Mirror Images and Division by Zero Calculus

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Abstract: Very classical results on the mirror images of the centers of circles and bolls should be the centers as the typical results of the division by zero calculus. For their importance, we would like to discuss them in a self-contained manner. Recall that David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Meanwhile,

Oliver Heaviside: *Mathematics is an experimental science, and definitions do not come first, but later on.*

Key Words: Division by zero (DBZ), division by zero calculus (DBZC), $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, $[(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$, mirror image, inversion, horn torus model.

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1 Introduction

In complex analysis, it is a very classical result that the inversion or mirror image of the center of a circle on the complex plane is the point at infinity.

This classical result is right in the sense of the limiting that when points approach to the center their images approach to the point at infinity. However, what is the image of the center itself? Surprisingly enough, the image is itself the center. This result will show our new idea on the universe. For its importance, here we would like to discuss the mirror images in a self contained manner.

One strong motivation to write this short paper is that our mathematical world do not accept this important new idea. For example, in Suugaku Tsuushin, MATHEMATICAL SOCIETY OF JAPAN, 24(2020), No. 4, page, 6, the author repeated the classical result and idea ([8]). The author stated directly our new opinion to the author and the editor of the journal, however, it seems that they do not like to change their idea.

Our new idea was presented many times in the Japanese Mathematical Society Meetings and some invited international conferences over these 6 years.

Since the problem will be serious for our mathematics and our general idea on the universe, the author wishes to make clear the problem.

Meanwhile, we would like to refer to the following interesting fact:

In the classical book entitled Classical Potential Theory, Springer Monographs in Mathematics (2001),

D. H. Armitage and S. J. Gardiner did not refer to the images of the centers of circles and bolls, not at all ([1]).

2 Simple introduction of the division by zero calculus

We will recall the simple background on the division by zero calculus (DBZC) for differentiable functions based on ([21, 22]).

For a function $y = f(x)$ which is $n(n > 0)$ order differentiable at $x = a$, we will **define** the value of the function

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for $1/0 = 0$ and $0/0 = 0$. Further, for the division by zero calculus (DBZC) we gave similarly several definitions. See, for example, [19].

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 **represents impossibility**. In particular, the value of the function $y = |x|/x$ at $x = 0$ is zero.

We will note its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a} \frac{F(x)}{x-a} = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a). \tag{2.2}$$

For many applications, see the papers cited in the reference.

The identity (2.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x\to a} \frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x - a$ of the function $f(x)$ at $x = a$ which is defined independently of $f(a)$ in (2.2). This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n = 0$ for the general formula; that is,

$$\frac{f(x)}{(x-a)^0}|_{x=a} = \frac{f^{(0)}(a)}{0!} = f(a).$$

In the expression (2.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{2.3}$$

in the left hand side, is represented by the dummy variable $x - a$ that represents the property of the function around the point $x = a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

When we apply the relation (2.1) to the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u + v)' = u' + v',$$

we have the result

$$\frac{u(x) + v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition. This is a result from the property of derivatives.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u\prime}{u} + \frac{v\prime}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u} \right).$$

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta-a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. See [22].

For the division by zero calculus, see the papers in the reference. Our division by zero is an assumption and definition, and so we are requested to show its power with many and many examples of over 1100 items. For its fundamental properties and importance, we may consider it as a new axiom, since Euclid.

3 Mirror images with respect to a circle

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\overline{z}} \tag{3.1}$$

for any point z, as is well-known. For the reflection point z^*, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the corresponding point as the zero point. The reflection is a one to one correspondence and onto mapping between the inside and the outside of the unit circle, by considering the point at infinity.

Are these correspondences, however, suitable? Does there exist the point at ∞, really? Is the point at infinity corresponding to the zero point, by the reflection? Is the point at ∞ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know and see pleasantly the point at infinity on the Riemann sphere, however, on the complex z-plane it seems that we can not find the corresponding point. When we approach the origin on a radial line on the complex z plane, it seems that the corresponding reflection points approach *the point at infinity* with the direction (of the radial line).

With the concept of the division by zero, there is no the point at infinity ∞ as numbers. For any point z such that $|z| > 1$, there exists the unique point z^* by (3.1). Meanwhile, for any point z such that $|z| < 1$ except $z = 0$, there exits the unique point z^* by (3.1). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z \to 0} z^* = \infty, \tag{3.2}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $W = \frac{1}{z}$ and (3.1) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

The result is a surprising one in a sense; indeed, by considering the geometrical correspondence of the mirror image, we will consider the center corresponds to the point at infinity that is represented by the origin $z = 0$. This will show that the mirror image is not followed by this concept; the correspondence seems to come from the concept of one-to-one and onto mapping.

Should we exclude the point at infinity, from numbers? We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically (geometrical optics). The division by zero gives a one to one and onto mapping of the reflection (3.1) from the whole complex plane onto the whole complex plane.

The infinity ∞ may be considered as in (3.2) **as the usual sense of limits,** however, the infinity ∞ is not a definite number.

We consider a circle on the complex z plane with its center z_0 and its radius r. Then, the mirror image relation p and q with respect to the circle is given by

$$p = z_0 + \frac{r^2}{\overline{q - z_0}}.$$

For $q = z_0$, we have, by the division by zero,

$$p = z_0,$$

For a circle

$$Az\overline{z} + \overline{\beta}z + \beta\overline{z} + D = 0; \quad A > 0, D: \text{ real number},$$

or

$$\left(z + \frac{\beta}{A}\right)\overline{\left(z + \frac{\beta}{A}\right)} = \frac{|\beta|^2 - AD}{A^2},$$

the points z and z_1 are in the relation of the mirror images with respect to the circle if and only if

$$Az_1\overline{z} + \overline{\beta}z_1 + \beta\overline{z} + D = 0,$$

or

$$\overline{z_1} = -\frac{\overline{\beta}z + D}{Az + \beta}$$

$$= -\frac{\overline{\beta}}{A} - \frac{1}{A}\left(D - \frac{|\beta|^2}{A}\right)\frac{1}{z - \left(-\frac{\beta}{A}\right)}.$$

The center $-\beta/A$ corresponds to the center itself, as we see from the division by zero.

On the x, y plane, we shall consider the inversion relation with respect to the circle with its radius R and with its center at the origin:

$$x' = \frac{xR^2}{x^2 + y^2}, \quad y' = \frac{yR^2}{x^2 + y^2}.$$

Then, the line

$$ax + by + c = 0$$

is transformed to the line

$$R^2(ax' + by') + c((x')^2 + (y')^2) = 0.$$

In particular, for $c = 0$, the line $ax + by = 0$ is transformed to the line $ax' + by' = 0$. This correspondence is one-to-one and onto, and so the origin $(0,0)$ has to correspond to the origin $(0,0)$.

For the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$$

and for the similar correspondences

$$x' = \frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}, \quad y' = \frac{a^2 b^2 y^2}{b^2 x^2 + a^2 y^2},$$

the origin corresponds to itself.

The pole (x_1, y_1) of the line

$$ax + by + c = 0$$

with respect to a circle with its radius R and with its center (x_0, y_0) is given by

$$x_1 = x_0 - \frac{aR^2}{ax_0 + by_0 + c}$$

and

$$y_1 = y_0 - \frac{bR^2}{ax_0 + by_0 + c}.$$

If $ax_0 + by_0 + c = 0$, then we have $(x_1, y_1) = (x_0, y_0)$.

Furthermore, for various higher dimensional cases the corresponding results are similar.

Anyhow, by the horn torus models of Puha and Däumler, we can see the whole situation of the reflection mappings or inversions clearly, because we can see that the zero point and the point at infinity are the same one point. See [2] for the beautiful model.

See also the beautiful world by W. W. Däumler:

Daeumler, Wolfgang, private publication (2019),
https://www.horntorus.com/manifolds/conformalmapping.pdf

4 In some simple situations

We would like to refer to the simple cases of the inversions.

1. On the real line, the points $P(p), Q(1), R(r), S(-1)$ form a harmonic range of points if and only if

   $$p = \frac{1}{r}.$$

   If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity (H. Okumura: 2017.12.27.)

2. We write a line by the polar coordinate

   $$r = \frac{d}{\cos(\theta - \alpha)},$$

   where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

   This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

3. For a function

   $$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{4.1}$$

   the radius R of the circle $S(x, y) = 0$ is given by

   $$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero. In this case, the circle is a line (degenerated).

The center of the circle (4.1) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

On the complex z plane, a circle containing a line is represented by the equation

$$az\overline{z} + \overline{\alpha}z + \alpha\overline{z} + c = 0,$$

for a, c : real and $ac \leq \overline{\alpha}\alpha$. Then the center and the radius are given by

$$-\frac{\alpha}{a}$$

and

$$\frac{\sqrt{\alpha\overline{\alpha} - ac}}{a},$$

respectively. If $a = 0$, then it is a line with center $(0, 0)$ with radius 0, by the division by zero. The curvature of the line is, of course, zero, by the division by zero.

4. We consider the unit circle with its center at the origin on the (x, y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x$ $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$. Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,
$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$
This example shows also that by the stereographic projection mapping of the unit sphere with its center at the origin $(0,0,0)$ onto the plane, the north pole corresponds to the origin $(0,0)$.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with its center $((\cos\theta)^{-1},0)$. Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1,0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero calculus, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

Note that from the expansion
$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^\nu \left(\frac{1}{z-(2\nu-1)\pi/2} + \frac{2}{(2\nu-1)\pi}\right),$$
$$\left(\frac{1}{\cos z}\right)\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1} = 0.$$
The points $(\cos\theta, 0)$ and $((\cos\theta)^{-1},0)$ are the symmetric points with respect to the unit circle, and the origin corresponds to the origin.

Of course, the division by zero and division by zero calculus show that the mirror images of the centers of circles are the centers. Therefore, we can find many and many concrete examples. See the papers listed in the references.

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