use a rectangular coordinates system with origin C such that the point B has coordinates $(2a, 0)$. Let $c = a + b$ and $d = \sqrt{ab}/c$. We use the next theorem.

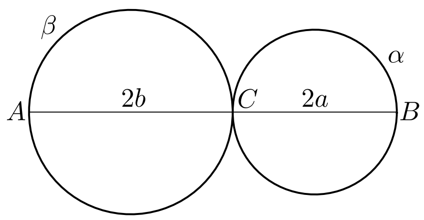

Figure 1.

Theorem 1. *A circle touches the circles α and β at points different form C if and only if its has radius r_z and center of coordinates (x_z, y_z) given by*

$$r_z = |q_z| \quad \text{and} \quad (x_z, y_z) = \left(\frac{b-a}{c} q_z, 2zq_z\right), \ \text{where } q_z = \frac{abc}{c^2z^2 - ab},$$

for a real number $z \neq \pm d$.

Proof. Let δ_z be the circle of radius r_z and center of coordinates (x_z, y_z). Then we have $(x_z - a)^2 + (y_z - 0)^2 = (a + q_z)^2$. Therefore δ_z and α touch internally or externally according as $q_z < 0$ or $q_z > 0$. Similarly δ_z and β touch internally or externally according as $q_z < 0$ or $q_z > 0$. Hence δ_z touches α and β at points different from C. Conversely we assume that a circle δ' of radius r touches α and β at points different from C. Then there is a real numbers z such that $|q_{\pm z}| = r$. Solving the equations for z, we get four solutions $z = z_i$ $(i = 1, 2, 3, 4)$ in general. Therefore we have $\delta' = \delta_{z_i}$ for some z_i. The proof is complete. □

Essentially the same formulas as Theorem 1 can be found in [5]. Simpler expression in the case z being an integer can be found in [1, 2]. We denote the circle of radius r_z and center of coordinates (x_z, y_z) by γ_z. Notice that γ_0 is the circle of diameter AB (see Figure 2). The external common tangents of α and β have following equations [3, 4], and denoted by $\gamma_{\pm d}$:

$$(a - b)x \mp 2\sqrt{ab}y + 2ab = 0.$$

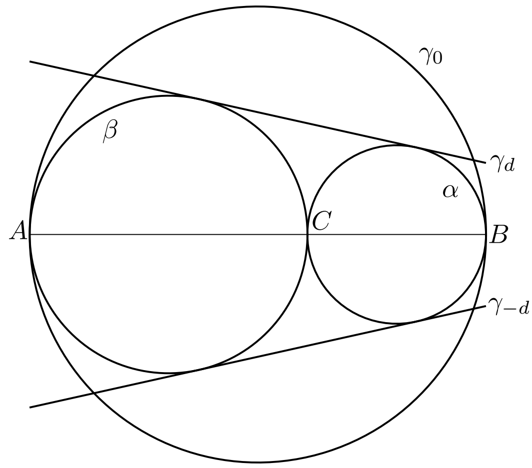

Figure 2.

Let $g_z(x, y) = (x - x_z)^2 + (y - y_z)^2 - (r_z)^2$. Then $g_z(x, y) = 0$ is an equation of the circle γ_z for $z \neq \pm d$. Let

$$g_z(x, y) = \cdots + C_{-2}(z - d)^{-2} + C_{-1}(z - d)^{-1} + C_0 + C_1(z - d) + \cdots$$

be the Laurent expansion of $g_z(x, y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$

$$C_{-1} = d((a - b)x - 2\sqrt{ab}y + 2ab),$$

$$C_0 = \left(x - \frac{a - b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2 + 18ab + b^2}}{4}\right)^2,$$

and

$$C_n = -\frac{1}{2}\left(\frac{-1}{2d}\right)^n ((a - b)x + 2\sqrt{ab}y + 2ab) \quad for\ n = 1, 2, 3, \cdots.$$

Therefore $C_{-1} = 0$ is an equation of the line γ_d. Also $C_n = 0$ is an equation of the line γ_{-d} for $n = 1, 2, 3, \cdots$.

Let ε be the circle given by the equation $C_0 = 0$. We have considered this circle in [3], which has the following properties (see Figure 3):
(i) The points, where γ_d touches α and β, lie on ε.

(ii) The radical center of the three circles α, β and ε has coordinates $(0, -\sqrt{ab})$, and lies on the line γ_{-d}.
Let E be the point of intersection of γ_d and γ_{-d}, which has coordinates $(2ab/(b-a), 0)$. We would like to state one more property here:
(iii) The radical axis of ε and γ_z passes through the point E and the two circles are orthogonal to the circle of center E passing through C for a real number z.

The y-axis meets γ_0 and $\gamma_{\pm d}$ in the points of coordinates $(0, \pm 2\sqrt{ab})$ and $(0, \pm\sqrt{ab})$, respectively, while the radical axis of γ_0 and ε passes through the point of coordinates $(0, 3\sqrt{ab})$. Hence the six points, where the y-axis meets γ_0, $\gamma_{\pm d}$, the line AB, the radical axis of γ_0 and ε, are evenly spaced. Those points are denoted in magenta in Figure 3. We also get similar results for the Laurent expansion of $g_z(x, y)$ around $z = -d$.

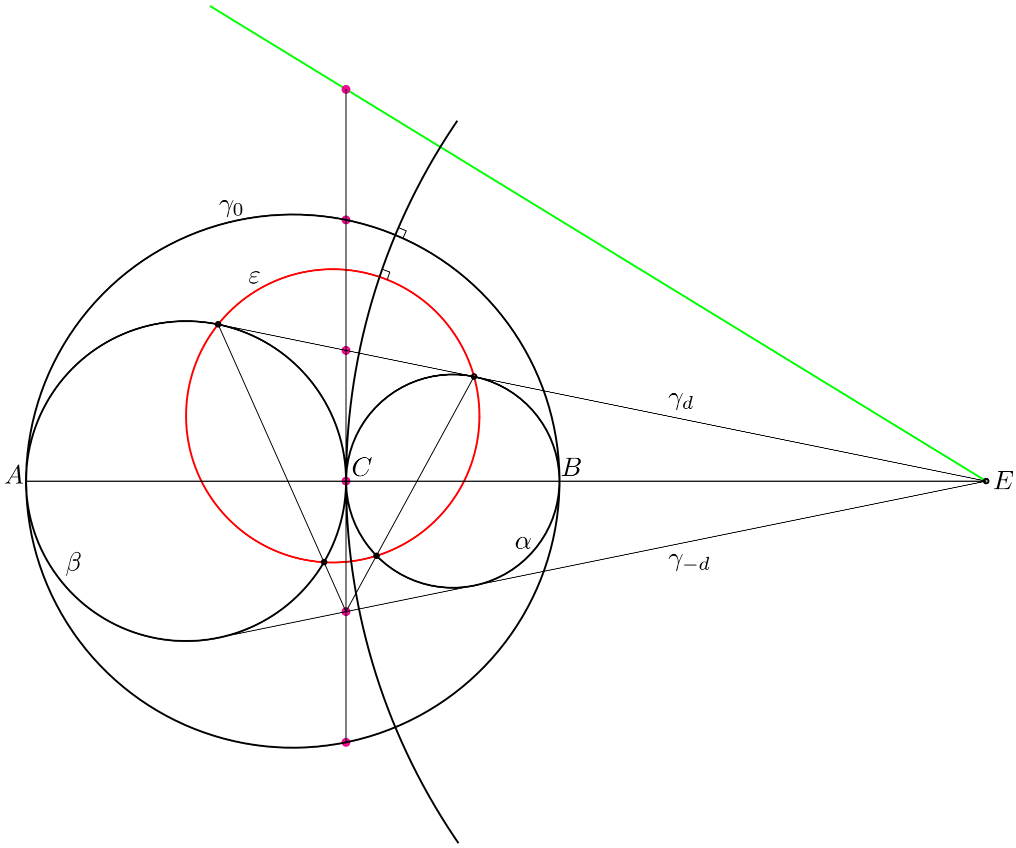

Figure 3: The green line denotes the radical axis of the circles γ_0 and ε.

We have seen that the equation $C_i = 0$ represents a meaningful figure for non zero C_i. There seems something new attractive theory around the fact. However it seems that we have no idea to logically explain this at the present time. Thereby the author would like to ask the readers how to understand the fact.

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