QUANTUM PERMUTATIONS AND QUANTUM REFLECTIONS

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Abstract. The permutation group S_N has a quantum analogue S_N^+, which is infinite at $N\geq 4$. We review the known facts regarding S_N^+, and its versions S_F^+, with F being a finite quantum space. We discuss then the structure of the closed subgroups $G\subset S_N^+$ and $G\subset S_F^+$, with particular attention to the quantum reflection groups.

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Introduction

The compact quantum Lie groups were introduced by Woronowicz in [147], [148]. The idea is very simple, namely looking at the compact Lie groups $G\subset U_N$, and removing the commutativity assumption on $C(G)$. Assume indeed $G\subset U_N$. The multiplication $m:G\times G\to G$, unit $u:\{.\}\to G$ and inversion map $i:G\to G$ are given by:

$$(UV)_{ij}=\sum_kU_{ik}V_{kj}$$

$$1_{ij}=\delta_{ij}$$

$$(U^{-1})_{ij}=\bar{U}_{ji}$$

Now let $C(G)$ be the algebra of continuous functions $f:G\to\mathbb C$, which by Stone-Weierstrass is generated by the coordinates $u_{ij}:U\to U_{ij}$. The transposes of m,u,i, denoted Δ,ε,S and called coumultiplication, counit and antipode, are given by:

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}$$

$$\varepsilon(u_{ij})=\delta_{ij}$$

$$S(u_{ij})=u_{ji}^*$$

With these observations in hand, Woronowicz considered pairs (A,u) consisting of an arbitrary C^*-algebra A, not necessarily commutative, and a unitary matrix $u\in M_N(A)$, whose coefficients generate A, having maps Δ,ε,S given by the above formulae. In this situation we can write $A=C(G)$, and call G a compact Lie quantum group. As explained in [147], [148], these quantum groups enjoy the existence of a Haar measure, a full analogue of the Peter-Weyl theory, and a Tannakian duality result as well.

Following Wang [140], we will be interested here in the quantum permutation group S_N^+. In order to construct this quantum group, let us look first at S_N. We can regard S_N as being the group of permutations of the N coordinate axes of $\mathbb R^N$, and we obtain an embedding $S_N\subset O_N$, with the coordinates of S_N being given by:

$$u_{ij}(\sigma)=\begin{cases}1&\text{if }\sigma(j)=i\\0&\text{otherwise}\end{cases}$$

These coordinates u_{ij} are projections ($p^2=p=p^*$), and the matrix $u=(u_{ij})$ that they form is "magic", in the sense that these projections sum up to 1, on each row and each column. With a bit more work, by using basic operator algebra theory, we are led in this way to a simple presentation result for the algebra $C(S_N)$, as follows:

$$C(S_N)=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

To be more precise, here C^*_{comm} means universal commutative C^*-algebra, and all this follows from the above remarks, and from the Gelfand theorem.

We can now go ahead, and construct S_N^+. The definition is very simple, just by lifting the commutativity property from the above picture of S_N, as follows:

$$C(S_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

Observe that S_N^+ is by definition a compact quantum group, and that we have an embedding $S_N\subset S_N^+$. Quite remarkably, this embedding is not an isomorphism at $N\geq 4$, where S_N^+ is non-classical, and infinite. This latter result can be proved as follows:

$\underline{N=2}$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+=S_2$.

$\underline{N=3}$. By using the same argument as in the $N=2$ case, and the symmetries of the problem, it is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}u_{11}u_{22}&=u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\&=u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\&=u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\&=u_{11}u_{22}u_{11}\end{aligned}$$

Indeed, by applying the involution to this formula, we obtain as well $u_{22}u_{11}=u_{11}u_{22}u_{11}$. Thus, we have $u_{11}u_{22}=u_{22}u_{11}$, and so the entries of u commute, as desired.

$\underline{N=4}$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose p,q as for the algebra $<p,q>$ to be non-commutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

$\underline{N\geq 5}$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well.

There has been a lot of work, in trying to understand what the quantum permutations really are. At $N=4$ one can prove that we have an isomorphism as follows:

$$S_4^+ = SO_3^{-1}$$

In the general $N\geq 4$ case, the quantum group S_N^+ still appears as a kind of deformation of the group SO_3, in the sense that the fusion rules for its irreducible representations coincide with the usual Clebsch-Gordan rules for SO_3, namely:

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+1}+\ldots+r_{k+l}$$

At $N=4$ the dimensions of these representations are the same as those for SO_3. In general they are bigger, the formula being as follows, with $q+q^{-1}=N-2$:

$$\dim(r_k)=\frac{q^{k+1}-q^{-k}}{q-1}$$

This latter result is of particular interest in connection with the Kesten amenability criterion, because it shows that at $N\geq 5$ the compact quantum group S_N^+ is not coamenable, in the sense that its discrete quantum group dual $\widehat{S_N^+}$ is not amenable.

All the above has deep connections with the Temperley-Lieb algebra [134], with Jones' subfactor theory [101], and with Voiculescu's free probability theory [135]. One way of viewing things, which is now something standard, coming from the work in [2], then [27], and then finally [43], is via the notion of "easiness". The idea here is that the partitions $\pi\in P(k,l)$ act on the vectors from the tensor powers of $\mathbb C^N$ as follows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

The point now is that for the symmetric group S_N, these maps span the intertwiners between the tensor powers of the fundamental representation:

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in P(k,l)\right)$$

Regarding now the quantum group S_N^+, the situation is perfectly similar, but this time with only the noncrossing partitions $\pi\in NC(k,l)$ being involved:

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in NC(k,l)\right)$$

Thus, the conclusion is that the liberation operation $S_N\to S_N^+$, which was constructed in [140] by a somewhat ad-hoc procedure, corresponds to something conceptual, namely the passage $P\to NC$, at the level of the associated Tannakian categories.

As a concrete result now, best explaining the liberation operation $S_N \to S_N^+$, let us look at the law of the main character, which is the following variable:

$$\chi = \sum_i u_{ii}$$

In the classical case χ counts the number of fixed points, and by a well-known result, coming from inclusion-exclusion, with $N \to \infty$ this variable becomes Poisson:

$$\chi \sim \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

In the free case the computation is more complicated, using the easiness methods above, and with $N \to \infty$ we obtain the Marchenko-Pastur, or free Poisson law:

$$\chi \sim \frac{1}{2\pi} \sqrt{4x^{-1} - 1}\, dx$$

Thus, as a conclusion, our liberation operation $S_N \to S_N^+$ corresponds to the standard liberation operation from free probability, in the $N \to \infty$ limit [46].

More generally now, we can talk about closed quantum subgroups $G \subset S_N^+$, in the obvious way, and a whole theory of "quantum permutation groups" can be developed. As a basic result here, selected from the massive number of known facts on the subject, let us mention that such quantum subgroups $G \subset S_N^+$ are in one-to-one correspondence with the planar subalgebras $P \subset \mathcal{S}_N$ of Jones' spin planar algebra [105].

Even more generally now, we can talk about the quantum symmetry groups S_F^+ of the finite noncommutative spaces F, and their closed subgroups $G \subset S_F^+$. Let us recall indeed that, according to the general operator algebra philosophy, a finite noncommutative space F is the dual of a finite dimensional C^*-algebra, and so appears via:

$$C(F) = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

The construction of S_N^+ can be generalized in this setting, and we obtain a quantum symmetry group of F, which appears as a certain closed subgroup, as follows:

$$S_F^+ \subset U_{|F|}^+$$

This generalization often provides good explanations for results regarding the quantum groups S_N^+ themselves. As an example, the above-mentioned result $S_N^+ \sim SO_3$ is best understood in the quantum symmetry group setting, via a pair of results, as follows:

(1) The fusion rules for S_F^+ with $|F| \geq 4$ are independent of F.

(2) For $F = M_2$, coming via $C(F) = M_2(\mathbb{C})$, we have $S_F^+ = SO_3$.

As it was the case with the previous results, this was just a basic example of what can be done with the quantum symmetry groups S_F^+, and their subgroups $G \subset S_F^+$.

Summarizing, we will be interested here in S_N^+ and its closed subgroups $G \subset S_N^+$, and more generally in S_F^+ with $|F| < \infty$, and its closed subgroups $G \subset S_F^+$. We will discuss the basic theory of such quantum groups, ranging from elementary to advanced, from theoretical to applied, from mathematical to more quantum and uncertain, and with results regarding algebraic, geometric, analytic and probabilistic aspects.

The present book is somewhat self-contained, but remains quite technical, with the main aim of being complete, rather than elementary, and the prior reading of a basic quantum group book is recommended. The organization is in 4 parts, as follows:

(1) Sections 1-4 contain quick operator algebra and quantum group preliminaries, then the basic algebraic theory of the quantum groups S_N^+, and their generalizations S_F^+, with emphasis on representation theory, easiness, and planar algebra aspects.

(2) Sections 5-8 discuss various probabilistic aspects of S_N^+ and S_F^+, with the Weingarten integration formula, standard character and truncated character computations, and then De Finetti theorems, hypergeometric and hyperspherical laws, and more.

(3) Sections 9-12 discuss the quantum automorphism groups $G^+(X) \subset S_N^+$ of the finite graphs X, and their generalizations $G^+(X) \subset S_F^+$, and then focus on the various types of quantum reflection groups, namely real, arithmetic, complex, twisted.

(4) Sections 13-16 contain more advanced theory, of analytic nature, regarding the transitive or quasi-transitive subgroups $G \subset S_N^+$ and $G \subset S_F^+$, and various matrix modelling questions for them, notably with results on the Weyl and Fourier models.

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1. Quantum groups

We first need a good formalism of "compact quantum spaces". There are several such formalisms, and a particularly simple and beautiful one, which is exactly what we need, is provided by the C^*-algebra theory. The starting definition here is as follows:

Definition 1.1. *A C^*-algebra is a complex algebra A, having a norm $||.||$ making it a Banach algebra, and an involution $*$, related to the norm by the formula*

$$||aa^*|| = ||a||^2$$

which must hold for any $a \in A$.

As a basic example, the usual matrix algebra $M_N(\mathbb{C})$ is a C^*-algebra, with the usual matrix norm and involution, namely $||M|| = \sup_{||x||=1} ||Mx||$ and $(M^*)_{ij} = \bar{M}_{ji}$. More generally, any $*$-subalgebra $A \subset M_N(\mathbb{C})$ is automatically closed, and so is a C^*-algebra. In fact, in finite dimensions, the situation is as follows:

Theorem 1.2. *The finite dimensional C^*-algebras are exactly the algebras of type*

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

with norm $||(a_1, \ldots, a_k)|| = \sup_i ||a_i||$, and involution $(a_1, \ldots, a_k)^* = (a_1^*, \ldots, a_k^*)$.

Proof. In one sense this is clear, either by standard direct sum arguments, or because with $N = N_1 + \ldots + N_k$ we have an embedding of $*$-algebras $A \subset M_N(\mathbb{C})$. In the other sense, this is something more subtle, coming by breaking the unit of our finite dimensional C^*-algebra A as a sum of central minimal projections, as follows:

$$1 = p_1 + \ldots + p_k$$

Indeed, when doing so, each of the $*$-algebras $A_i = p_i A p_i$ follows to be a matrix algebra, $A_i \simeq M_{N_i}(\mathbb{C})$, and this gives the direct sum decomposition in the statement. □

In general now, the main theoretical result about the C^*-algebras, due to Gelfand, Naimark and Segal, and called GNS representation theorem, is as follows:

Theorem 1.3. *Given a Hilbert space H, the algebra $B(H)$ of linear bounded operators $T : H \to H$ is a C^*-algebra, with norm and involution given by:*

$$||T|| = \sup_{||x||=1} ||Tx||$$

$$< Tx, y >=< x, T^*y >$$

More generally, and norm closed $*$-subalgebra of this full operator algebra

$$A \subset B(H)$$

is a C^*-algebra. Any C^*-algebra appears in this way, for a certain Hilbert space H.

Proof. There are several statements here, with the first ones being standard operator theory, and with the last one being the GNS theorem, the idea being as follows:

(1) First of all, the full operator algebra $B(H)$ is a Banach algebra. Indeed, given a Cauchy sequence $\{T_n\}$ inside $B(H)$, we can set $Tx = \lim_{n\to\infty} T_n x$, for any $x \in H$. It is then routine to check that we have $T \in B(H)$, and that $T_n \to T$ in norm.

(2) Regarding the involution, the point is that we must have $< Tx, y >=< x, T^*y >$, for a certain vector $T^*y \in H$. But this can serve as a definition for T^*, and the fact that T^* is indeed linear, and bounded, with the bound $||T^*|| = ||T||$, is routine. As for the formula $||TT^*|| = ||T||^2$, this is elementary as well, coming by double inequality.

(3) The assertion about the subalgebras $A \subset B(H)$ which are closed under the norm and the involution is clear from definitions.

(4) Finally, the fact that any C^*-algebra appears as $A \subset B(H)$, for a certain Hilbert space H, is advanced. The idea is that each $a \in A$ acts on A by multiplication, $T_a(b) = ab$. Thus, we are more or less led to the result, provided that we are able to convert our algebra A, regarded as a complex vector space, into a Hilbert space $H = L^2(A)$. But this latter conversion can be done, by using advanced functional analysis techniques. □

As a third and last basic result about the C^*-algebras, which will be of particular interest for us, we have the following well-known theorem of Gelfand:

Theorem 1.4. *Given a compact space X, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with norm and involution as follows:*

$$||f|| = \sup_{x\in X} |f(x)|$$

$$f^*(x) = \overline{f(x)}$$

This algebra is commutative, and any commutative C^*-algebra A is of this form, with $X = Spec(A)$ appearing as the space of Banach algebra characters $\chi : A \to \mathbb{C}$.

Proof. Once again, there are several statements here, some of them being trivial, and some of them being advanced, the idea being as follows:

(1) First of all, the fact that $C(X)$ is indeed a Banach algebra is clear, because a uniform limit of continuous functions must be continuous.

(2) Regarding now for the formula $||ff^*|| = ||f||^2$, this is something trivial for functions, because on both sides we obtain $\sup_{x\in X} |f(x)|^2$.

(3) Given a commutative C^*-algebra A, the character space $X = \{\chi : A \to \mathbb{C}\}$ is compact, and we have an evaluation morphism $ev : A \to C(X)$.

(4) The tricky point, which follows from basic spectral theory in Banach algebras, is to prove that ev is indeed isometric. This gives the last assertion. □

In what follows, we will be mainly using Definition 1.1 and Theorem 1.4, as general theory. To be more precise, in view of Theorem 1.4, let us formulate:

Definition 1.5. *Given an arbitrary C^*-algebra A, we agree to write*

$$A = C(X)$$

and call the abstract space X a compact quantum space.

In other words, we can define the category of compact quantum spaces X as being the category of the C^*-algebras A, with the arrows reversed. A morphism $f : X \to Y$ corresponds by definition to a morphism $\Phi : C(Y) \to C(X)$, a product of spaces $X \times Y$ corresponds by definition to a product of algebras $C(X) \otimes C(Y)$, and so on.

Finally, no discussion here would be complete without a word about von Neumann algebras. These are operator algebras of more advanced type, that we will use later on, in connection with more advanced questions. Their basic theory is as follows:

Theorem 1.6. *For a $*$-algebra $A \subset B(H)$ the following conditions are equivalent, and if they are satisfied, we say that A is a von Neumann algebra:*

1. *A is closed with respect to the weak topology, making each $T \to Tx$ continuous.*
2. *A is equal to its algebraic bicommutant, $A = A''$, computed inside $B(H)$.*

As basic examples, we have the algebras $A = L^\infty(X)$, acting on $H = L^2(X)$. Such algebras are commutative, any any commutative von Neumann algebra is of this form.

Proof. There are several assertions here, the idea being as follows:

(1) The equivalence (1) $\iff$ (2) is the well-known bicommutant theorem of von Neumann, which can be proved by using an amplification trick, $H \to \mathbb{C}^N \otimes H$.

(2) Given a measured space X, we have indeed an emdedding $L^\infty(X) \subset B(L^2(X))$, with weakly closed image, given by $T_f : g \to fg$, as in the proof of the GNS theorem.

(3) Given a commutative von Neumann algebra $A \subset B(H)$ we can write $A =< T >$ with T being a normal operator, and the Spectral Theorem gives $A \simeq L^\infty(X)$. $\square$

In the context of a C^*-algebra representation $A \subset B(H)$ we can consider the weak closure, or bicommutant $A'' \subset B(H)$, which is a von Neumann algebra. In the commutative case, $C(X) \subset B(L^2(X))$, the weak closure is $L^\infty(X)$. In general, we agree to write:

$$A'' = L^\infty(X)$$

We are ready now to introduce the compact quantum groups. The axioms here, due to Woronowicz [147], and slightly modified for our purposes, are as follows:

Definition 1.7. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$$

$$\varepsilon(u_{ij})=\delta_{ij}$$

$$S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras as follows,

$$\Delta:A\to A\otimes A$$

$$\varepsilon:A\to\mathbb{C}$$

$$S:A\to A^{opp}$$

called comultiplication, counit and antipode.

In the above definition the tensor product used in the definition of Δ can be any C^*-algebra tensor product. In order to get rid of redundancies, coming from this and from amenability issues, we will divide everything by an equivalence relation, as follows:

Definition 1.8. *We agree to identify two Woronowicz algebras, $(A,u)=(B,v)$, when we have an isomorphism of $*$-algebras as follows,*

$$<u_{ij}>\simeq<v_{ij}>$$

mapping standard coordinates to standard coordinates, $u_{ij}\to v_{ij}$.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. We have then the following key result, from [147], providing us with examples:

Proposition 1.9. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi)=(g,h)\to\varphi(gh)$$

$$\varepsilon(\varphi)=\varphi(1)$$

$$S(\varphi)=g\to\varphi(g^{-1})$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g)=g\otimes g$$

$$\varepsilon(g)=1$$

$$S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. In both cases, we have to exhibit a certain matrix u, and then prove that we have indeed a Woronowicz algebra. The constructions are as follows:

(1) For the first assertion, we can use the matrix $u = (u_{ij})$ formed by the standard matrix coordinates of G, which is by definition given by:

$$g = \begin{pmatrix} u_{11}(g) & \ldots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \ldots & u_{NN}(g) \end{pmatrix}$$

(2) For the second assertion, we can use the diagonal matrix formed by generators:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, regarding the last assertion, in the commutative case this follows from the Gelfand result, Theorem 1.4 above. In the cocommutative case this is something more complicated, requiring as well an amenability discussion. We will be back to this. □

In order to get now to quantum groups, we will need as well:

Proposition 1.10. *Assuming that $G \subset U_N$ is abelian, we have an identification of Woronowicz algebras $C(G) = C^*(\Gamma)$, with Γ being the Pontrjagin dual of G:*

$$\Gamma = \{\chi : G \to \mathbb{T}\}$$

Conversely, assuming that $F_N \to \Gamma$ is abelian, we have an identification of Woronowicz algebras $C^*(\Gamma) = C(G)$, with G being the Pontrjagin dual of Γ:

$$G = \{\chi : \Gamma \to \mathbb{T}\}$$

Thus, the Woronowicz algebras which are both commutative and cocommutative are exactly those of type $A = C(G) = C^*(\Gamma)$, with G, Γ being abelian, in Pontrjagin duality.

Proof. All this follows from Gelfand duality, Theorem 1.4 above, because the characters of a group algebra are in correspondence with the characters of the group. □

We have the following definition, complementing Definition 1.7 and Definition 1.8:

Definition 1.11. *Given a Woronowicz algebra, we write it as follows, and call G a compact quantum Lie group, and Γ a finitely generated discrete quantum group:*

$$A = C(G) = C^*(\Gamma)$$

Also, we say that G, Γ are dual to each other, and write $G = \widehat{\Gamma}, \Gamma = \widehat{G}$.

Summarizing, we have a nice framework, for both the compact and discrete quantum groups. Let us discuss now some tools for studying the Woronowicz algebras, and the underlying quantum groups. First, we have the following result:

Proposition 1.12. *Let (A,u) be a Woronowicz algebra.*

(1) *Δ,ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

(2) *S satisfies the antipode axiom, on the $*$-algebra generated by entries of u:*

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2=id$.*

Proof. As a first observation, the result holds in the commutative case, $A=C(G)$ with $G\subset U_N$. Indeed, here we know from Proposition 1.9 that Δ,ε,S appear as functional analytic transposes of the multiplication, unit and inverse maps m,u,i:

$$\Delta=m^t\quad,\quad\varepsilon=u^t\quad,\quad S=i^t$$

With these remark in hand, the various conditions in the statement on Δ,ε,S come by transposition from the group axioms satisfied by m,u,i, namely:

$$m(m\times id)=m(id\times m)$$

$$m(u\times id)=m(id\times u)=id$$

$$m(i\times id)\delta=m(id\times i)\delta=1$$

Observe that the condition $S^2=id$ is satisfied too, coming by transposition from the formula $i^2=id$, which corresponds to the following formula, for group elements:

$$(g^{-1})^{-1}=g$$

The result holds as well in the cocommutative case, $A=C^*(\Gamma)$ with $F_N\to\Gamma$, trivially. In general now, the two comultiplication axioms follow from:

$$(\Delta\otimes id)\Delta(u_{ij})=(id\otimes\Delta)\Delta(u_{ij})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

$$(\varepsilon\otimes id)\Delta(u_{ij})=(id\otimes\varepsilon)\Delta(u_{ij})=u_{ij}$$

As for the antipode axiom, the verification here is similar. First, we have the following computation, by using the fact that the matrix $u=(u_{ij})$ is unitary:

$$m(S\otimes id)\Delta(u_{ij})=\sum_k u_{ki}^*u_{kj}=(u^*u)_{ij}=\delta_{ij}$$

On the other hand, we have as well the following computation:

$$m(id\otimes S)\Delta(u_{ij})=\sum_k u_{ik}u_{jk}^*=(uu^*)_{ij}=\delta_{ij}$$

Finally, we have $S^2(u_{ij})=u_{ij}$, and so $S^2=id$ everywhere, as claimed. $\square$

In the compact Lie group case, in order to reach to advanced results, one must do either representation theory, or Lie algebras [145]. In what regards the compact quantum Lie groups, there is no Lie algebra that can be defined, at least in some elementary sense, and we are left with doing representation theory. Following [147], let us start with:

Definition 1.13. *Given a Woronowicz algebra A, we call corepresentation of it any unitary matrix $v\in M_n(A)$ satisfying the same conditions are those satisfied by u, namely:*

$$\Delta(v_{ij})=\sum_k v_{ik}\otimes v_{kj}\quad,\quad \varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

We also say that v is a representation of the underlying compact quantum group G, and a corepresentation of the underlying discrete quantum group Γ.

In the commutative case, $A=C(G)$ with $G\subset U_N$, we obtain in this way the finite dimensional unitary smooth representations $v:G\to U_n$, as follows:

$$v(g)=\begin{pmatrix}v_{11}(g)&\ldots&v_{1n}(g)\\ \vdots&&\vdots\\ v_{n1}(g)&\ldots&v_{nn}(g)\end{pmatrix}$$

In the cocommutative case, $A=C^*(\Gamma)$ with $F_N\to\Gamma$, we will see in a moment that we obtain in this way the formal sums of elements of Γ, possibly rotated by a unitary. As a first result now regarding the representations, we have:

Proposition 1.14. *The corepresentations are subject to the following operations:*

1. *Making sums, $v+w=diag(v,w)$.*
2. *Making tensor products, $(v\otimes w)_{ia,jb}=v_{ij}w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij}=v_{ij}^*$.*
4. *Rotating by a unitary, $v\to UvU^*$.*

Proof. We first check the fact that the matrices in the statement are unitaries:

(1) The fact that $v+w$ is unitary is clear.

(2) Regarding now $v\otimes w$, this can be written in standard leg-numbering notation as $v\otimes w=v_{13}w_{23}$, and with this interpretation in mind, the unitarity is clear as well.

(3) In order to check that $\bar{v}$ is unitary, we can use the antipode. Indeed, by regarding the antipode as an antimultiplicative map $S:A\to A$, we have:

$$(\bar{v}v^t)_{ij}=\sum_k v_{ik}^*v_{jk}=\sum_k S(v_{kj}^*v_{ki})=S((v^*v)_{ji})=\delta_{ij}$$

We have as well the following computation:

$$(v^t\bar{v})_{ij}=\sum_k v_{ki}v_{kj}^*=\sum_k S(v_{jk}v_{ik}^*)=S((vv^*)_{ji})=\delta_{ij}$$

(4) Finally, the fact that UvU^* is unitary is clear. As for the verification of the comultiplicativity axioms, involving Δ,ε,S, this is elementary and routine, in all cases. $\square$

As a consequence of the above result, we can formulate:

Definition 1.15. *We denote by $u^{\otimes k}$, with $k=\circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u,\bar{u}$, indexed according to the rules*

$$u^{\otimes\emptyset}=1\quad,\quad u^{\otimes\circ}=u\quad,\quad u^{\otimes\bullet}=\bar{u}$$

and multiplicativity, $u^{\otimes kl}=u^{\otimes k}\otimes u^{\otimes l}$, and call them Peter-Weyl corepresentations.

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes\circ\circ}=u\otimes u\quad,\quad u^{\otimes\circ\bullet}=u\otimes\bar{u}$$

$$u^{\otimes\bullet\circ}=\bar{u}\otimes u\quad,\quad u^{\otimes\bullet\bullet}=\bar{u}\otimes\bar{u}$$

In order to do representation theory, we first need to know how to integrate over G. And we have here the following key result, due to Woronowicz [147]:

Theorem 1.16. *Any Woronowicz algebra $A=C(G)$ has a unique Haar integration,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

Proof. Following [147], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, by linearity we can assume that a is the coefficient of certain corepresentation, $a=(\tau\otimes id)v$. But in this case, an elementary computation gives the following formula, with P_φ being the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb C^n\Big|v\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi=P$. Thus our linear form $\int_\varphi$ is independent of φ, and is given on coefficients $a=(\tau\otimes id)v$ by the following formula:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [147]. □

With these integration results in hand, we can now develop a Peter-Weyl type theory for the corepresentations, in analogy with the theory from the classical case. We will need a number of straightforward definitions and results. Let us begin with:

Definition 1.17. *Given two corepresentations $v\in M_n(A),w\in M_m(A)$, we set*

$$Hom(v,w)=\left\{T\in M_{m\times n}(\mathbb C)\Big|Tv=wT\right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v)=Hom(1,v)$, and $End(v)=Hom(v,v)$.*
2. *We write $v\sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v\in Irr(G)$, when $End(v)=\mathbb C1$.*

In the classical case, where $A=C(G)$ with $G\subset U_N$ being a closed subgroup, we obtain the usual notions concerning the representations. Observe also that in the group dual case we have $g\sim h$ when $g=h$. Finally, observe that $v\sim w$ means that v,w are conjugated by an invertible matrix. Here are a few basic results, regarding the above Hom spaces:

Proposition 1.18. *We have the following results:*

1. $T\in Hom(u,v),S\in Hom(v,w)\implies ST\in Hom(u,w)$.
2. $S\in Hom(p,q),T\in Hom(v,w)\implies S\otimes T\in Hom(p\otimes v,q\otimes w)$.
3. $T\in Hom(v,w)\implies T^*\in Hom(w,v)$.

In other words, the Hom spaces form a tensor $*$-category.

Proof. The proofs are all elementary, as follows:

(1) Assume indeed that we have $Tu=vT$, $Sv=Ws$. We obtain, as desired:

$$STu=SvT=wST$$

(2) Assume indeed that we have $Sp=qS$, $Tv=wT$. We have then:

$$(S\otimes T)(p\otimes v)=S_1T_2p_{13}v_{23}=(Sp)_{13}(Tv)_{23}$$

On the other hand, we have as well the following computation:

$$(q \otimes w)(S \otimes T) = q_{13}w_{23}S_1T_2 = (qS)_{13}(wT)_{23}$$

The quantities on the right being equal, this gives the result.

(3) By conjugating, and then using the unitarity of v, w, we obtain, as desired:

$$\begin{aligned} Tv = wT &\implies v^*T^* = T^*w^* \\ &\implies vv^*T^*w = vT^*w^*w \\ &\implies T^*w = vT^* \end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units. □

Finally, in order to formulate the Peter-Weyl results, we will need as well:

Proposition 1.19. *The characters of the corepresentations, given by*

$$\chi_v = \sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

In addition, given two equivalent corepresentations, $v \sim w$, we have $\chi_v = \chi_w$.

Proof. The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have $v = T^{-1}wT$, we obtain:

$$\chi_v = Tr(v) = Tr(T^{-1}wT) = Tr(w) = \chi_w$$

We conclude that $v \sim w$ implies $\chi_v = \chi_w$, as claimed. □

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

With this convention, we have the following fundamental result, from [147]:

Theorem 1.20. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. $\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$*, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from [147], the idea being as follows:

(1) Given a corepresentation $v\in M_n(A)$, consider its interwiner algebra:

$$End(v)=\{T\in M_n(\mathbb C)|Tv=vT\}$$

We know from Proposition 1.18 that this is a finite dimensional C^*-algebra, and by using Theorem 1.2 above, we have a decomposition as follows:

$$End(v)=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_r}(\mathbb C)$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1=p_1+\ldots+p_r$$

But this decomposition allows us to define subcorepresentations $v_i\subset v$, which are irreducible, so we obtain, as desired, a decomposition as follows:

$$v=v_1+\ldots+v_r$$

(2) Consider indeed the Peter-Weyl corepresentations, $u^{\otimes k}$ with k colored integer, defined by $u^{\otimes\emptyset}=1$, $u^{\otimes\circ}=u$, $u^{\otimes\bullet}=\bar u$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal A$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is technically a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id\otimes\int_G)v$ is the orthogonal projection P_v onto the space $Fix(v)$, for any corepresentation v.

(4) Since the character $\chi_v=Tr(v)$ is a coefficient of v, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id\otimes\int_G)v=P_v$. $\square$

Observe that in the cocommutative case, we obtain from (4) above that we must have $A=C^*(\Gamma)$ for some discrete group Γ, as mentioned in Proposition 1.9. As another consequence of the above results, following [147] and then [59], we have the following result, dealing with amenability and functional analysis aspects:

Theorem 1.21. *Let A_{full} be the enveloping C^*-algebra of $\mathcal A$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full}\to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon:A_{full}\to\mathbb C$ factorizes through A_{red}.*
4. *We have $N\in\sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

Proof. This is well-known in the group dual case, $A=C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $\Phi : A_{red} \to A_{full}$, via a formula of type $\Phi = (\varepsilon \otimes id)\Delta'$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some functional analysis. See [59], [147]. $\square$

With these results in hand, we can formulate, as a refinement of Definition 1.11:

Definition 1.22. *Given a Woronowicz algebra A, we formally write as before*

$$A = C(G) = C^*(\Gamma)$$

and by GNS construction with respect to the Haar functional, we write as well

$$A'' = L^\infty(G) = L(\Gamma)$$

with G being a compact quantum group, and Γ being a discrete quantum group.

Now back to Theorem 1.21, as in the discrete group case, the most interesting criterion for amenability, leading to some interesting mathematics and physics, is the Kesten one, from Theorem 1.21 (4). This leads us into computing character laws:

Proposition 1.23. *Given a Woronowicz algebra (A, u), consider its main character:*

$$\chi = \sum_i u_{ii}$$

1. *The moments of χ are the numbers $M_k = \dim(Fix(u^{\otimes k}))$.*
2. *When $u \sim \bar{u}$ the law of χ is a real measure, supported by $\sigma(\chi)$.*
3. *The notion of coamenability of A depends only on $law(\chi)$.*

Proof. All this is elementary, the idea being as follows:

(1) This follows indeed from Peter-Weyl theory.

(2) When $u \sim \bar{u}$ we have $\chi = \chi^*$, which gives the result.

(3) This follows from from Theorem 1.21 (4), and from (2) applied to $u + \bar{u}$. $\square$

All this is quite interesting, because it tells us that, regardless on whether we want to understand the representation theory of our compact quantum group G, or the analytic aspects of its discrete dual Γ, we must compute the fixed point spaces $Fix(u^{\otimes k})$.

The computation of these spaces is a delicate algebra problem, related to results of Schur-Weyl, Brauer and Tannaka. In order to get started, the first idea is to replace the series of fixed point spaces $F_k = Fix(u^{\otimes k})$ by the double series of Hom spaces:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

Indeed, by Frobenius duality, computing the sequence of spaces $\{F_k\}$ is the same as computing the family of spaces $\{C_{kl}\}$. But computing the spaces $\{C_{kl}\}$ is simpler than computing the spaces $\{F_k\}$, because these former spaces form a category. And we can use here the following version of Tannakian duality, due to Woronowicz [148]:

Theorem 1.24. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to any tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

Proof. This is something quite deep, going back to [148] in a slightly different form, and to [116] in the simplified form presented above. The idea is that we have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces. We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement. Regarding now the bijection claim, some elementary algebra shows that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing a lot of algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [116]. □

As a last piece of general theory, let us discuss fusion rules, and Cayley graphs:

Proposition 1.25. *Let (A, u) be a Woronowicz algebra, and assume, by enlarging if necessary u, that we have $1 \in u = \bar{u}$. The formula*

$$d(v, w) = \min\left\{k \in \mathbb{N}\Big|1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

defines then a distance on $Irr(A)$, which coincides with the geodesic distance on the associated Cayley graph. Moreover, the moments of the main character,

$$\int_G \chi^k = \dim\left(Fix(u^{\otimes k})\right)$$

count the loops based at 1, having lenght k, on the corresponding Cayley graph.

Proof. Observation first the result holds indeed in the group dual case, where $A = C^*(\Gamma)$ with $\Gamma =< S >$ being a finitely generated discrete group. In general, the fact that the lengths are finite follows from Peter-Weyl theory. The symmetry axiom is clear as well, and the triangle inequality is elementary to establish as well. Finally, the last assertion, regarding the moments, is elementary as well. □

Let us discuss now the basic examples of compact and discrete quantum groups. We know so far that the compact quantum groups include the usual compact Lie groups, $G \subset U_N$, and the abstract duals $G = \widehat{\Gamma}$ of the finitely generated groups $F_N \to \Gamma$. Equivalently, we know that the discrete quantum groups include the finitely generated groups $F_N \to \Gamma$, and the abstract duals $\Gamma = \widehat{G}$ of the compact Lie groups, $G \subset U_N$. We can combine these examples by performing basic operations, as follows:

Proposition 1.26. *The class of Woronowicz algebras is stable under taking:*

(1) *Tensor products, $A = A' \otimes A''$, with $u = u' + u''$. At the quantum group level we obtain usual products, $G = G' \times G''$ and $\Gamma = \Gamma' \times \Gamma''$.*
(2) *Free products, $A = A' * A''$, with $u = u' + u''$. At the quantum group level we obtain dual free products $G = G' \,\hat{*}\, G''$ and free products $\Gamma = \Gamma' * \Gamma''$.*

Proof. Everything here is clear from definitions. In addition to this, let us mention as well that we have $\int_{A' \otimes A''} = \int_{A'} \otimes \int_{A''}$ and $\int_{A' * A''} = \int_{A'} * \int_{A''}$. Also, the corepresentations of the products can be explicitely computed. See [139]. $\square$

Here are some further basic operations, once again from [139]:

Proposition 1.27. *The class of Woronowicz algebras is stable under taking:*

(1) *Subalgebras $A' =< u'_{ij} >\subset A$, with u' being a corepresentation of A. At the quantum group level we obtain quotients $G \to G'$ and subgroups $\Gamma' \subset \Gamma$.*
(2) *Quotients $A \to A' = A/I$, with I being a Hopf ideal, $\Delta(I) \subset A \otimes I + I \otimes A$. At the quantum group level we obtain subgroups $G' \subset G$ and quotients $\Gamma \to \Gamma'$.*

Proof. Once again, everything is clear, and we have as well some straightforward supplementary results, regarding integration and corepresentations. See [139]. $\square$

Finally, here are two more operations, which are of key importance:

Proposition 1.28. *The class of Woronowicz algebras is stable under taking:*

(1) *Projective versions, $PA =< w_{ia,jb} >\subset A$, where $w = u \otimes \bar{u}$. At the quantum group level we obtain projective versions, $G \to PG$ and $P\Gamma \subset \Gamma$.*
(2) *Free complexifications, $\tilde{A} =< zu_{ij} >\subset C(\mathbb{T}) * A$. At the quantum group level we obtain free complexifications, denoted $\widetilde{G}$ and $\widetilde{\Gamma}$.*

Proof. This is clear from the previous results. For details here, we refer to [139]. $\square$

Once again following [139], as well as [43], let us discuss now a number of truly "new" quantum groups, obtained by liberating and half-liberating. We first have:

Theorem 1.29. *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u}, u^t = u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^* = u^{-1}, u^t = \bar{u}^{-1}\right) \end{aligned}$$

and the same goes for the following quotient algebras,

$$\begin{aligned} C(O_N^*) &= C(O_N^+)\Big/\left\langle abc = cba\Big|\forall a,b,c \in \{u_{ij}\}\right\rangle \\ C(U_N^*) &= C(U_N^+)\Big/\left\langle abc = cba\Big|\forall a,b,c \in \{u_{ij}, u_{ij}^*\}\right\rangle \end{aligned}$$

so the underlying spaces O_N^+, U_N^+ and O_N^*, U_N^* are compact quantum groups.

Proof. The first assertion follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Thus, we can define morphisms Δ, ε, S as in Definition 1.7, by using the universality property of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, the proof here is similar, based on the fact that if the entries of u satisfy $abc = cba$, then so do the entries of $u^\Delta, u^\varepsilon, u^S$. $\square$

Our first task is to verify that Theorem 1.29 provides us indeed with new quantum groups. For this purpose, we can use the notion of diagonal torus, from [45]:

Proposition 1.30. *Given a closed subgroup $G \subset U_N^+$, consider its diagonal torus, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. $\square$

We can now distinguish between our various quantum groups, as follows:

Theorem 1.31. *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{tikzcd}
U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\
O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

are the following discrete group duals,

$$\begin{tikzcd}
\widehat{\mathbb{Z}^N} \arrow[r] & \widehat{\mathbb{Z}^{\circ N}} \arrow[r] & \widehat{\mathbb{Z}^{*N}} \\
\widehat{\mathbb{Z}_2^N} \arrow[r] \arrow[u] & \widehat{\mathbb{Z}_2^{\circ N}} \arrow[r] \arrow[u] & \widehat{\mathbb{Z}_2^{*N}} \arrow[u]
\end{tikzcd}$$

with $\circ$ standing for the half-classical product operation for groups.

Proof. This is clear for U_N^+, where on the diagonal we obtain the biggest group dual, namely $\widehat{F_N}$. For the other quantum groups this follows by taking quotients, which corresponds to taking quotients as well, at the level of the diagonal torus dual $\Lambda = \widehat{T}$. $\square$

Let us discuss now the representation theory of these quantum groups. In order to formulate our results, we use the modern notion of "easiness", from [43]:

Definition 1.32. *A closed subgroup $G \subset U_N^+$ is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

for any colored integers k,l, for certain sets of partitions $D(k,l) \subset P(k,l)$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

To be more precise here, let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. Our claim is that given $N \in \mathbb{N}$, any partition $\pi \in P(k,l)$ produces a linear map between tensor powers of $\mathbb{C}^N$, as follows:

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$$

Indeed, if we denote by $e_1, \ldots, e_N$ the standard basis of $\mathbb{C}^N$, we can define T_π by the formula in Definition 1.32, with the Kronecker symbols appearing there being computed by putting the multi-indices i,j on the legs of π, in the obvious way. If all the blocks of π contain equal indices we set $\delta_\pi = 1$, and if not, we set $\delta_\pi = 0$. We have then:

Theorem 1.33. *The basic unitary quantum groups are all easy, with*

$$\begin{matrix} U_N & \rightarrow & U_N^* & \rightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \rightarrow & O_N^* & \rightarrow & O_N^+ \end{matrix} \quad : \quad \begin{matrix} \mathcal{P}_2 & \leftarrow & \mathcal{P}_2^* & \leftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \leftarrow & P_2^* & \leftarrow & NC_2 \end{matrix}$$

being the associated categories of partitions $D \subset P$.

Proof. This is something that requires some work, the idea being as follows:

(1) O_N^+. Consider the set NC_2 of all noncrossing pairings. It is routine to check that $span(T_\pi|\pi \in NC_2)$ is a Tannakian category, and also that this category is the smallest possible one allowed by the Tannakian axioms, in the $u = \bar{u}$ setting. Thus, the associated quantum group must be the biggest subgroup $G \subset O_N^+$, which is O_N^+ itself.

(2) O_N. Since $O_N \subset O_N^+$ appears by adding the commutation relations $ab = ba$ between coordinates, which are implemented by the linear map $T_{\times}$ coming from the basic crossing $\times$, we obtain here the category $< NC_2, \times >= P_2$ of all pairings.

(3) O_N^*. Here we obtain the category $< NC_2, \slashed{\times} >= P_2^*$ of pairings having the property that, when legs are labelled clockwise $\circ \bullet \circ \bullet \ldots$, each string connects $\circ - \bullet$.

(4) U_N^+, U_N, U_N^*. The situation is similar here, but due to $u \neq \bar{u}$ everything is now colored, and we obtain in all cases pairings which are "matching", in the sense that the vertical strings connect $\circ - \circ$ or $\bullet - \bullet$, and the horizontal ones connect $\circ - \bullet$. □

Here are some concrete consequences of the above result, following [1], [26]:

Theorem 1.34. *The quantum groups O_N^+, U_N^+ have the following properties:*

(1) *We have an isomorphism as follows, up to the standard equivalence relation:*

$$\widetilde{O}_N^+ = U_N^+$$

(2) *We have as well an isomorphism as follows, once again up to equivalence:*

$$PO_N^+ = PU_N^+$$

(3) *The fusion rules for O_N^+ are the same as the Clebsch-Gordan rules for SU_2:*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

(4) *Those for U_N^+ are as follows, with the representations being indexed by $\mathbb{N} * \mathbb{N}$:*

$$r_k \otimes r_l = \sum_{k=xy, l=\bar{y}z} r_{xz}$$

(5) *The main characters follow the Wigner semicircle and Voiculescu circular law:*

$$\chi \sim \begin{cases} \gamma_1 & \text{for } O_N^+ \ , \ N \geq 2 \\ \Gamma_1 & \text{for } U_N^+ \ , \ N \geq 2 \end{cases}$$

(6) *With $N \to \infty$, the truncated characters follow the t-versions of these laws:*

$$\chi_t \sim \begin{cases} \gamma_t & \text{for } O_N^+ \ , \ N \to \infty \\ \Gamma_t & \text{for } U_N^+ \ , \ N \to \infty \end{cases}$$

Proof. All this follows from our Brauer type results, via standard techniques. There is actually quite some work ot be done here, the idea being as follows:

(1) As a first observation, by using the universal property of U_N^+, as being the biggest $N \times N$ compact matrix quantum group, we have an inclusion as follows:

$$\widetilde{O}_N^+ \subset U_N^+$$

Now by using the easiness results from Theorem 1.33, we can compute the tensor categories for both these quantum groups, with the conclusion that these tensor categories are equal. Thus, our inclusion is an isomorphism, up to the equivalence relation.

(2) This follows from (1) above, via the following computation:

$$PU_N^+=P\widetilde{O}_N^+=PO_N^+$$

(3) This is something more complicated, the idea being that from the Brauer result for O_N^+ we obtain, after some work, the following formula, valid at any $N\geq 2$:

$$\begin{aligned}\dim(End(u^{\otimes k}))&=|NC_2(k,k)|\\&=\frac{1}{k+1}\binom{2k}{k}\end{aligned}$$

Now since the same formula is well-known to hold for the fundamental representation of SU_2, we obtain the same combinatorics, as claimed. We will be back to this.

(4) This follows from (1) and (3). Indeed, the fusion rules for the quantum group $U_N^+=\widetilde{O}_N^+$ can be computed starting from the knowledge of those of O_N^+, and we end up with a "free complexification" of the Clebsch-Gordan rules, namely:

$$r_k\otimes r_l=\sum_{k=xy,l=\bar{y}z}r_{xz}$$

(5) This follows once again from (1) and (3). Indeed, in what regards O_N^+, we can convert our combinatorial results into a moment formula, as follows:

$$\int_{O_N^+}\chi^{2k}=\frac{1}{k+1}\binom{2k}{k}$$

But this shows precisely that χ must follow the Wigner semicircle law γ_1, and by complexifying, we obtain the result for U_N^+ as well. We will be back to this.

(6) This is something more technical. Given a parameter $t\in(0,1]$, we can define a truncation of the main character, as follows:

$$\chi_t=\sum_{i=1}^{[tN]}u_{ii}$$

The point now is that our Brauer theorems allow us to explicitely integrate over O_N^+,U_N^+, via a combinatorial formula, and in the $N\to\infty$ limit the combinatorics simplifies, and in what regards χ_t, we obtain the laws γ_t,Γ_t. We will be back to this. $\square$

The above presentation was of course quite short, but all this can be found in any good quantum group book. Some similar results regarding O_N^*,U_N^* are available as well, and we can twist everything at $q=-1$ too. We will be back to this.

2. Quantum permutations

Welcome to quantum permutations. The rest of this book is dedicated to them. And, good news, the presentation will be far less intense than that in the previous section, which was meant to be a quick introduction to the quantum groups, survey style. In order to get started, let us look at the usual symmetric group S_N. We have:

Proposition 2.1. *Consider the symmetric group S_N, viewed as the permutation group of the N coordinate axes of $\mathbb{R}^N$. The coordinate functions on $S_N \subset O_N$ are then given by*

$$u_{ij} = \chi\left(\sigma \in G\Big|\sigma(j) = i\right)$$

and the matrix $u = (u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column.

Proof. Everything here follows from definitions. The formula of the coordinates u_{ij} is obviously the good one, and the fact that $u = (u_{ij})$ is magic is clear too. □

With a bit more effort, we obtain the following nice characterization of S_N:

Theorem 2.2. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

Proof. This is something elementary as well. Indeed, the universal algebra A in the statement is a commutative C^*-algebra, so by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates from a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 on each row and each column, and so:

$$X = S_N$$

Thus we have proved the first assertion, and the second assertion is clear as well, by using the general theory from section 1. To be more precise, the multiplication, unit and inverse map of $S_N \subset O_N$ are the standard ones for the orthogonal matrices, namely:

$$(gh)_{ij} = \sum_k g_{ik}h_{kj} \quad , \quad 1_{ij} = \delta_{ij} \quad , \quad (g^{-1})_{ij} = g_{ji}$$

Now by transposing, we obtain the formulae of Δ, ε, S in the statement. □

Following now Wang [140], we can liberate S_N, simply by lifting the commutativity condition in Theorem 2.2. To be more precise, we have the following result:

Theorem 2.3. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

and so the underlying compact quantum space S_N^+ is a compact quantum group, called quantum permutation group.

Proof. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$\begin{aligned} (u_{ij}^\Delta)^2 &= \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} \\ &= \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} \\ &= u_{ij}^\Delta \end{aligned}$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u_{ij}^\Delta)^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^\Delta$$

The row sums for the matrix u^Δ can be computed as follows:

$$\sum_j u_{ij}^\Delta = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

As for the computation of the column sums, this is similar, as follows:

$$\sum_i u_{ij}^\Delta = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, and so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$.

As a conclusion, the algebra $C(S_N^+)$ satisfies Woronowicz's axioms from section 1 above, and so its abstract spectrum S_N^+ is a compact quantum group, as claimed. □

Our first task now is to make sure that Theorem 2.3 produces indeed a new quantum group, which does not collapse to S_N. Following [140], we have here:

Theorem 2.4. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma\Big|\sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

Proof. The fact that we have indeed an embedding as above follows from Theorem 2.2. Observe that in fact more is true, because Theorem 2.2 and Theorem 2.3 give:

$$C(S_N) = C(S_N^+)\Big/\Big\langle ab = ba\Big\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

<u>Case $N = 2$</u>. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+ = S_2$.

<u>Case $N = 3$</u>. By using the same argument as in the $N = 2$ case, and the symmetries of the problem, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this that we have as well $u_{22}u_{11} = u_{11}u_{22}u_{11}$. Thus, we obtain $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N=4$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose p,q as for the algebra $<p,q>$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. □

The above result is quite surprising, and understanding all this will be our next goal. As a first observation, we are not wrong with our formalism, because as explained once again in [140], we have as well the following alternative picture for S_N^+:

Theorem 2.5. *The quantum permutation group S_N^+ acts on the set $X=\{1,\ldots,N\}$, the corresponding coaction map $\Phi:C(X)\to C(X)\otimes C(S_N^+)$ being given by:*

$$\Phi(e_i)=\sum_j e_j\otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(tr\otimes id)\Phi=tr(.)1$, where $tr(e_i)=\frac{1}{N},\forall i$.

Proof. Our claim is that given a compact matrix quantum group G, the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u=(u_{ij})$ is a magic corepresentation of $C(G)$:

$$\Phi(e_i)=\sum_j e_j\otimes u_{ji}$$

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(e_i)\Phi(e_k)=\sum_{jl}e_je_l\otimes u_{ji}u_{lk}=\sum_j e_j\otimes u_{ji}u_{jk}$$

On the other hand, we have as well the following computation:

$$\Phi(e_ie_k)=\delta_{ik}\Phi(e_i)=\delta_{ik}\sum_j e_j\otimes u_{ji}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji}u_{jk}=\delta_{ik}u_{ji}\quad,\quad\forall i,j,k$$

Regarding now the unitality of Φ, we have the following formula:

$$\Phi(1)=\sum_i\Phi(e_i)=\sum_{ij}e_j\otimes u_{ji}=\sum_j e_j\otimes\left(\sum_i u_{ji}\right)$$

Thus Φ is unital when $\sum_i u_{ji}=1,\forall j$. Finally, the fact that Φ is a $*$-morphism translates into $u_{ij}=u_{ij}^*,\forall i,j$. Summing up, in order for $\Phi(e_i)=\sum_j e_j\otimes u_{ji}$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u. Regarding now the preservation of the trace condition, observe that we have:

$$(tr\otimes id)\Phi(e_i)=\frac{1}{N}\sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u. We conclude from this that $\Phi(e_i)=\sum_j e_j\otimes u_{ji}$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. □

In order to study now S_N^+, we can use our various methods developed in section 1 above. Let us begin with some basic algebraic results, as follows:

Theorem 2.6. *The quantum groups S_N^+ have the following properties:*

(1) *We have $S_N^+\,\hat{*}\,S_M^+\subset S_{N+M}^+$, for any N,M.*
(2) *In particular, we have an embedding $\widehat{D_\infty}\subset S_4^+$.*
(3) *$S_4\subset S_4^+$ are distinguished by their spinned diagonal tori.*
(4) *The half-classical version $S_N^*=S_N^+\cap O_N^*$ collapses to S_N.*

Proof. These results are all elementary, the proofs being as follows:

(1) If we denote by u,v the fundamental corepresentations of $C(S_N^+),C(S_M^+)$, the fundamental corepresentation of $C(S_N^+\,\hat{*}\,S_M^+)$ is by definition:

$$w=\begin{pmatrix}u&0\\0&v\end{pmatrix}$$

But this matrix is magic, because both u,v are magic. Thus by universality of $C(S_{N+M}^+)$ we obtain a quotient map as follows, as desired:

$$C(S_{N+M}^+)\to C(S_N^+\,\hat{*}\,S_M^+)$$

(2) This result, which refines our $N=4$ trick from the proof of Theorem 2.4, follows from (1) with $N=M=2$. Indeed, we have the following computation:

$$S_2^+\,\hat{*}\,S_2^+=S_2\,\hat{*}\,S_2=\mathbb{Z}_2\,\hat{*}\,\mathbb{Z}_2\simeq\widehat{\mathbb{Z}_2}\,\hat{*}\,\widehat{\mathbb{Z}_2}=\widehat{\mathbb{Z}_2*\mathbb{Z}_2}=\widehat{D_\infty}$$

(3) As a first observation here, the quantum groups $S_4\subset S_4^+$ are not distinguished by their diagonal torus, which is $\{1\}$ for both of them. However, according to the general results of Woronowicz in [147], the group dual $\widehat{D_\infty}\subset S_4^+$ that we found in (2) must be a subgroup of the diagonal torus of the following compact quantum group, with the standard unitary representations being spinned by a certain unitary $F\in U_4$:

$$(S_4^+,FuF^*)$$

Now since this group dual $\widehat{D_\infty}$ is not classical, it cannot be a subgroup of the diagonal torus of (S_4,FuF^*). Thus, the diagonal torus spinned by F distinguishes $S_4\subset S_4^+$.

(4) Consider the following compact quantum group, with the intersection operation being taken inside U_N^+, whose coordinates satisfy $abc=cba$:

$$S_N^*=S_N^+\cap O_N^*$$

In order to prove that we have $S_N^*=S_N$, it is enough to prove that S_N^* is classical. And here, we can use the fact that for a magic matrix, the entries in each row sum up to 1. Indeed, by making c vary over a full row of u, we obtain $abc=cba\implies ab=ba$. $\square$

Let us discuss now the representation theory of S_N^+, which will eventually lead to a clarification of all this. Our main result here, which is quite conceptual, will be the fact that $S_N\subset S_N^+$ is a liberation of easy quantum groups. Following [43], let us formulate:

Definition 2.7. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A collection of sets*

$$D=\bigsqcup_{k,l}D(k,l)$$

with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under the vertical concatenation, $(\pi,\sigma)\to[^\sigma_\pi]$.*
3. *Stability under the upside-down turning, $\pi\to\pi^*$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The set $P(0,2)$ contains the semicircle partition $\cap$.*

As a basic example, we have the category of all partitions P itself. Other basic examples include the category of pairings P_2, or the categories NC,NC_2 of noncrossing partitions, and pairings. There are many other examples, and we will be back to this. Following [43], the relation with the Tannakian categories and duality comes from:

Proposition 2.8. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. This follows from the elementary computations, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
&T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
=&\sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p} < T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}), e_{i_1}\otimes\ldots\otimes e_{i_p} > e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix} e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In relation with the quantum groups, we have the following result, from [43]:

Theorem 2.9. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and the Tannakian duality correspondence.

Proof. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [116], as explained in section 1 above. Indeed, let us set:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By using the axioms in Definition 2.7, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 2.8 above, we deduce that $C=(C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

We can now formulate the following key definition:

Definition 2.10. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

for any colored integers k,l, for a certain category of partitions $D\subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool.

Observe that the category D is not unique, for instance because at $N=1$ all the categories of partitions produce the same easy quantum group, namely $G_1=\{1\}$. We will be back to this issue on several occasions, with various results about it.

In relation now with our quantum permutation groups, and with the orthogonal quantum groups too, here is our main result, coming from [1], [2] and then [26], [27], [43]:

Theorem 2.11. *The quantum permutation and rotation groups are all easy,*

$$\begin{matrix} S_N^+ & \to & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \to & O_N \end{matrix} \quad : \quad \begin{matrix} NC & \leftarrow & NC_2 \\ \downarrow & & \downarrow \\ P & \leftarrow & P_2 \end{matrix}$$

with the corresponding categories of partitions being those on the right.

Proof. This is something quite fundamental, the proof being as follows:

(1) O_N^+. Consider the Tannakian category of O_N^+, formed by the following spaces:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

By using Proposition 2.8, consider as well the following Tannakian category:

$$D = span\left(T_\pi \Big| \pi \in NC_2\right)$$

We want to prove that we have $C = D$. In one sense, this follows from:

$$\begin{aligned} u^t = u^{-1} &\implies T_\cap \in C \\ &\implies < T_\cap > \subset C \\ &\implies span\left(T_\pi \Big| \pi \in < \cap >\right) \subset C \\ &\implies D \subset C \end{aligned}$$

In the other sense, Tannakian duality tells us that associated to D is a certain closed subgroup $G \subset O_N^+$. But since Tannakian duality is contravariant, at the level of categories $G \subset O_N^+$ translates into $C \subset D$. Thus we have $C = D$, and we are done.

(2) O_N. Since $O_N \subset O_N^+$ appears by adding the commutation relations $ab = ba$ between coordinates, which are implemented by the linear map $T_{\slash\!\!\!\backslash}$ coming from the basic crossing $\slash\!\!\!\backslash$, this group is indeed easy, coming from the following category:

$$< NC_2, \slash\!\!\!\backslash >= P_2$$

Alternatively, if this argument was too fast, the above proof for O_N^+ can be simply rewritten, by adding at each step the basic crossing $\slash\!\!\!\backslash$, next to the semicircle $\cap$.

(3) S_N^+. We know that the algebra $C(S_N^+)$ appears as follows:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle u = \text{magic} \Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2, 1)$$

The linear map associated to this fork partition Y is then given by:

$$T_Y(e_i \otimes e_j) = \delta_{ij} e_i$$

Thus, in usual matrix notation, this linear map is given by:

$$T_Y = (\delta_{ijk})_{i,jk}$$

Now given a corepresentation u, we have the following formula:

$$(T_Y u^{\otimes 2})_{i,jk} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

On the other hand, we have as well the following formula:

$$(u T_Y)_{i,jk} = \sum_l u_{il} (T_Y)_{l,jk} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i,j,k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_Y \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D =< Y >= NC$$

(4) S_N. Here there is no need for new computations, because we have:

$$S_N = S_N^+ \cap O_N$$

At the categorial level means that S_N is easy, coming from:

$$< NC, P_2 >= P$$

Alternatively, we can rewrite if we want the proof for S_N^+ or O_N, by adding at each step the basic crossing/fork next respectively to the fork/basic crossing. □

As explained in section 1, in the context of the unitary quantum groups, this kind of easiness result has a massive number of applications. We will explore these applications in what follows, gradually. Let us start with something philosophical:

Theorem 2.12. *The inclusions $O_N \subset O_N^+$ and $S_N \subset S_N^+$ are liberation operations in the easy quantum group sense, given by*

$$D_{G^+} = D_G \cap NC$$

at the level of the associated categories of partitions.

Proof. This is clear indeed from Theorem 2.11 above, and from the following trivial equalities connecting the categories found there:

$$NC_2 = P_2 \cap NC \quad , \quad NC = P \cap NC$$

Indeed, these equalities correspond to the formulae in the statement. $\square$

Let us get now into the real thing, namely classification of the irreducible representations, fusion rules, Cayley graphs, laws of characters, and other probabilistic questions. As explained in section 1, all these problems are related, and their solution basically requires the knowledge of the associated Tannakian category, given by:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

But in the easy case, where our quantum group G comes from a category of partitions D, and which covers our 4 main examples, this problem is half-solved, because:

$$C_{kl} = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

The remaining half-problem to be solved is that of investigating the linear independence properties of the maps T_π. Let us begin with some standard combinatorics:

Definition 2.13. *Let $P(k)$ be the set of partitions of $\{1,\ldots,k\}$, and let $\pi,\sigma \in P(k)$.*

1. *We write $\pi \leq \sigma$ if each block of π is contained in a block of σ.*
2. *We let $\pi \vee \sigma \in P(k)$ be the partition obtained by superposing π,σ.*

Also, we denote by $|.|$ the number of blocks of the partitions $\pi \in P(k)$.

As an illustration here, at $k=2$ we have $P(2) = \{||, \sqcap\}$, and we have:

$$|| \leq \sqcap$$

Also, at $k=3$ we have $P(3) = \{|||, \sqcap|, \sqcap\!\!\!\!|\,, |\sqcap, \sqcap\sqcap\}$, and the order relation is as follows:

$$||| \ \leq \ \sqcap|\ ,\ \sqcap\!\!\!\!|\,,\ |\sqcap \ \leq \ \sqcap\sqcap$$

Observe also that we have $\pi,\sigma \leq \pi\vee\sigma$, and that $\pi\vee\sigma$ is the smallest partition with this property. Due to this fact, $\pi\vee\sigma$ is called supremum of π,σ.

Now back to quantum groups, and to the questions that we want to solve, by Frobenius duality it is enough to study the partitions having no upper legs. We have:

Proposition 2.14. *The vectors $\xi_\pi = T_\pi$ with $\pi \in P(k)$ are given by*

$$\xi_\pi = \sum_{i_1\ldots i_k} \delta_\pi(i_1,\ldots,i_k)\, e_{i_1} \otimes \ldots \otimes e_{i_k}$$

and their scalar products are given by the formula

$$<\xi_\pi, \xi_\sigma> = N^{|\pi\vee\sigma|}$$

where $\vee$ is the superposition operation, and $|.|$ is the number of blocks.

Proof. According to the formula of the vectors ξ_π, we have:

$$\begin{aligned}
< \xi_\pi, \xi_\sigma > &= \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k)\delta_\sigma(i_1, \ldots, i_k) \\
&= \sum_{i_1 \ldots i_k} \delta_{\pi \vee \sigma}(i_1, \ldots, i_k) \\
&= N^{|\pi \vee \sigma|}
\end{aligned}$$

Thus, we have obtained the formula in the statement. $\square$

In order to study the Gram matrix $G_k(\pi, \sigma) = N^{|\pi \vee \sigma|}$, and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

Definition 2.15. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi \leq \tau < \sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions, $P(2) = \{||, \sqcap\}$. We have by definition:

$$\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$$

Also, we know that we have $|| < \sqcap$, with no intermediate partition in between, and so the above recurrence procedure gives:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, and so we have as well the following formula:

$$\mu(\sqcap, ||) = 0$$

Thus, as a conclusion, we have computed the Möbius matrix $M_2(\pi, \sigma) = \mu(\pi, \sigma)$ of the lattice $P(2) = \{||, \sqcap\}$, the formula being as follows:

$$M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The computation for $P(3) = \{|||, \sqcap|, \sqcap\!\!\!\!\!{}_{|}\ , |\sqcap, \sqcap\!\sqcap\}$ is similar, and leads to the following formula for the associated Möbius matrix:

$$M_3 = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\sigma)=\sum_{\pi\leq\sigma}g(\pi)\quad\implies\quad g(\sigma)=\sum_{\pi\leq\sigma}\mu(\pi,\sigma)f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

Theorem 2.16. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1&\text{if }\pi\leq\sigma\\0&\text{if }\pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

Proof. This is well-known, coming for instance from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 2.15. $\square$

As a first illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Also, for $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!|\,|,|\sqcap,\sqcap\!\sqcap\}$ the formula $M_3=A_3^{-1}$ reads:

$$\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}^{-1}$$

In general, the inversion formula $M_k=A_k^{-1}$ is something of similar nature. Now back to our Gram matrix considerations, we have the following key result:

Proposition 2.17. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1)&\text{if }\sigma\leq\pi\\0&\text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

Proof. We have the following computation, using Proposition 2.14:

$$\begin{aligned}
G_k(\pi,\sigma) &= N^{|\pi\vee\sigma|}\\
&= \#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\
&= \sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\
&= \sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)
\end{aligned}$$

According now to the definition of A_k,L_k, this formula reads:

$$\begin{aligned}
G_k(\pi,\sigma) &= \sum_{\tau\geq\pi}L_k(\tau,\sigma)\\
&= \sum_{\tau}A_k(\pi,\tau)L_k(\tau,\sigma)\\
&= (A_kL_k)(\pi,\sigma)
\end{aligned}$$

Thus, we are led to the formula in the statement. $\square$

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\N&N\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\N&N\end{pmatrix}$$

At $k=3$ now, we have $P(3)=\{|||,\sqcap|,\sqcap\!\!\!|\,,|\sqcap,\sqcap\!\sqcap\}$, and the Gram matrix is:

$$G_3=\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^2&N&N&N\\N^2&N&N^2&N&N\\N^2&N&N&N^2&N\\N&N&N&N&N\end{pmatrix}$$

Regarding L_3, this can be computed by writing down the matrix $E_3(\pi,\sigma)=\delta_{\sigma\leq\pi}|\pi|$, and then replacing each entry by the corresponding polynomial in N. We reach to the conclusion that the product A_3L_3 is as follows, producing the above matrix G_3:

$$A_3L_3=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}N^3-3N^2+2N&0&0&0&0\\N^2-N&N^2-N&0&0&0\\N^2-N&0&N^2-N&0&0\\N^2-N&0&0&N^2-N&0\\N&N&N&N&N\end{pmatrix}$$

In general, the formula $G_k=A_kL_k$ appears a bit in the same way, with A_k being binary and upper triangular, and with L_k depending on N, and being lower triangular.

We are led in this way to the following formula, due to Lindstöm [114]:

Theorem 2.18. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

with the convention that in the case $N < k$ we obtain 0.

Proof. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, then A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned}
\det(G_k) &= \det(A_k)\det(L_k) \\
&= \det(L_k) \\
&= \prod_\pi L_k(\pi, \pi) \\
&= \prod_\pi N(N-1)\ldots(N - |\pi| + 1)
\end{aligned}$$

Thus, we are led to the formula in the statement. □

Now back to the laws of characters, we can formulate:

Theorem 2.19. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$, the asymptotic moments of the main character are given by*

$$\lim_{N \to \infty} \int_{G_N} \chi^k = |D(k)|$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k-th term.

Proof. According to the Peter-Weyl theory, and to the definition of easiness, the moments of the main character are given by the following formula:

$$\begin{aligned}
\int_{G_N} \chi^k &= \int_{G_N} \chi_{u^{\otimes k}} \\
&= \dim\left(Fix(u^{\otimes k})\right) \\
&= \dim\left(span\left(\xi_\pi \Big| \pi \in D(k)\right)\right)
\end{aligned}$$

Now since by Theorem 2.18 the vectors ξ_π are linearly independent with $N \geq k$, and in particular with $N \to \infty$, we obtain the formula in the statement. □

In order to work out consequences, we will need the following standard result:

Theorem 2.20. *The Catalan numbers $C_k = |NC_2(2k)|$ satisfy*

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

their generating series $f(z) = \sum_{k\geq 0} C_k z^k$ satisfies $zf^2 - f + 1 = 0$, and we have:

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

Proof. We must count the noncrossing pairings of $\{1, \ldots, 2k\}$. Now observe that such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing of $\{2, \ldots, 2a\}$, and a noncrossing pairing of $\{2a+2, \ldots, 2k\}$. We conclude from this that we have the following recurrence formula for the Catalan numbers:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

But this gives $zf^2 - f + 1 = 0$, and by solving this equation and choosing the solution which is bounded at $z = 0$, we obtain the following formula for f:

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

By using now the Taylor formula for $\sqrt{x}$, we obtain the following formula:

$$f(z) = \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} z^k$$

But this gives the formula in the statement, for the coefficients C_k. $\square$

Now back to quantum groups, we have the following result, from [26], [27]:

Theorem 2.21. *The asymptotic k-moments for the main quantum permutation and rotation groups are the double factorials, and Bell and Catalan numbers,*

$$\begin{matrix} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{matrix} \quad : \quad \begin{matrix} C_k & - & C_{k/2} \\ | & & | \\ B_k & - & k!! \end{matrix}$$

the precise formulae being as follows,

(1) $k!! = 1.3.5\ldots(k-3)(k-1)$,
(2) $B_k = |P(k)|$ *are the Bell numbers,*
(3) $C_k = \frac{1}{k+1}\binom{2k}{k}$ *are the Catalan numbers,*

with the conventions $k!! = 0$ and $C_{k/2} = 0$ for $k \notin 2\mathbb{N}$.

Proof. Consider indeed the quantum groups in the statement. According to the easiness result from Theorem 2.11, and to the character formula in Theorem 2.19, the asymptotic moments in question appear by counting the following sets of partitions:

$$\begin{tikzcd} NC(k) \arrow[d] & NC_2(k) \arrow[l] \arrow[d] \\ P(k) & P_2(k) \arrow[l] \end{tikzcd}$$

But these counting questions are all standard, as follows:

(1) Regarding $k!! = |P_2(k)|$, this formula is clear, because we have $k-1$ choices for the pair of 1, then $k-3$ choices for the pair of the next number, and so on.

(2) Regarding $B_k = |P(k)|$, there is nothing much to be done here, because these numbers, called Bell numbers, cannot be explicitely computed.

(3) Regarding now the numbers $C_{k/2} = |NC_2(k)|$, which are the Catalan numbers, these can be explicitely computed by recurrence, as explained in Theorem 2.20.

(4) Regarding $C_k = |NC(k)|$, this can be established either by recurrence, or deduced from (3), via fattening/shrinking. Indeed, by fattening the pairings into partitions, and shrinking the partitions into pairings, we have a correspondence as follows:

$$NC_2(2k) \simeq NC(k)$$

We conclude from this that we have $|NC(k)| = C_k$, as claimed. □

Once again following [26], [27], we have as well the following result:

Theorem 2.22. *The asymptotic laws of characters for the quantum permutation and rotation groups are the Gaussian, Poisson, Wigner and Marchenko-Pastur laws,*

$$\begin{tikzcd} S_N^+ \arrow[r] & O_N^+ \\ S_N \arrow[u] \arrow[r] & O_N \arrow[u] \end{tikzcd} \quad : \quad \begin{tikzcd} \pi_1 \arrow[r, dash] \arrow[d, dash] & \gamma_1 \arrow[d, dash] \\ p_1 \arrow[r, dash] & g_1 \end{tikzcd}$$

the precise formulae being as follows:

(1) $g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ *is the Gaussian law of parameter* 1.
(2) $p_1 = \frac{1}{e}\sum_p \frac{\delta_p}{p!}$ *is the Poisson law of parameter* 1.
(3) $\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$ *is the Wigner semicircle law of parameter* 1.
(4) $\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$ *is the Marchenko-Pastur law of parameter* 1.

Proof. This follows indeed from Theorem 2.21, by doing some calculus:

(1) By partial integration, we have the following formula:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^k e^{-x^2/2}\,dx = (k-1)\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^{k-2}e^{-x^2/2}\,dx$$

Thus the moments of g_1 satisfy the same recurrence as the numbers $k!!$.

(2) The moments of the Poisson law p_1 are the following numbers:

$$M_k = \frac{1}{e}\sum_{p\in\mathbb{N}}\frac{p^k}{k!}$$

Computations show that the recurrence is the same as for the Bell numbers B_k.

(3) The moment generating function for the semicircle law γ_1 is given by:

$$f(z) = \frac{1}{2\pi}\int_{-2}^{2}\frac{\sqrt{4-x^2}}{1-zx}\,dx$$

By doing some computations, the coefficients of f are the Catalan numbers.

(4) The moment generating function for the Marchenko-Pastur law π_1 is:

$$f(z) = \frac{1}{2\pi}\int_0^4\frac{\sqrt{4x^{-1}-1}}{1-zx}\,dx$$

By computation, we obtain the generating series of the Catalan numbers. □

The above proof was of course quite short, but all this is standard material, and we will be back to it with full details in section 5 below, when doing analysis.

As a conclusion now, the representation theory of our basic quantum groups is something extremely simple and fundamental, in the $N\to\infty$ limit.

We will see in the next section that the results in the free case can be improved, with the convergences there being actually stationary, starting from $N=2$. Also, we will see in section 5 below that the above results can be extended to the case of truncated characters, with the limiting $N\to\infty$ measures being p_t,g_t,π_t,γ_t, with $t\in(0,1]$.

3. Representation theory

We have seen so far that the inclusion $S_N \subset S_N^+$, as well as its companion inclusion $O_N \subset O_N^+$, are liberations in the sense of easy quantum groups, and that some interesting representation theory consequences, in the $N \to \infty$ limit, can be derived from this. We discuss here the case where $N \in \mathbb{N}$ is fixed. Let us first discuss the representation theory of O_N^+. Here the result, from [1], which is elementary, is as follows:

Theorem 3.1. *The quantum groups O_N^+ with $N \geq 2$ have the following properties:*

(1) *The odd moments of the main character vanish, and the even moments are:*

$$\int_{O_N^+} \chi^{2k} = C_k$$

(2) *The main character follows the Wigner semicircle law of parameter 1:*

$$\chi \sim \gamma_1$$

(3) *The fusion rules for irreducible representations are the same as for SU_2:*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

(4) *The dimensions of the representations are as follows, with $q + q^{-1} = N$:*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}$$

Proof. There are several proofs for this fact, the simplest one being via purely algebraic methods, based on the easiness property of O_N^+, as follows:

(1) Our claim is that we can define, by recurrence on $k \in \mathbb{N}$, a sequence $r_0, r_1, r_2, \ldots$ of irreducible, self-adjoint and distinct representations of O_N^+, satisfying:

$$r_0 = 1$$

$$r_1 = u$$

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(2) Indeed, at $k = 0$ this is clear, and at $k = 1$ this is clear as well, with the irreducibility of $r_1 = u$ coming from the embedding $O_N \subset O_N^+$. So assume now that $r_0, \ldots, r_{k-1}$ as above are constructed, and let us construct r_k. We have, by recurrence:

$$r_{k-1} + r_{k-3} = r_{k-2} \otimes r_1$$

In particular we have an inclusion of representations, as follows:

$$r_{k-1} \subset r_{k-2} \otimes r_1$$

Now since r_{k-2} is irreducible, by Frobenius reciprocity we have:

$$r_{k-2} \subset r_{k-1} \otimes r_1$$

Thus, there exists a certain representation r_k such that:

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(3) As a first observation, this representation r_k is self-adjoint. Indeed, our recurrence formula $r_k + r_{k-2} = r_{k-1} \otimes r_1$ for the representations $r_0, r_1, r_2, \ldots$ shows that the characters of these representations are polynomials in χ_u. Now since χ_u is self-adjoint, all the characters that we can obtain via our recurrence are self-adjoint as well.

(4) It remains to prove that r_k is irreducible, and non-equivalent to $r_0, \ldots, r_{k-1}$. For this purpose, observe that according to our recurrence formula, $r_k + r_{k-2} = r_{k-1} \otimes r_1$, we can now split $u^{\otimes k}$, as a sum of the following type, with positive coefficients:

$$u^{\otimes k} = c_k r_k + c_{k-2} r_{k-2} + \ldots$$

We conclude by Peter-Weyl that we have an inequality as follows, with equality precisely when r_k is irreducible, and non-equivalent to the other summands r_i:

$$\sum_i c_i^2 \leq \dim(End(u^{\otimes k}))$$

(5) Now let us use the easiness property of O_N^+. This gives us an upper bound for the number on the right, that we can add to our inequality, as follows:

$$\sum_i c_i^2 \leq \dim(End(u^{\otimes k})) \leq C_k$$

The point now is that the coefficients c_i come straight from the Clebsch-Gordan rules, and their combinatorics shows that $\sum_i c_i^2$ equals the Catalan number C_k, with the remark that this follows as well from the known theory of SU_2. Thus, we have global equality in the above estimate, and in particular we have equality at left, as desired.

(6) In order to finish the proof of our claim, observe that r_k is non-equivalent to $r_{k-1}, r_{k-3}, \ldots$, for instance because of the embedding $O_N \subset O_N^+$, which shows that the even and odd tensor powers of u cannot have common irreducible components.

(7) Since by Peter-Weyl any irreducible representation of O_N^+ must appear in some tensor power $u^{\otimes k}$, and we know how to decomposing each $u^{\otimes k}$ into sums of representations r_k, these representations r_k are all the irreducible representations of O_N^+.

(8) In what regards now the law of the main character, we obtain here the Wigner law γ_1, as stated, due to the fact that the equality in (5) gives us the even moments of this law, and that the observation in (6) tells us that the odd moments vanish.

(9) Finally, from the Clebsch-Gordan rules we have $r_k r_1 = r_{k-1} + r_{k+1}$, and we obtain from this, by recurrence, with $q > 0$ being such that $q + q^{-1} = N$:

$$\dim r_k = q^k + q^{k-2} + \ldots + q^{-k+2} + q^{-k}$$

But this gives the dimension formula in the statement, and we are done. $\square$

The above result has some interesting combinatorial consequences, as follows:

Proposition 3.2. *The following are linearly independent, for any $N \geq 2$:*

(1) *The linear maps $\left\{T_\pi \middle| \pi \in NC_2(k,l)\right\}$, with $k+l \in 2\mathbb{N}$.*
(2) *The vectors $\left\{\xi_\pi \middle| \pi \in NC_2(2k)\right\}$, with $k \in \mathbb{N}$.*
(3) *The linear maps $\left\{T_\pi \middle| \pi \in NC_2(k,k)\right\}$, with $k \in \mathbb{N}$.*

Proof. All this follows from the dimension equalities established in the proof of Theorem 3.1, because in all cases, the number of partitions is a Catalan number. □

In order to pass now to quantum permutations, we can use the following well-known trick, relating noncrossing pairings to arbitrary noncrossing partitions:

Proposition 3.3. *We have a bijection $NC(k) \simeq NC_2(2k)$, constructed by fattening and shrinking, as follows:*

(1) *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
(2) *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

At the level of the associated Gram matrices, the result is as follows:

Proposition 3.4. *The Gram matrices of the sets of partitions*

$$NC_2(2k) \simeq NC(k)$$

are related by the following formula, where $\pi \to \pi'$ is the shrinking operation,

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi',\sigma')$$

and where Δ_{kn} is the diagonal of G_{kn}.

Proof. In the context of the general fattening and shrinking bijection from Proposition 3.3 above, it is elementary to see that we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we obtain the formula in the statement. Now by applying the determinant, we obtain from this a formula of the following type, with $C > 0$ being a constant:

$$\det(G_{2k,n}) = C \cdot \det(G_{k,n^2})$$

Thus, we are led to the formula in the statement. □

We can now formulate a "projective" version of Proposition 3.2, as follows:

Proposition 3.5. *The following are linearly independent, for any $N = n^2$ with $n \geq 2$:*

(1) *The linear maps $\left\{T_\pi \middle| \pi \in NC(k,l)\right\}$, with $k,l \in 2\mathbb{N}$.*
(2) *The vectors $\left\{\xi_\pi \middle| \pi \in NC(k)\right\}$, with $k \in \mathbb{N}$.*
(3) *The linear maps $\left\{T_\pi \middle| \pi \in NC(k,k)\right\}$, with $k \in \mathbb{N}$.*

Proof. This follows indeed from the various results from Proposition 3.2, by using the Gram determinant formula from Proposition 3.4. □

Following [2], we can now work out the representation theory of the quantum group S_4^+, and more generally of any S_N^+ with $N = n^2$ and $n \geq 2$, as follows:

Theorem 3.6. *The quantum groups S_N^+ with $N = n^2$ and $n \geq 2$ are as follows:*

(1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+} \chi^k = C_k$$

(2) *The main character follows the Marchenko-Pastur law of parameter 1:*

$$\chi \sim \pi_1$$

(3) *The fusion rules for irreducible representations are the same as for SO_3:*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

(4) *The dimensions of the representations are as follows, with $q + q^{-1} = N - 2$:*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q - 1}$$

Proof. This is quite similar to the proof of Theorem 3.1 above, by using the linear independence result from Proposition 3.5 as main ingredient, as follows:

(1) We have indeed the following computation, based on the above:

$$\int_{S_N^+} \chi^k = \dim(Fix(u^{\otimes k})) = \#NC(k) = \#NC_2(2k) = C_k$$

(2) This follows from (1), as explained in section 1 above.

(3) This is standard, by using the moment formula in (1), and the known theory of SO_3. Let indeed $A = span(\chi_k | k \in \mathbb{N})$ be the algebra of characters of SO_3. We can define a morphism as follows, f being the character of the fundamental representation of S_N^+:

$$\Psi : A \to C(S_N^+) \quad , \quad \chi_1 \to f - 1$$

The elements $f_k = \Psi(\chi_k)$ verify then the Clebsch-Gordan rules, namely:

$$f_k f_l = f_{|k-l|} + f_{|k-l|+1} + \ldots + f_{k+l}$$

We prove now by recurrence that each f_k is the character of an irreducible corepresentation r_k of $C(S_N^+)$, non-equivalent to $r_0,\ldots,r_{k-1}$. At $k=0,1$ this is clear. So, assume now that the result holds at $k-1$. By integrating characters we have, as for SO_3:

$$r_{k-2},r_{k-1}\subset r_{k-1}\otimes r_1$$

Thus there exists a corepresentation r_k such that:

$$r_{k-1}\otimes r_1=r_{k-2}+r_{k-1}+r_k$$

Once again by integrating characters, we conclude that r_k is irreducible, and non-equivalent to $r_1,\ldots,r_{k-1}$, as for SO_3, and this proves our claim. Finally, since any irreducible representation of S_N^+ must appear in some tensor power of u, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations r_l, we conclude that these representations r_l are all the irreducible representations of S_N^+.

(4) Finally, the dimension formula there is clear by recurrence. □

Let us discuss now the extension of the above result, to all the quantum groups S_N^+ with $N\geq 4$. For this purpose we need an extension of the linear independence results from Proposition 3.5. There are several approaches here, none being trivial, and we will use in what follows a method which is long, but elementary and rock-solid, namely getting the linear independence by computing the associated Gram determinant.

We already know, from section 2 above, that for the group S_N the formula of the corresponding Gram matrix determinant, due to Lindstöm [114], is as follows:

Theorem 3.7. *The determinant of the Gram matrix of S_N is given by*

$$\det(G_{kN})=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0.

Proof. This is something that we know from section 2, the idea being that G_{kN} naturally decomposes as a product of an upper triangular and lower triangular matrix. □

Although we will not need this here, let us discuss as well, for the sake of completness, the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. The result is then as follows:

Theorem 3.8. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN})=\prod_{|\lambda|=k/2}f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

Proof. This follows from the results of Collins and Matsumoto [71] and Zinn-Justin [150]. Indeed, it is known from there that the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the result. See [31] and [71], [150]. □

For the free orthogonal group, the result, from [81], is as follows:

Theorem 3.9. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k+1,r}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb{Z}$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb{Z}$.

Proof. We present here a short proof, from [31]. As a first observation, the result holds when k is odd, all the exponents being 0. So, we assume that k is even.

Step 1. We establish a formula of the following type:

$$G_{kN}(\pi,\sigma)=<f_\pi,f_\sigma>$$

For this purpose, let Γ be a bipartite graph, with distinguished vertex 0 and adjacency matrix A, and let μ be an eigenvector of A, with eigenvalue N.

Let L_k be the set of length k loops $l=l_1\ldots l_k$ based at 0, and set:

$$H_k=span(L_k)$$

For $\pi\in NC_2(k)$ define $f_\pi\in H_k$ by:

$$f_\pi=\sum_{l\in L_k}\left(\prod_{i\sim_\pi j}\delta(l_i,l_j^o)\gamma(l_i)\right)l$$

Here $e\to e^o$ is the edge reversing, and the "spin factor" is as follows, as in [106], where s,t are the source and target of the edges:

$$\gamma=\sqrt{\mu(t)/\mu(s)}$$

The point now is that we have the following formula:

$$G_{kN}(\pi,\sigma)=< f_\pi,f_\sigma >$$

We refer to [106] for details regarding all this.

Step 2. With a suitable choice of (Γ,μ), we obtain a fomula of type:

$$G_{kN}=T_{kN}T_{kN}^t$$

Indeed, let us choose $\Gamma=\mathbb N$ to be the Cayley graph of O_N^+, and the eigenvector entries $\mu(r)$ to be the Chebycheff polynomials $P_r(N)$, i.e. the orthogonal polynomials for O_N^+.

In this case, we have a bijection $NC_2(k)\to L_k$, constructed as follows. For $\pi\in NC_2(k)$ and $0\leq i\leq k$ we define $h_\pi(i)$ to be the number of $1\leq j\leq i$ which are joined by π to a number strictly larger than i. We then define a loop $l(\pi)=l(\pi)_1\ldots l(\pi)_k$, where $l(\pi)_i$ is the edge from $h_\pi(i-1)$ to $h_\pi(i)$. Consider now the following matrix:

$$T_{kN}(\pi,\sigma)=\prod_{i\sim_\pi j}\delta(l(\sigma)_i,l(\sigma)_j^o)\gamma(l(\sigma)_i)$$

We have then the following formula:

$$f_\pi=\sum_\sigma T_{kn}(\pi,\sigma)\cdot l(\sigma)$$

Thus we obtain, as desired:

$$G_{kN}=T_{kN}T_{kN}^t$$

Step 3. We show that, with suitable conventions, T_{kN} is lower triangular.

Indeed, consider the partial order on $NC_2(k)$ given by $\pi\leq\sigma$ if $h_\pi(i)\leq h_\sigma(i)$ for $i=1,\ldots,k$. Our claim is that $\sigma\not\leq\pi$ implies:

$$T_{kN}(\pi,\sigma)=0$$

Indeed, suppose that $\sigma\not\leq\pi$, and let j be the least number with $h_\sigma(j)>h_\pi(j)$. Note that we must have $h_\sigma(j-1)=h_\pi(j-1)$ and $h_\sigma(j)=h_\pi(j)+2$. It follows that we have $i\sim_\pi j$ for some $i<j$. From the definitions of T_{kn} and $l(\sigma)$, if $T_{kn}(\pi,\sigma)\neq 0$ then we must have $h_\sigma(i-1)=h_\sigma(j)=h_\pi(j)+2$. But we also have $h_\pi(i-1)=h_\pi(j)$, so that $h_\sigma(i-1)=h_\pi(i-1)+2$, which contradicts the minimality of j.

Step 4. End of the proof, by computing the determinant of T_{kN}.

Since T_{kN} is lower triangular we have:

$$\begin{aligned}\det(T_{kN}) &= \prod_\pi T_{kN}(\pi,\pi)\\ &= \prod_\pi\prod_{i\sim_\pi j}\sqrt{\frac{P_{h_{\pi(i)}}}{P_{h_{\pi(i)-1}}}}\\ &= \prod_{r=1}^{k/2} P_r^{e_{kr}/2}\end{aligned}$$

Here the exponents appearing on the right are by definition as follows:

$$e_{kr}=\sum_\pi\sum_{i\sim_\pi j}\delta_{h_\pi(i),r}-\delta_{h_\pi(i),r+1}$$

Our claim now, which finishes the proof, is that for $1\leq r\leq k/2$ we have:

$$\sum_\pi\sum_{i\sim_\pi j}\delta_{h_\pi(i)r}=f_{k/2,r}$$

Indeed, note that the left term counts the number of times that the edge $(r,r+1)$ appears in all loops in L_k. Define a shift operator S on the edges of Γ by:

$$S(s,t)=(s+1,t+1)$$

Given a loop $l=l_1\ldots l_k$ and $1\leq s\leq k$ with $l_s=(r,r+1)$, define a path:

$$S^r(l_s)\ldots S^r(l_k)l_{s-1}^o\ldots l_1^o$$

Observe that this is a path in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and first reaches $r-1$ after $k-s+1$ steps.

Conversely, given a path $f_1\ldots f_k$ in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and first reaches $r-1$ after s steps, define a loop:

$$f_k^o\ldots f_s^oS^{-r}(f_1)\ldots S^{-r}(f_{s-1})$$

Observe that this is a loop in Γ based at 0 whose $k-s+1$ edge is $(r,r+1)$.

These two operations are inverse to each other, so we have established a bijection between k-loops in Γ based at 0 whose s-th edge is $(r,r+1)$ and k-paths in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and which first reach $r-1$ after $k-s+1$ steps.

It follows that the left hand side is equal to the number of paths in $\Gamma=\mathbb N$ from $2r$ to 0 whose first edge is $(2r,2r+1)$. By the usual reflection trick, this is the difference of binomials defining $f_{k/2,r}$, and we are done. $\square$

Regarding now the quantum group S_N^+, we have here the following formula, also established by Di Francesco and collaborators in [81], [82], [83], [84]:

Theorem 3.10. *The determinant of the Gram matrix for S_N^+ is given by*

$$\det(G_{kN})=(\sqrt{N})^{a_k}\prod_{r=1}^k P_r(\sqrt{N})^{d_{kr}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k+1,r}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb{Z}$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb{Z}$, and where $a_k=\sum_{\pi\in\mathcal{P}(k)}(2|\pi|-k)$.

Proof. According to Proposition 3.4 above, if we denote by G' the Gram matrix for O_N^+, we have the following formula, with $D_{kN}=diag(N^{|\widetilde{\pi}|/2-k/4})$:

$$G_{kN}=D_{kN}G'_{2k,\sqrt{N}}D_{kN}$$

With this formula in hand, the result follows from Theorem 3.9. $\square$

We refer to [31] and to [38], [81], [82], [83], [84], [94] for a discussion here. Now with the above result in hand, we have the following extension of Theorem 3.6:

Theorem 3.11. *The quantum groups S_N^+ with $N\geq 4$ have the following properties:*

(1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+}\chi^k=C_k$$

(2) *The main character follows the Marchenko-Pastur law of parameter 1:*

$$\chi\sim\pi_1$$

(3) *The fusion rules for irreducible representations are the same as for SO_3:*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+1}+\ldots+r_{k+l}$$

(4) *The dimensions of the representations are as follows, with $q+q^{-1}=N-2$:*

$$\dim(r_k)=\frac{q^{k+1}-q^{-k}}{q-1}$$

Proof. The above statement is exactly the statement of Theorem 3.6, with the assumption $N=n^2$ lifted. As for the proof, this is identical to the proof of Theorem 3.6, using this time the linear independence result coming from Theorem 3.10 as technical ingredient. $\square$

Summarizing, we have now full representation theory results for both O_N^+,S_N^+. These results are quite surprising, and there are many things that can be said about O_N^+,S_N^+, in analogy with the known facts about SU_2,SO_3. However, all this is quite technical, and we defer the discussion to section 4 below. Let us record, however:

Theorem 3.12. *The quantum groups O_N^+,S_N^+ have the following properties:*

(1) *O_2^+,S_4^+ are coamenable, and of polynomial growth.*
(2) *O_N^+,S_N^+ with $N\geq 3,5$ are not coamenable, and have exponential growth.*

Proof. The various coamenability assertions follow from the Kesten criterion from section 1 above, the support of the spectral measure of χ being respectively:

$$supp(\gamma_1)=[-2,2]$$

$$supp(\pi_1)=[0,4]$$

As for the growth assertions, which can be of course improved with explicit exponents and so on, these follow from the fact that the corresponding Cayley graphs are $\mathbb N$. □

In the remainder of this section we keep developing some useful general theory for O_N^+,S_N^+ and their subgroups. We will present among others a general result from [7], refining the Tannakian duality for the quantum permutation groups $G\subset S_N^+$, stating that the following spaces form a planar algebra in the sense of Jones [105]:

$$P_k=Fix(u^{\otimes k})$$

To be more precise, we will show that these spaces form a planar subalgebra $P=(P_k)$ of the Jones spin planar algebra $\mathcal S_N$, and that any planar subalgebra $P\subset\mathcal S_N$ appears in this way, so that we have a refined Tannakian correspondence, as follows:

$$G\subset S_N^+\ \longleftrightarrow\ P\subset\mathcal S_N$$

In order to get started, we need a lot of preliminaries, the lineup being von Neumann algebras, II_1 factors, subfactors, and finally planar algebras. We already met von Neumann algebras, in section 1 above. The fundamental result regarding them is as follows:

Theorem 3.13. *Any von Neumann algebra $A\subset B(H)$ decomposes as*

$$A=\int_X A_x\,dx$$

with X being the measured space appearing as spectrum of the center, $Z(A)=L^\infty(X)$, and with the fibers A_x being "factors", in the sense that $Z(A_x)=\mathbb C$.

Proof. The decomposition result definitely holds in finite dimensions, where von Neumann algebra is the same as C^*-algebra, and where the algebras are as follows:

$$A=M_{N_1}(\mathbb C)\oplus\ldots\oplus M_{N_k}(\mathbb C)$$

Indeed, as explained in section 1 above, this decomposition is obtained by writing $Z(A) = \mathbb{C}^k$. In general, the decomposition result in the statement is von Neumann's "reduction theory" result, based on advanced functional analysis. $\square$

At an even more advanced level now, we know from Theorem 3.13 that, at least in theory, things basically reduce to "factors". And, regarding these factors, we have:

Theorem 3.14. *The von Neumann factors, $Z(A) = \mathbb{C}$, have the following properties:*

1. *They can be fully classified in terms of II_1 factors, which are by definition those satisfying $\dim A = \infty$, and having a faithful trace $tr : A \to \mathbb{C}$.*
2. *The II_1 factors enjoy the "continuous dimension geometry" property, in the sense that the traces of their projections can take any values in $[0, 1]$.*
3. *Among the II_1 factors, the smallest one is the Murray-von Neumann hyperfinite factor R, obtained as an inductive limit of matrix algebras.*

Proof. This is one again something heavy, the idea being as follows:

(1) This comes from results of Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions.

(2) This is subtle functional analysis, with the rational traces being relatively easy to obtain, and with the irrational ones coming from limiting arguments.

(3) Once again, heavy results, by Murray-von Neumann and Connes, the idea being that any finite dimensional construction always leads to the same factor, called R. $\square$

Let us discuss now subfactor theory, following Jones' fundamental paper [101]. Jones looked at the inclusions of II_1 factors $A \subset B$, called subfactors, which are quite natural objects physics. Given such an inclusion, we can talk about its index:

Definition 3.15. *The index of an inclusion of II_1 factors $A \subset B$ is the quantity*

$$[B : A] = \dim_A B \in [1, \infty]$$

constructed by using the Murray-von Neumann continuous dimension theory.

In order to explain Jones' result [101], it is better to relabel our subfactor as:

$$A_0 \subset A_1$$

We can construct the orthogonal projection $e_1 : A_1 \to A_0$, and set:

$$A_2 =< A_1, e_1 >$$

This remarkable procedure, called "basic construction", can be iterated, and we obtain in this way a whole tower of II_1 factors, as follows:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

Quite surprisingly, this construction leads to a link with the Temperley-Lieb algebra $TL_N = span(NC_2)$, and with many other things, which can be summarized as follows:

Theorem 3.16. *Let $A_0\subset A_1$ be an inclusion of II_1 factors.*

(1) *The sequence of projections $e_1,e_2,e_3,\ldots\in B(H)$ produces a representation of the Temperley-Lieb algebra of index $N=[A_1,A_0]$, as follows:*
$$TL_N\subset B(H)$$

(2) *The index $N=[A_1,A_0]$, which is a Murray-von Neumann continuous quantity $N\in[1,\infty]$, must satisfy the following condition:*
$$N\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n\in\mathbb N\right\}\cup[4,\infty]$$

Proof. This result, from [101], is something quite tricky, the idea being as follows:

(1) The idea here is that the functional analytic study of the basic construction leads to the conclusion that the sequence of projections $e_1,e_2,e_3,\ldots\in B(H)$ behaves algebrically, when rescaled, exactly as the sequence of diagrams $\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots\in TL_N$ given by:
$$\varepsilon_1={}^{\cup}_{\cap}\quad,\quad\varepsilon_2=|{}^{\cup}_{\cap}\quad,\quad\varepsilon_3=||{}^{\cup}_{\cap}\quad,\quad\ldots$$

But these diagrams generate TL_N, and so we have an embedding $TL_N\subset B(H)$, where H is the Hilbert space where our subfactor $A_0\subset A_1$ lives, as claimed.

(2) This is something quite surprising, which follows from (1), via some clever positivity considerations, involving the Perron-Frobenius theorem. In fact, the subfactors having index $N\in[1,4]$ can be classified by ADE diagrams, and the obstruction $N=4\cos^2(\frac{\pi}{n})$ itself comes from the fact that N must be the squared norm of such a graph. □

Quite remarkably, Theorem 3.16 is just the "tip of the iceberg". One can prove indeed that the planar algebra structure of TL_N, taken in an intuitive sense, extends to a planar algebra structure on the following sequence of commutants:
$$P_k=A_0'\cap A_k$$

In order to discuss this key result, from [105], that we will need as well, in connection with our quantum permutation group problems, let us start with:

Definition 3.17. *The planar algebras are defined as follows:*

(1) *A k-tangle, or k-box, is a rectangle in the plane, with $2k$ marked points on its boundary, containing r small boxes, each having $2k_i$ marked points, and with the $2k+\sum 2k_i$ marked points being connected by noncrossing strings.*

(2) *A planar algebra is a sequence of finite dimensional vector spaces $P=(P_k)$, together with linear maps $P_{k_1}\otimes\ldots\otimes P_{k_r}\to P_k$, one for each k-box, such that the gluing of boxes corresponds to the composition of linear maps.*

As basic example of a planar algebra, we have the Temperley-Lieb algebra TL_N. Indeed, putting $TL_N(k_i)$ diagrams into the small r boxes of a k-box clearly produces a $TL_N(k)$ diagram, and so we have indeed a planar algebra, of somewhat "trivial" type. In general, the planar algebras are more complicated than this, and we will be back later with some explicit examples. However, the idea is very simple, namely "the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams".

In relation now with subfactors, the result, which extends Theorem 3.16 (1) above, and which was found by Jones in [105], almost 20 years after [101], is as follows:

Theorem 3.18. *Given a subfactor $A_0 \subset A_1$, the collection $P = (P_k)$ of linear spaces*

$$P_k = A_0' \cap A_k$$

has a planar algebra structure, extending the planar algebra structure of TL_N.

Proof. As a first observation, since $e_1 : A_1 \to A_0$ commutes with A_0 we have $e_1 \in P_2'$. By translation we obtain $e_1, \ldots, e_{k-1} \in P_k$ for any k, and so:

$$TL_N \subset P$$

The point now is that the planar algebra structure of TL_N, obtained by composing diagrams, can be shown to extend into an abstract planar algebra structure of P. This is something quite heavy, and we will not get into details here. See [105]. □

Getting back now to quantum groups, all this machinery is very interesting for us. We will need the construction of the spin planar algebra $\mathcal{S}_N$. Let us start with:

Definition 3.19. *We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2 \times k$ matrix form,*

$$e_{i_1 \ldots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \ldots & \ldots \\ i_k & i_k & i_{k-1} & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra $\mathcal{S}_N$, also from [105], as follows:

Definition 3.20. *The spin planar algebra $\mathcal{S}_N$ is the sequence of vector spaces*

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

written as above, with the multilinear maps associated to the various k-tangles

$$T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$$

being given by the following formula, in multi-index notation,

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

Here are some illustrating examples for the spin planar algebra calculus:

(1) The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y)=1$ being the pairs of the form (x,x), this tangle 1_k acts by the identity:

$$1_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(2) The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k\left(\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}\otimes\begin{pmatrix}l_1&\ldots&l_k\\ m_1&\ldots&m_k\end{pmatrix}\right)=\delta_{j_1m_1}\ldots\delta_{j_km_k}\begin{pmatrix}l_1&\ldots&l_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(3) The inclusion I_k is the $(k,k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y)=1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_l\begin{pmatrix}j_1&\ldots&j_k&l\\ i_1&\ldots&i_k&l\end{pmatrix}$$

Observe that I_k is an inclusion of algebras, and that the various I_k are compatible with each other. The inductive limit of the algebras $\mathcal{S}_N(k)$ is a graded algebra, denoted $\mathcal{S}_N$.

(4) The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

Observe that U_k is a bimodule morphism with respect to I_k.

(5) The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have the following formula:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

(6) The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap&|&|&|&\|\\ \|&&&&\|\\ \|&|&|&|&\Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1\ldots i_k})=e_{i_2i_3\ldots i_ki_1}$$

Thus R_k acts by an order k linear automorphism of $\mathcal{S}_N(k)$, also called rotation.

There are many other interesting examples of k-tangles, but in view of our present purposes, we can actually stop here, due to the following useful fact:

Theorem 3.21. *The multiplications, inclusions, expectations, Jones projections, and rotations generate the set of all tangles, via the gluing operation.*

Proof. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list. See [105]. □

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}^*=\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}$$

Summarizing, the sequence of vector spaces $\mathcal{S}_N(k)=C(X^k)$ has a planar $*$-algebra structure, called spin planar algebra of index $N=|X|$. See [105].

Following [7], we have the following result:

Theorem 3.22. *Given $G\subset S_N^+$, consider the tensor powers of the associated coaction map on $C(X)$, where $X=\{1,\ldots,N\}$, which are the folowing linear maps:*

$$\Phi^k:C(X^k)\to C(X^k)\otimes C(G)$$

$$e_{i_1\ldots i_k}\to\sum_{j_1\ldots j_k}e_{j_1\ldots j_k}\otimes u_{j_1i_1}\ldots u_{j_ki_k}$$

The fixed point spaces of these coactions, which are by definition the spaces

$$P_k=\left\{x\in C(X^k)\Big|\Phi^k(x)=1\otimes x\right\}$$

are given by $P_k=Fix(u^{\otimes k})$, and form a subalgebra of the spin planar algebra $\mathcal{S}_N$.

Proof. Since the map Φ is a coaction, coming from the corepresentation u, its tensor powers Φ^k are coactions too, coming from the corepresentations $u^{\otimes k}$, and at the level of the fixed point algebras we have the following formula, which is standard:

$$P_k=Fix(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 3.21.

Consider the rotation R_k. Rotating, then applying Φ^k, and rotating backwards by R_k^{-1} is the same as applying Φ^k, then rotating each k-fold product of coefficients of Φ.

Thus the elements obtained by rotating, then applying Φ^k, or by applying Φ^k, then rotating, differ by a sum of Dirac masses tensored with commutators in $A = C(G)$:

$$\Phi^k R_k(x) - (R_k \otimes id)\Phi^k(x) \in C(X^k) \otimes [A, A]$$

Now let $\int_A$ be the Haar functional of A, and consider the conditional expectation onto the fixed point algebra P_k, which is given by the following formula:

$$\phi_k = \left(id \otimes \int_A \right) \Phi^k$$

The square of the antipode being the identity, the Haar integration $\int_A$ is a trace, so it vanishes on commutators. Thus R_k commutes with ϕ_k:

$$\phi_k R_k = R_k \phi_k$$

The commutation relation $\phi_k T = T\phi_l$ holds in fact for any (l, k)-tangle T. These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle T:

$$\phi_k T \phi_l = T\phi_l$$

We conclude from this that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ of the planar operad consisting of tangles T satisfying the following condition, where $\phi = (\phi_k)$, and where $i(.)$ is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of Φ^k gives $M_k \in \mathcal{P}'$. Since $\mathcal{P}$ is generated by multiplications and annular tangles, it follows that we have $\mathcal{P}' = P$.

Thus for any tangle T the corresponding multilinear map between spaces $P_k(X)$ restricts to a multilinear map between spaces P_k. In other words, the action of the planar operad $\mathcal{P}$ restricts to P, and makes it a subalgebra of $\mathcal{S}_N$, as claimed. $\square$

As a second result now, also from [7], completing our study, we have:

Theorem 3.23. *Given an arbitrary planar subalgebra $Q \subset \mathcal{S}_N$, there is a unique quantum permutation group $G \subset S_N^+$ whose associated planar algebra is Q.*

Proof. The idea is that this will follow by applying Tannakian duality to the annular category over Q. Let n, m be positive integers. To any element $T_{n+m} \in Q_{n+m}$ we can associate a linear map $L_{nm}(T_{n+m}) : P_n(X) \to P_m(X)$ in the following way:

$$L_{nm} \begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \to \begin{pmatrix} |\ \ |\ \ \cap \\ \ \ T_{n+m}| \\ ||\ \ |\ | \\ a_n|\ \ |\ | \\ \cup\ \ |\ | \end{pmatrix}$$

That is, we consider the planar $(n,n+m,m)$-tangle having an small input n-box, a big input $n+m$-box and an output m-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X)\otimes P_{n+m}(X)\to P_m(X)$$

Now let us put the element T_{n+m} in the big input box. We obtain in this way a certain linear map $P_n(X)\to P_m(X)$, that we call L_{nm}. To be more precise:

(1) The above picture corresponds to $n=1$ and $m=2$. This is illustrating whenever $n\leq m$, it suffices to imagine that in the general case all strings are multiple.

(2) If $n>m$ there are $n+m$ strings of a_n which connect to the $n+m$ lower strings of T_{n+m}, and the remaining $n-m$ ones go to the upper right side and connect to the $n-m$ strings on top right of T_{n+m}. Here is the picture for $n=2$ and $m=1$:

$$L_{nm}\begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} |\,| \\ a_n \\ |\,| \end{pmatrix} \to \begin{pmatrix} \;\;| \quad \Cap \\ T_{n+m}\| \\ \quad ||\;\; ||| \\ \quad a_n||| \\ \qquad \Cup| \\ \qquad\quad | \end{pmatrix}$$

Consider now the linear spaces formed by the maps constructed above:

$$Q_{nm}=\left\{L_{nm}(T_{n+m}):P_n(X)\to P_m(X)\Big|T_{n+m}\in Q_{n+m}\right\}$$

These spaces form a Tannakian category, and so by [148] we obtain a Woronowicz algebra (A,u), such that the following equalities hold, for any m,n:

$$Hom(u^{\otimes m},u^{\otimes n})=Q_{mn}$$

We prove that u is a magic unitary. We have $Hom(1,u^{\otimes 2})=Q_{02}=Q_2$, so the unit of Q_2 must be a fixed vector of $u^{\otimes 2}$. But $u^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$\begin{aligned} u^{\otimes 2}(1) &= u^{\otimes 2}\left(\sum_i \begin{pmatrix} i & i \\ i & i \end{pmatrix}\right) \\ &= \sum_{ikl}\begin{pmatrix} k & k \\ l & l \end{pmatrix}\otimes u_{ki}u_{li} \\ &= \sum_{kl}\begin{pmatrix} k & k \\ l & l \end{pmatrix}\otimes (uu^t)_{kl} \end{aligned}$$

From $u^{\otimes 2}(1)=1\otimes 1$ ve get that uu^t is the identity matrix. Together with the unitarity of u, this gives the following formulae:

$$u^t=u^*=u^{-1}$$

Consider the Jones projection $E_1\in Q_3$. After isotoping, $L_{21}(E_1)$ looks as follows:

$$L_{21}\begin{pmatrix}|\cup\\ |\cap\end{pmatrix}:\begin{pmatrix}|&|\\ i&i\\ j&j\\ |&|\end{pmatrix}\to\begin{pmatrix}|&&\\ i&i&\\ j&j&\supset\\ |&&\end{pmatrix}=\delta_{ij}\begin{pmatrix}|\\ i\\ i\\ |\end{pmatrix}$$

In other words, the linear map $M=L_{21}(E_1)$ is the multiplication $\delta_i\otimes\delta_j\to\delta_{ij}\delta_i$:

$$M\begin{pmatrix}i&i\\ j&j\end{pmatrix}=\delta_{ij}\begin{pmatrix}i\\ i\end{pmatrix}$$

Consider now the following element of $C(X)\otimes A$:

$$\begin{aligned}(M\otimes id)u^{\otimes 2}\left(\begin{pmatrix}i&i\\ j&j\end{pmatrix}\otimes 1\right)&=(M\otimes id)\left(\sum_{kl}\begin{pmatrix}k&k\\ l&l\end{pmatrix}\otimes u_{ki}u_{lj}\right)\\ &=\sum_k\begin{pmatrix}k\\ k\end{pmatrix}\delta_k\otimes u_{ki}u_{kj}\end{aligned}$$

Since $M\in Q_{21}=Hom(u^{\otimes 2},u)$, this equals the following element of $C(X)\otimes A$:

$$\begin{aligned}u(M\otimes id)\left(\begin{pmatrix}i&i\\ j&j\end{pmatrix}\otimes 1\right)&=u\left(\delta_{ij}\begin{pmatrix}i\\ i\end{pmatrix}\delta_i\otimes 1\right)\\ &=\sum_k\begin{pmatrix}k\\ k\end{pmatrix}\delta_k\otimes\delta_{ij}u_{ki}\end{aligned}$$

Thus $u_{ki}u_{kj}=\delta_{ij}u_{ki}$ for any i,j,k. With $i=j$ we get $u_{ki}^2=u_{ki}$, and together with the formula $u^t=u^*$ this shows that all entries of u are self-adjoint projections. With $i\neq j$ we get $u_{ki}u_{kj}=0$, so the projections on each row of u are orthogonal to each other. Together with $u^t=u^{-1}$, this shows that each row of u is a partition of unity with self-adjoint projections. The antipode is given by the formula $(id\otimes S)u=u^*$. But u^* is the transpose of u, so we can apply S to the formulae saying that rows of u are partitions of unity, and we get that columns of u are also partitions of unity. Thus u is a magic unitary.

Now if P is the planar algebra associated to u, we have $Hom(1,v^{\otimes n})=P_n=Q_n$, as desired. As for the uniqueness, this is clear from the Peter-Weyl theory. $\square$

The results established above, regarding the subgroups $G\subset S_N^+$, have several generalizations, to the subgroups $G\subset O_N^+$ and $G\subset U_N^+$, as well as subfactor versions, going beyond the purely combinatorial level. We refer here to [5] and related papers, and we will be back to some of these questions in section 12 below.

4. Twisted permutations

In this section we investigate the quantum permutation groups S_F^+ of the finite quantum spaces F. Besides providing a useful generalization of our results regarding S_N^+, this will eventually explain the connection with SO_3, in an elegant way. As a bonus, we will obtain as well a conceptual result on the connection between S_N^+ and O_N^+.

In order to get started, let us first talk about finite quantum spaces. We have:

Definition 4.1. *A finite quantum space F is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(F) = B$$

The number of elements of such a space is by definition the number $|F| = \dim B$. By decomposing the algebra B, we have a formula of the following type:

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \ldots = n_k = 1$ we obtain in this way the space $F = \{1, \ldots, k\}$. Also, when $k = 1$ the equation is $C(F) = M_n(\mathbb{C})$, and the solution will be denoted $F = M_n$.

In order to talk now about the quantum symmetry group S_F^+, we must use universal coactions. As in section 2, we must endow our space F with its counting measure:

Definition 4.2. *We endow each finite quantum space F with its counting measure, corresponding as the algebraic level to the integration functional*

$$tr : C(F) \to B(l^2(F)) \to \mathbb{C}$$

obtained by applying the regular representation, and then the normalized matrix trace.

To be more precise, consider the algebra $B = C(F)$, which is by definition finite dimensional. We can make act B on itself, by left multiplication:

$$\pi : B \to \mathcal{L}(B) \quad , \quad a \to (b \to ab)$$

The target of π being a matrix algebra, $\mathcal{L}(B) \simeq M_N(\mathbb{C})$ with $N = \dim B$, we can further compose with the normalized matrix trace, and we obtain tr:

$$tr = \frac{1}{N} Tr \circ \pi$$

As basic examples, for both $F = \{1, \ldots, N\}$ and $F = M_N$ we obtain the usual trace. In general, with $C(F) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, the weights of tr are:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Let us also mention that the canonical trace is precisely the one making $\mathbb{C} \subset B$ a Markov inclusion. Equivalently, the counting measure is the one making $F \to \{.\}$ a Markov fibration. For a discussion of these facts, see [2], and also [5], [22].

Let us study now the quantum group actions $G \curvearrowright F$. It is convenient here to use, in order to get started, the no basis approach from [2]. If we denote by μ, η the multiplication and unit map of the algebra $C(F)$, we have the following result, from [2]:

Proposition 4.3. *Consider a linear map $\Phi : C(F) \to C(F) \otimes C(G)$, written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

with $\{e_i\}$ being a linear space basis of $C(F)$, orthonormal with respect to tr.

1. *Φ is a linear space coaction $\iff$ u is a corepresentation.*
2. *Φ is multiplicative $\iff$ $\mu \in Hom(u^{\otimes 2}, u)$.*
3. *Φ is unital $\iff$ $\eta \in Hom(1, u)$.*
4. *Φ leaves invariant tr $\iff$ $\eta \in Hom(1, u^*)$.*
5. *If these conditions hold, Φ is involutive $\iff$ u is unitary.*

Proof. This is a bit similar to the proof for S_N^+ from section 2, as follows:

(1) There are two axioms to be processed here. First, we have:

$$\begin{aligned}(id \otimes \Delta)\Phi = (\Phi \otimes id)\Phi \quad &\iff \quad \sum_j e_j \otimes \Delta(u_{ji}) = \sum_k \Phi(e_k) \otimes u_{ki} \\ &\iff \quad \sum_j e_j \otimes \Delta(u_{ji}) = \sum_{jk} e_j \otimes u_{jk} \otimes u_{ki} \\ &\iff \quad \Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki}\end{aligned}$$

As for the axiom involving the counit, here we have as well, as desired:

$$(id \otimes \varepsilon)\Phi = id \iff \sum_j \varepsilon(u_{ji}) e_j = e_i \iff \varepsilon(u_{ji}) = \delta_{ji}$$

(2) We have the following formula:

$$\Phi(e_i) = \left(\sum_{ij} e_{ji} \otimes u_{ji} \right) (e_i \otimes 1) = u(e_i \otimes 1)$$

By using this formula, we obtain the following identity:

$$\Phi(e_i e_k) = u(e_i e_k \otimes 1) = u(\mu \otimes id)(e_i \otimes e_k \otimes 1)$$

On the other hand, we have as well the following identity, as desired:

$$\begin{aligned}
\Phi(e_i)\Phi(e_k) &= \sum_{jl} e_je_l \otimes u_{ji}u_{lk}\\
&= (\mu\otimes id)\sum_{jl} e_j\otimes e_l\otimes u_{ji}u_{lk}\\
&= (\mu\otimes id)\left(\sum_{ijkl} e_{ji}\otimes e_{lk}\otimes u_{ji}u_{lk}\right)(e_i\otimes e_k\otimes 1)\\
&= (\mu\otimes id)u^{\otimes 2}(e_i\otimes e_k\otimes 1)
\end{aligned}$$

(3) The formula $\Phi(e_i)=u(e_i\otimes 1)$ found above gives by linearity $\Phi(1)=u(1\otimes 1)$. But this shows that Φ is unital precisely when $u(1\otimes 1)=1\otimes 1$, as desired.

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned}
(tr\otimes id)\Phi(e_i)=tr(e_i)1 &\iff \sum_j tr(e_j)u_{ji}=tr(e_i)1\\
&\iff \sum_j u_{ji}^*1_j=1_i\\
&\iff (u^*1)_i=1_i\\
&\iff u^*1=1
\end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that Φ is involutive, we have:

$$\begin{aligned}
(u^*u)_{ik} &= \sum_l u_{li}^*u_{lk}\\
&= \sum_{jl} tr(e_j^*e_l)u_{ji}^*u_{lk}\\
&= (tr\otimes id)\sum_{jl} e_j^*e_l\otimes u_{ji}^*u_{lk}\\
&= (tr\otimes id)(\Phi(e_i)^*\Phi(e_k))\\
&= (tr\otimes id)\Phi(e_i^*e_k)\\
&= tr(e_i^*e_k)1\\
&= \delta_{ik}
\end{aligned}$$

Thus $u^*u=1$, and since we know from (1) that u is a corepresentation, it follows that u is unitary. The proof of the converse is standard too, by using similar tricks. $\square$

Following now [2], we have the following result, extending the basic theory of S_N^+ from the previous section to the present finite quantum space setting:

Theorem 4.4. *Given a finite quantum space F, there is a universal compact quantum group S_F^+ acting on F, leaving the counting measure invariant. We have*

$$C(S_F^+)=C(U_N^+)\Big/\Big\langle\mu\in Hom(u^{\otimes 2},u),\eta\in Fix(u)\Big\rangle$$

where $N=|F|$ and where μ,η are the multiplication and unit maps of $C(F)$. For $F=\{1,\ldots,N\}$ we have $S_F^+=S_N^+$. Also, for the space $F=M_2$ we have $S_F^+=SO_3$.

Proof. This result is from [2], the idea being as follows:

(1) This follows from Proposition 4.3 above, by using the standard fact that the complex conjugate of a corepresentation is a corepresentation too.

(2) Regarding now the main example, for $F=\{1,\ldots,N\}$ we obtain indeed the quantum permutation group S_N^+, due to the results in section 2 above.

(3) In order to do now the computation for $F=M_2$, we use some standard facts about SU_2,SO_3. We have an action by conjugation $SU_2\curvearrowright M_2(\mathbb C)$, and this action produces, via the canonical quotient map $SU_2\to SO_3$, an action $SO_3\curvearrowright M_2(\mathbb C)$. On the other hand, it is routine to check, by using arguments like those from the proof of $S_N^+=S_N$ at $N=2,3$, from section 2 above, that any action $G\curvearrowright M_2(\mathbb C)$ must come from a classical group. We conclude that the action $SO_3\curvearrowright M_2(\mathbb C)$ is universal, as claimed. □

In practice, for our purposes here it will be very useful to have bases and indices. We will use a single index approach, based on the following formalism:

Definition 4.5. *Given a finite quantum space F, we let $\{e_i\}$ be the standard multimatrix basis of $B=C(F)$, so that the multiplication, involution and unit of B are given by*

$$e_ie_j=e_{ij}\quad,\quad e_i^*=e_{\bar i}\quad,\quad 1=\sum_{i=\bar i}e_i$$

where $(i,j)\to ij$ is the standard partially defined multiplication on the indices, with the convention $e_\emptyset=0$, and where $i\to\bar i$ is the standard involution on the indices.

To be more precise, let $\{e_{ab}^r\}\subset B$ be the multimatrix basis. We set then $i=(abr)$, and with this convention, the multiplication, coming from $e_{ab}^re_{cd}^p=\delta_{rp}\delta_{bc}e_{ad}^r$, is given by:

$$(abr)(cdp)=\begin{cases}(adr)&\text{if }r=p,\ b=c\\ \emptyset&\text{otherwise}\end{cases}$$

As for the involution, coming from $(e_{ab}^r)^*=e_{ba}^r$, this is given by:

$$\overline{(a,b,r)}=(b,a,r)$$

Finally, the unit formula comes from the following formula for the unit $1\in B$:

$$1=\sum_{ar}e_{aa}^r$$

We can now convert the main assertion in Theorem 4.4, namely the no indices formula of S_F^+ from there into something more concrete, as follows:

Theorem 4.6. *Given a finite quantum space F, with basis $\{e_i\}\subset C(F)$ as above, the algebra $C(S_F^+)$ is generated by variables u_{ij} with the following relations,*

$$\sum_{ij=p}u_{ik}u_{jl}=u_{p,kl}\quad,\quad\sum_{kl=p}u_{ik}u_{jl}=u_{ij,p}$$

$$\sum_{i=\bar{i}}u_{ij}=\delta_{j\bar{j}}\quad,\quad\sum_{j=\bar{j}}u_{ij}=\delta_{i\bar{i}}$$

$$u_{ij}^*=u_{\bar{i}\bar{j}}$$

with the fundamental corepresentation being the matrix $u=(u_{ij})$. We call a matrix $u=(u_{ij})$ satisfying the above relations "generalized magic".

Proof. This can be deduced from any of the known presentations of $C(S_F^+)$:

(1) If we take the triple index presentation of $C(S_F^+)$ from [140], and replace there the triple indices by single indices, we obtain the relations in the statement.

(2) Alternatively, if we take the double index presentation of $C(S_F^+)$ from [5], and replace there the double indices by single indices, we obtain the relations in the statement.

(3) Also, when using Theorem 4.4, $\mu\in Hom(u^{\otimes 2},u)$ and $\eta\in Fix(u)$ produce the 1st and 4th relations, then the biunitarity of u gives the 5th relation, and finally the 2nd and 3rd relations follow from the 1st and 4th relations, by using the antipode. $\square$

As an illustration, consider the case $F=\{1,\ldots,N\}$. Here the index multiplication is $ii=i$ and $ij=\emptyset$ for $i\neq j$, and the involution is $\bar{i}=i$. Thus, our relations read:

$$u_{ik}u_{il}=\delta_{kl}u_{ik}\quad,\quad u_{ik}u_{jk}=\delta_{ij}u_{ik}$$

$$\sum_i u_{ij}=1\quad,\quad\sum_j u_{ij}=1$$

$$u_{ij}^*=u_{ij}$$

We recognize here the standard magic conditions on a matrix $u=(u_{ij})$.

Let us develop now some basic theory for the quantum symmetry groups S_F^+, and their closed subgroups $G\subset S_F^+$. Some of the results here are well-known, some other are folklore, and some other are new. We first have the following result, from [2]:

Theorem 4.7. *The quantum groups S_F^+ have the following properties:*

1. *The associated Tannakian categories are $TL(N)$, with $N=|F|$.*
2. *The main character follows the Marchenko-Pastur law π_1, when $N\geq 4$.*
3. *The fusion rules for S_F^+ with $|F|\geq 4$ are the same as for SO_3.*

Proof. This result is from [2], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint, $u \sim \bar{u}$. Indeed, let us go back to the coaction formula from Proposition 4.3:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

We can pick our orthogonal basis $\{e_i\}$ to be the stadard multimatrix basis of $C(F)$, so that we have, for a certain involution $i \to i^*$ on the index set:

$$e_i^* = e_{i^*}$$

With this convention made, by conjugating the above formula of $\Phi(e_i)$, we obtain:

$$\Phi(e_{i^*}) = \sum_j e_{j^*} \otimes u_{ji}^*$$

Now by interchanging $i \leftrightarrow i^*$ and $j \leftrightarrow j^*$, this latter formula reads:

$$\Phi(e_i) = \sum_j e_j \otimes u_{j^*i^*}^*$$

We therefore conclude, by comparing with the original formula, that we have:

$$u_{ji}^* = u_{j^*i^*}$$

But this shows that we have an equivalence as follows, as claimed:

$$u \sim \bar{u}$$

Now with this result in hand, the proof goes as for the proof for S_N^+, from the previous section. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of the algebra $C(F)$ that we are interested in here, can be modelled by the following two diagrams:

$$m = |\cup| \qquad , \qquad u = \cap$$

Indeed, this is certainly true algebrically, and this is something well-known. As in what regards the $*$-structure, things here are fine too, because our choice for the trace from Definition 4.2 leads to the following formula, which must be satisfied as well:

$$\mu\mu^* = N \cdot id$$

But the above diagrams m, u generate the Temperley-Lieb algebra $TL(N)$, as stated.

(2) The proof here is exactly as for S_N^+, by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of π_1.

(3) Once again same proof as for S_N^+, by using the fact that the moments of χ are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules. □

It is quite clear now that our present formalism, and the above results, provide altogether a good and conceptual explanation for our SO_3 result regarding S_N^+. To be more precise, we can merge and reformulate our main results so far in the following way:

Theorem 4.8. *The quantun groups S_F^+ have the following properties:*

1. *For $F = \{1, \ldots, N\}$ we have $S_F^+ = S_N^+$.*
2. *For the space $F = M_N$ we have $S_F^+ = PO_N^+ = PU_N^+$.*
3. *In particular, for the space $F = M_2$ we have $S_F^+ = SO_3$.*
4. *The fusion rules for S_F^+ with $|F| \geq 4$ are independent of F.*
5. *Thus, the fusion rules for S_F^+ with $|F| \geq 4$ are the same as for SO_3.*

Proof. This is basically a compact form of what has been said above, with a new result added, and with some technicalities left aside:

(1) This is something that we know from Theorem 4.4.

(2) This is new, the idea being as follows. First of all, we know from section 1 above that the inclusion $PO_N^+ \subset PU_N^+$ is an isomorphism, with this coming from the free complexification formula $\widetilde{O}_N^+ = U_N^+$, but we will actually reprove this result. Consider indeed the standard vector space action of the free unitary group:

$$U_N^+ \curvearrowright \mathbb{C}^N$$

We associate to this action its adjoint action:

$$PU_N^+ \curvearrowright M_N(\mathbb{C})$$

By universality of $S_{M_N}^+$, we must have inclusions as follows:

$$PO_N^+ \subset PU_N^+ \subset S_{M_N}^+$$

On the other hand, the main character of O_N^+ with $N \geq 2$ being semicircular, the main character of PO_N^+ must be Marchenko-Pastur. Thus the inclusion $PO_N^+ \subset S_{M_N}^+$ has the property that it keeps fixed the law of main character, and by Peter-Weyl theory we conclude that this inclusion must be an isomorphism, as desired.

(3) This is something that we know from Theorem 4.4, and that can be deduced as well from (2), by using the formula $PO_2^+ = SO_3$, which is something elementary.

(4) This is something that we know from Theorem 4.7.

(5) This follows from (3,4), as already pointed out in Theorem 4.7. □

Summarizing, we have now a good explanation for the occurrence of SO_3, in connection with quantum permutation questions. Philosophically, the idea is that S_F^+ does not depend that much on F, and so in order to obtain results, it is enough to take $F = M_2$, where the corresponding symmetry group is simply $S_F^+ = SO_3$, and then to conclude.

As another application of our extended formalism, the Cayley theorem for the finite quantum groups, which fails in the S_N^+ setting, due to some subtle reasons, as explained in [23], holds in the S_F^+ setting. We have indeed the following result:

Theorem 4.9. *Any finite quantum group G has a Cayley embedding, as follows:*

$$G \subset S_G^+$$

However, there are finite quantum groups which are not quantum permutation groups.

Proof. There are two statements here, the idea being as follows:

(1) We have an action $G \curvearrowright G$, which leaves invariant the Haar measure. Now since the counting measure is left and right invariant, so is the Haar measure, we conclude that $G \curvearrowright G$ leaves invariant the counting measure, and so we have $G \subset S_G^+$, as claimed.

(2) Regarding the second assertion, this is something non-trivial, from [23], the simplest counterexample being a certain quantum group G appearing as a split abelian extension associated to the exact factorization $S_4 = \mathbb{Z}_4 S_3$, and having cardinality $|G| = 24$. □

Getting back now to the quantum groups S_F^+ themselves, and to Theorem 4.8 above, it is quite hard to go beyond this result, with results truly matching the known theory of S_N^+. Some simplifications, however, appear is the "homogeneous" case:

Definition 4.10. *We call homogeneous the finite quantum spaces of the following type:*

$$F = M_K \times \{1 \ldots, L\}$$

That is, the algebra $B = C(F)$ must be a finite dimensional random matrix algebra:

$$B = M_K(\mathbb{C}^L)$$

The corresponding quantum permutation groups S_F^+ are called homogeneous too.

Observe that the above spaces generalize both the spaces $X = \{1, \ldots, K\}$ and $X = M_L$, where most of the known theory lies. The "random matrix" terminology comes from the fact that the random matrix algebras, in general, are the von Neumann algebras of type $B = M_K(L^\infty(X))$, with X being a measured space, and for such an algebra to be finite dimensional, we must have $X = \{1, \ldots, L\}$. Thus, we are led to the above definition, up to changing the given measure $X = \{1, \ldots, L\}$ into the counting measure.

As a first result regarding such spaces, which is well-known, we have:

Theorem 4.11. *The symmetry group of $F = M_K \times \{1 \ldots, L\}$ is given by*

$$G(F) = PU_K \wr S_L$$

with on the right a wreath product, equal by definition to $PU_K^L \rtimes S_L$.

Proof. The fact that we have an inclusion $PU_K \wr S_L \subset G(F)$ is standard, and this follows as well by taking the classical version of the inclusion $PU_K^+ \wr_* S_L^+ \subset G^+(F)$, established below. As for the fact that this inclusion $PU_K \wr S_L \subset G(F)$ is an isomorphism, this can be proved by picking an arbitrary element $g \in G^+(F)$, and decomposing it. □

In order to discuss the quantum analogue of the above result, we will need a notion of free wreath product. The basic theory here, coming from [50], is as follows:

Proposition 4.12. *Given closed subgroups $G \subset U_N^+$, $H \subset S_k^+$, with fundamental corepresentations u, v, the following construction produces a closed subgroup of U_{Nk}^+:*

$$C(G \wr_* H) = (C(G)^{*k} * C(H)) / < [u_{ij}^{(a)}, v_{ab}] = 0 >$$

In the case where G, H are classical, the classical version of $G \wr_* H$ is the usual wreath product $G \wr H$. Also, when G is a quantum permutation group, so is $G \wr_* H$.

Proof. Consider the matrix $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$, over the quotient algebra in the statement. It is routine to check that w is unitary, and in the case $G \subset S_N^+$, our claim is that this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row sums are as follows:

$$\sum_{jb} w_{ia,jb} = \sum_{jb} u_{ij}^{(a)} v_{ab} = \sum_b v_{ab} \sum_j u_{ij}^{(a)} = 1$$

As for the column sums, these are as follows:

$$\sum_{ia} w_{ia,jb} = \sum_{ia} u_{ij}^{(a)} v_{ab} = \sum_a v_{ab} \sum_i u_{ij}^{(a)} = 1$$

With these observations in hand, it is routine to check that $G \wr_* H$ is indeed a quantum group, with fundamental corepresentation w, by constructing maps Δ, ε, S as in section 1, and in the case $G \subset S_N^+$, we obtain in this way a closed subgroup of S_{Nk}^+. Finally, the assertion regarding the classical version is standard as well. See [50]. □

We refer to [13], [50], [132] for more details regarding the above construction. With this notion in hand, we can now formulate the following result:

Theorem 4.13. *The quantum symmetry group of $F = M_K \times \{1 \ldots, L\}$ satisfies*

$$PU_K^+ \wr_* S_L^+ \subset G^+(F)$$

but this inclusion is not an isomorphism at $K, L \geq 2$.

Proof. We have two assertions to be proved, the idea being as follows:

(1) The fact that we have $PU_K^+ \wr_* S_L^+ \subset G^+(F)$ is well-known and routine, by checking the fact that the matrix $w_{ija,klb} = u_{ij,kl}^{(a)} v_{ab}$ is a generalized magic unitary.

(2) The inclusion $PU_K^+ \wr_* S_L^+ \subset G^+(F)$ is not an isomorphism, for instance by using [132], along with the fact that $\pi_1 \boxtimes \pi_1 \neq \pi_1$ where π_1 is the Marchenko-Pastur distribution. □

Let us focus now on the case $N=4$. According to our previous philosophical considerations, the link between S_4^+ and SO_3 comes as follows:

$$\{1,2,3,4\}\sim M_2\implies S_4^+\sim SO_3$$

It is possible to get beyond this, with a very precise result, stating that S_4^+ is a twist of SO_3. Let us start with the following definition, from [15]:

Definition 4.14. *$C(SO_3^{-1})$ is the universal C^*-algebra generated by the entries of a 3×3 orthogonal matrix $a=(a_{ij})$, with the following relations:*

1. *Skew-commutation: $a_{ij}a_{kl}=\pm a_{kl}a_{ij}$, with sign $+$ if $i\neq k,j\neq l$, and $-$ otherwise.*
2. *Twisted determinant condition: $\Sigma_{\sigma\in S_3}a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}=1$.*

Normally, our first task would be to prove that $C(SO_3^{-1})$ is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following result, from [15]:

Theorem 4.15. *We have an isomorphism of compact quantum groups*

$$S_4^+=SO_3^{-1}$$

given by the Fourier transform over the Klein group $K=\mathbb Z_2\times\mathbb Z_2$.

Proof. Consider indeed the following matrix, corresponding to the standard vector space action of SO_3^{-1} on $\mathbb C^4$:

$$a^+=diag(1,a)$$

We apply to this matrix the Fourier transform over the Klein group $K=\mathbb Z_2\times\mathbb Z_2$:

$$u=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over K converts the relations in Definition 4.14 into the magic relations. Thus, we obtain the identification from the statement. □

We have the following classification result, also from [15]:

Theorem 4.16. *The closed subgroups of $S_4^+=SO_3^{-1}$ are as follows:*

1. *Infinite quantum groups: S_4^+, O_2^{-1}, $\widehat{D}_\infty$.*
2. *Finite groups: S_4, and its subgroups.*
3. *Finite group twists: S_4^{-1}, A_5^{-1}.*
4. *Series of twists: D_{2n}^{-1} $(n\geq 3)$, DC_{2n}^{-1} $(n\geq 2)$.*
5. *A group dual series: $\widehat{D}_n$, with $n\geq 3$.*

Moreover, these quantum groups are subject to an ADE classification result, with the graphs coming from the representation theory of the quantum groups.

Proof. The idea here is that the classification can be obtained by taking some inspiration from the McKay classification of the subgroups of SO_3, by twising everything using the cocycle twisting method. As for the last assertion, the idea here is that the moments of the main character count the loops based at 1 on the graph. See [15]. $\square$

An interesting extension of the $S_4^+ = SO_3^{-1}$ result comes by looking at the general case $N = n^2$, with $n \in \mathbb{N}$. We will prove that we have a twisting result, as follows:

$$PO_n^+ = (S_N^+)^\sigma$$

In order to explain this material, from [22], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

Proposition 4.17. *Given a finite group F, the algebra $C(S_{\widehat{F}}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g, h \in F$, with the following relations:*

$$x_{1g} = x_{g1} = \delta_{1g}$$

$$x_{s,gh} = \sum_{t\in F} x_{st^{-1},g} x_{th}$$

$$x_{gh,s} = \sum_{t\in F} x_{gt^{-1}} x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh}) = \sum_{s\in F} x_{gs} \otimes x_{sh}$$

$$\varepsilon(x_{gh}) = \delta_{gh}$$

$$S(x_{gh}) = x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. This follows indeed from a direct verification, based either on Theorem 4.4 above, or on its equivalent formulation from Wang's paper [139]. $\square$

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on F, which is by definition a map $\sigma : F \times F \to \mathbb{C}^*$ satisfying:

$$\sigma_{gh,s}\sigma_{gh} = \sigma_{g,hs}\sigma_{hs}$$

$$\sigma_{g1} = \sigma_{1g} = 1$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{F}_\sigma)$, as being the vector space $C(\widehat{F}) = C^*(F)$, with product as follows:

$$e_g e_h = \sigma_{gh} e_{gh}$$

We have then the following generalization of Proposition 4.17:

Proposition 4.18. *The algebra $C(S_{\widehat{F}_\sigma}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h\in G$, with the relations $x_{1g}=x_{g1}=\delta_{1g}$ and:*

$$\sigma_{gh}x_{s,gh}=\sum_{t\in F}\sigma_{st^{-1},t}x_{st^{-1},g}x_{th}$$

$$\sigma_{gh}^{-1}x_{gh,s}=\sum_{t\in F}\sigma_{t^{-1},ts}^{-1}x_{gt^{-1}}x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh})=\sum_{s\in F}x_{gs}\otimes x_{sh}$$

$$\varepsilon(x_{gh})=\delta_{gh}$$

$$S(x_{gh})=\sigma_{h^{-1}h}\sigma_{g^{-1}g}^{-1}x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain $\sigma_{gg^{-1}}=\sigma_{g^{-1}g}$, needed in the proof. □

In what follows, we will prove that the quantum groups $S_{\widehat{F}}^+$ and $S_{\widehat{F}_\sigma}^+$ are related by a cocycle twisting operation. Let H be a Hopf algebra. We recall that a left 2-cocycle is a convolution invertible linear map $\sigma:H\otimes H\to\mathbb C$ satisfying:

$$\sigma_{x_1y_1}\sigma_{x_2y_2,z}=\sigma_{y_1z_1}\sigma_{x,y_2z_2}$$

$$\sigma_{x1}=\sigma_{1x}=\varepsilon(x)$$

Note that σ is a left 2-cocycle if and only if σ^{-1}, the convolution inverse of σ, is a right 2-cocycle, in the sense that we have:

$$\sigma_{x_1y_1,z}^{-1}\sigma_{x_1y_2}^{-1}=\sigma_{x,y_1z_1}^{-1}\sigma_{y_2z_2}^{-1}$$

$$\sigma_{x1}^{-1}=\sigma_{1x}^{-1}=\varepsilon(x)$$

Given a left 2-cocycle σ on H, one can form the 2-cocycle twist H^σ as follows. As a coalgebra, $H^\sigma=H$, and an element $x\in H$, when considered in H^σ, is denoted $[x]$. The product in H^σ is defined, in Sweedler notation, by:

$$[x][y]=\sum\sigma_{x_1y_1}\sigma_{x_3y_3}^{-1}[x_2y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism $H\to H^\sigma$ given by $x\to[x]$ commutes with the respective Haar integrals, as soon as H has a Haar integral.

We can now state and prove a main theorem from [22], as follows:

Theorem 4.19. *If F is a finite group and σ is a 2-cocycle on F, the Hopf algebras*
$$C(S^+_{\widehat{F}})\quad,\quad C(S^+_{\widehat{F}_\sigma})$$
are 2-cocycle twists of each other, in the above sense.

Proof. In order to prove this result, we use the following Hopf algebra map:
$$\pi : C(S^+_{\widehat{F}}) \to C(\widehat{F})$$
$$x_{gh} \to \delta_{gh}e_g$$
Our 2-cocycle $\sigma : F \times F \to \mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:
$$\sigma : C(\widehat{F}) \otimes C(\widehat{F}) \to \mathbb{C}$$
Consider now the following composition:
$$\alpha = \sigma(\pi \otimes \pi) : C(S^+_{\widehat{F}}) \otimes C(S^+_{\widehat{F}}) \to C(\widehat{F}) \otimes C(\widehat{F}) \to \mathbb{C}$$
Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1}. Thus we can construct the twisted algebra $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, and inside this algebra we have the following computation:
$$\begin{aligned}[x_{gh}][x_{rs}] &= \alpha^{-1}(x_g, x_r)\alpha(x_h, x_s)[x_{gh}x_{rs}]\\ &= \sigma^{-1}_{gr}\sigma_{hs}[x_{gh}x_{rs}]\end{aligned}$$
By using this, we obtain the following formula:
$$\begin{aligned}\sum_{t\in F}\sigma_{st^{-1},t}[x_{st^{-1},g}][x_{th}] &= \sum_{t\in F}\sigma_{st^{-1},t}\sigma^{-1}_{st^{-1},t}\sigma_{gh}[x_{st^{-1},g}x_{th}]\\ &= \sigma_{gh}[x_{s,gh}]\end{aligned}$$
Similarly, we have the following formula:
$$\sum_{t\in F}\sigma^{-1}_{t^{-1},ts}[x_{g,t^{-1}}][x_{h,ts}] = \sigma^{-1}_{gh}[x_{gh,s}]$$
We deduce from this that there exists a Hopf algebra map, as follows:
$$\Phi : C(S^+_{\widehat{F}_\sigma}) \to C(S^+_{\widehat{F}})^{\alpha^{-1}}$$
$$x_{gh} \to [x_{g,h}]$$
This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. $\square$

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

Proposition 4.20. *If F is a finite group and σ is a 2-cocycle on F, then*

$$\Phi(x_{g_1h_1}\ldots x_{g_mh_m})=\Omega(g_1,\ldots,g_m)^{-1}\Omega(h_1,\ldots,h_m)x_{g_1h_1}\ldots x_{g_mh_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1,\ldots,g_m)=\prod_{k=1}^{m-1}\sigma_{g_1\ldots g_k,g_{k+1}}$$

is a coalgebra isomorphism $C(S^+_{\widehat{F}_\sigma})\to C(S^+_{\widehat{F}})$, commuting with the Haar integrals.

Proof. This is indeed just a technical reformulation of Theorem 4.19. □

Here is another useful result from [22], that we will need in what follows:

Theorem 4.21. *Let $X\subset F$ be such that $\sigma_{gh}=1$ for any $g,h\in X$, and consider the subalgebra*

$$B_X\subset C(S^+_{\widehat{F}_\sigma})$$

generated by the elements x_{gh}, with $g,h\in X$. Then we have an injective algebra map

$$\Phi_0:B_X\to C(S^+_{\widehat{F}})$$

given by $x_{g,h}\to x_{g,h}$.

Proof. With the notations in the proof of Theorem 4.19, we have the following equality in $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, for any $g_i,h_i,r_i,s_i\in X$:

$$[x_{g_1h_1}\ldots x_{g_ph_p}]\cdot[x_{r_1s_1}\ldots x_{r_qs_q}]=[x_{g_1h_1}\ldots x_{g_ph_p}x_{r_1s_1}\ldots x_{r_qs_q}]$$

Now Φ_0 can be defined to be the composition of $\Phi_{|B_X}$ with the linear isomorphism $C(S^+_{\widehat{F}})^{\alpha^{-1}}\to C(S^+_{\widehat{F}})$ given by $[x]\to x$, and is clearly an injective algebra map. □

Let us discuss now some concrete applications of the general results established above. Consider the group $F=\mathbb{Z}_n^2$, let $w=e^{2\pi i/n}$, and consider the following map:

$$\sigma:F\times F\to\mathbb{C}^*$$

$$\sigma_{(ij)(kl)}=w^{jk}$$

It is easy to see that σ is a bicharacter, and hence a 2-cocycle on F. Thus, we can apply our general twisting result, to this situation. In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i,j\in\mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. Following [22], we first have the following result:

Proposition 4.22. *The linear map given by*

$$\psi(e_{(i,j)})=\sum_{k=0}^{n-1}w^{ki}E_{k,k+j}$$

defines an isomorphism of algebras $\psi:C(\widehat{F}_\sigma)\simeq M_n(\mathbb{C})$.

Proof. Consider indeed the following linear map:

$$\psi'(E_{ij})=\frac{1}{n}\sum_{k=0}^{n-1}w^{-ik}e_{(k,j-i)}$$

It is routine to check that both ψ,ψ' are morphisms of algebras, and that these maps are inverse to each other. In particular, ψ is an isomorphism of algebras, as stated. □

Next in line, we have the following result:

Proposition 4.23. *The algebra map given by*

$$\varphi(u_{ij}u_{kl})=\frac{1}{n}\sum_{a,b=0}^{n-1}w^{ai-bj}x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi:C(S_{M_n}^+)\simeq C(S_{\widehat{F}_\sigma}^+)$.

Proof. Consider the universal coactions on the two algebras in the statement:

$$\begin{aligned}\alpha:M_n(\mathbb{C})&\to M_n(\mathbb{C})\otimes C(S_{M_n}^+)\\ \beta:C(\widehat{F}_\sigma)&\to C(\widehat{F}_\sigma)\otimes C(S_{\widehat{F}_\sigma}^+)\end{aligned}$$

In terms of the standard bases, these coactions are given by:

$$\begin{aligned}\alpha(E_{ij})&=\sum_{kl}E_{kl}\otimes u_{ki}u_{lj}\\ \beta(e_{(i,j)})&=\sum_{kl}e_{(k,l)}\otimes x_{(k,l),(i,j)}\end{aligned}$$

We use now the identification $C(\widehat{F}_\sigma)\simeq M_n(\mathbb{C})$ from Proposition 4.22. This identification produces a coaction map, as follows:

$$\gamma:M_n(\mathbb{C})\to M_n(\mathbb{C})\otimes C(S_{\widehat{F}_\sigma}^+)$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij})=\frac{1}{n}\sum_{ab}E_{ab}\otimes\sum_{kr}w^{ar-ik}x_{(r,b-a),(k,j-i)}$$

By comparing with the formula of α, we obtain the isomorphism in the statement. □

We will need one more result of this type, as follows:

Proposition 4.24. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)})=\frac{1}{n^2}\sum_{klrs}w^{ki+lj-ra-sb}p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho:C(S_{\widehat{F}}^+)\simeq C(S_F^+)$.

Proof. We have a Fourier transform isomorphism, as follows:

$$C(\widehat{F}) \simeq C(F)$$

Thus the algebras in the statement are indeed isomorphic. □

As a conclusion to all this, we have the following result, from [22]:

Theorem 4.25. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n}\sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO_n^+) \to C(S_{n^2}^+)$$

commuting with the Haar integrals.

Proof. We recall from Theorem 4.23 (2) that we have identifications as follows, where the projective version of (A,u) is the pair (PA,v), with $PA =< v_{ij} >$ and $v = u \otimes \bar{u}$:

$$PO_n^+ = PU_n^+ = S_{M_n}^+$$

With this in hand, the result follows from Theorem 4.18 and Proposition 4.20, by combining them with the various isomorphisms established above. □

Here is a useful version of the above result, that we will need later on:

Theorem 4.26. *The following two algebras are isomorphic, via $u_{ij}^2 \to X_{ij}$:*

(1) *The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.*
(2) *The algebra generated by $X_{ij} = \frac{1}{n}\sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

Proof. We have $\Theta(u_{ij}^2) = X_{ij}$, so it remains to prove that if B is the subalgebra of $C(S_{M_n}^+)$ generated by the variables u_{ij}^2, then $\Theta_{|B}$ is an algebra morphism. Let us set:

$$X = \{(i,0) | i \in \mathbb{Z}_n\} \subset \mathbb{Z}_n^2$$

Then X satisfies the assumption in Theorem 4.20, and $\varphi(B) \subset B_X$. Thus by Theorem 4.20, the map $\Theta_{|B} = \rho F_0 \varphi_{|B}$ is indeed an algebra morphism. □

We will be back to this in section 8 below, with some probabilistic consequences.

As an overall conclusion, the twisting formula $S_4^+ = SO_3^{-1}$ ultimately comes from something of type $X_4 \simeq M_2$, where $X_4 = \{1,2,3,4\}$ and $M_2 = Spec(M_2(\mathbb{C}))$, and at $N \geq 5$ there are some extensions of this, and notably when $N = n^2$ with $n \geq 3$.

Finally, let us go back to the small index classification results.

In order to obtain the classification at $N=5$, we will use the recent progress in subfactor theory [108], concerning the classification of the small index subfactors. For our purposes, the most convenient formulation of the result in [108] is as follows:

Theorem 4.27. *The principal graphs of the irreducible index 5 subfactors are:*

(1) *A_∞, and a non-extremal perturbation of $A_\infty^{(1)}$.*
(2) *The McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$.*
(3) *The twists of the McKay graphs of A_5, S_5.*

Proof. This is a heavy result, and we refer to [108] for the whole story. The above formulation is the one from [108], with the subgroup subfactors there replaced by fixed point subfactors [2], and with the cyclic groups denoted as usual by $\mathbb{Z}_N$. □

In the quantum permutation group setting, this result becomes:

Theorem 4.28. *The set of principal graphs of the transitive subgroups $G\subset S_5^+$ coincide with the set of principal graphs of the subgroups $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5, S_5^+$.*

Proof. We must take the list of graphs in Theorem 4.27, and exclude some of the graphs, on the grounds that the graph cannot be realized by a transitive subgroup $G\subset S_5^+$.

We have 3 cases here to be studied, as follows:

(1) The graph A_∞ corresponds to S_5^+ itself. As for the perturbation of $A_\infty^{(1)}$, this dissapears, because our notion of transitivity requires the subfactor extremality.

(2) For the McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$ there is nothing to be done, all these graphs being solutions to our problem.

(3) The possible twists of A_5, S_5, coming from the graphs in Theorem 4.27 (3) above, cannot contain S_5, because their cardinalities are smaller or equal than $|S_5|=120$. □

With a little more work, the above considerations give in principle the full list of transitive subgroups $G\subset S_5^+$. To be more precise, the only piece of work left is that of classifying the twists of A_5, S_5, appearing in (3) in the above proof.

As an interesting consequence of the above results, we have:

Theorem 4.29. *The following quantum group inclusions are both maximal, in the sense that there is no quantum group in between:*

$$S_4\subset S_4^+$$

$$S_5\subset S_5^+$$

In addition, the inclusion $S_N\subset S_N^+$ is maximal, at any $N\in\mathbb{N}$, when restricting the attention to the class of the easy quantum groups.

Proof. There are several statements here, the idea being as follows:

(1) The $N=4$ assertion follows from the ADE classification from Theorem 4.16.

(2) The $N=5$ assertion follows from Theorem 4.28, with the remark that S_5 being transitive, so must be any intermediate subgroup $S_5\subset G\subset S_5^+$.

(3) As for the last assertion, this is something elementary, obtained by doing some combinatorics, as explained in [45]. $\square$

The above results suggest the conjecture that $S_N\subset S_N^+$ should be maximal, at any $N\in\mathbb N$. For a discussion and some related results here, we refer to [21].

5. **Laws of characters**

In this second part of this book, this section and the next 3 ones, we discuss a number of analytic questions, for the most in relation with free probability. We will be mainly interested in S_N, S_N^+, but in view of the subtle relationship between S_N^+, O_N^+, we will include O_N, O_N^+ as well in our discussion. We will comment on S_F^+ extensions, too.

Let us begin with some character basics. We have the following result:

Theorem 5.1. *Given a Woronowicz algebra (A, u), with fundamental corepresentation $u \in M_N(A)$, the law of the main character*

$$\chi = \sum_{i=1}^N u_{ii}$$

with respect to the Haar integration has the following properties:

1. *The moments of χ are the numbers $M_k = \dim(Fix(u^{\otimes k}))$.*
2. *M_k counts as well the lenght p loops at 1, on the Cayley graph of A.*
3. *$law(\chi)$ is the Kesten measure of the associated discrete quantum group.*
4. *When $u \sim \bar{u}$ the law of χ is a usual measure, supported on $[-N, N]$.*
5. *The algebra A is amenable precisely when $N \in supp(law(Re(\chi)))$.*
6. *Any morphism $f : (A, u) \to (B, v)$ must increase the numbers M_k.*
7. *Such a morphism f is an isomorphism when $law(\chi_u) = law(\chi_v)$.*

Proof. All this is very standard, basically coming from the Peter-Weyl theory developed in [147], and explained in section 1 above, the idea being as follows:

(1) This comes from the Peter-Weyl type theory, which tells us the number of fixed points of $v = u^{\otimes k}$ can be recovered by integrating the character $\chi_v = \chi_u^k$.

(2) This is something true, and well-known, for $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, the proof is quite similar.

(3) This is actually the definition of the Kesten measure, in the case $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, this follows from (2).

(4) The equivalence $u \sim \bar{u}$ translates into $\chi_u = \chi_u^*$, and this gives the first assertion. As for the support claim, this follows from $uu^* = 1 \implies ||u_{ii}|| \leq 1$, for any i.

(5) This is the Kesten amenability criterion, which can be established as in the classical case, $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group.

(6) This is something elementary, which follows from (1) above, and from the fact that the morphisms of Woronowicz algebras increase the spaces of fixed points.

(7) This follows by using (6), and the Peter-Weyl type theory, the idea being that if f is not injective, then it must strictly increase one of the spaces $Fix(u^{\otimes k})$. $\square$

Summarizing, regardless of our precise motivations and philosophy, computing the law of $\chi = \sum_i u_{ii}$ is a central question, and the "main problem" to be solved.

In what regards the quantum rotation and permutation groups, that we are interested in here, we have already solved this problem for them, our result being as follows:

Theorem 5.2. *The main character laws for the basic quantum groups are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws* $p_1, g_1, \pi_1, \gamma_1$

$$\begin{array}{ccccccc} S_N^+ & \to & O_N^+ & & \pi_1 & - & \gamma_1 \\ \uparrow & & \uparrow & : & | & & | \\ S_N & \to & O_N & & p_1 & - & g_1 \end{array}$$

in the $N \to \infty$ *limit. Moreover, the convergence is stationary starting from* $N = 2$ *for* O_N^+*, starting from* $N = 4$ *for* S_N^+*, and is not stationary for* O_N, S_N*.*

Proof. This is something that we know from sections 1-4, as follows:

(1) For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k, l))$, the asymptotic moments of the main character are given by:

$$\begin{aligned} \lim_{N\to\infty} \int_{G_N} \chi^k &= \lim_{N\to\infty} \dim\left(Fix(u^{\otimes k})\right) \\ &= \lim_{N\to\infty} \dim\left(span\left(\xi_\pi \Big| \pi \in D(k)\right)\right) \\ &= |D(k)| \end{aligned}$$

(2) This result applies to our 4 quantum groups, which are all easy, the corresponding categories of partitions, and asymptotic moments of χ, being as follows:

$$\begin{array}{ccccccc} NC & \leftarrow & NC_2 & & C_k & - & C_{k/2} \\ \downarrow & & \downarrow & : & | & & | \\ P & \leftarrow & P_2 & & B_k & - & k!! \end{array}$$

But these numbers being the moments of $p_1, g_1, \pi_1, \gamma_1$, we obtain the result.

(3) Regarding the stationarity claims, these are more advanced, and come for O_N^+, S_N^+ from the results in section 3 above. As for the non-stationarity claims for O_N, S_N, these come either via direct computations, or from the Kesten amenability criterion. □

All this is very nice, but the lack of symmetry between the classical and quantum results, in what concerns the stationarity, remains an issue. As a piece of an answer here, standard free probability, based on partitions as above, shows that π_1, γ_1 are the free analogues of p_1, g_1. However, at a more advanced level, that of the Bercovici-Pata bijection [46], the correct statement is that the free convolution semigroups $\{\pi_t\}, \{\gamma_t\}$ are the free analogues of the convolution semigroups $\{p_t\}, \{g_t\}$.

Thus, in order to fix things, we need a parameter $t > 0$. The idea will be that of looking at truncated characters, with respect to a parameter $t \in (0,1]$:

Definition 5.3. *Associated to any Woronowicz algebra (A,u) are the variables*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

depending on a parameter $t \in (0,1]$, called truncations of the main character.

We will see that for our basic quantum groups, the asymptotic laws of these variables are respectively $p_t, g_t, \pi_t, \gamma_t$, and that the convergence at generic $t \in (0,1]$ is not stationary. Thus, we will have our fix for Theorem 5.2. Also, as a bonus, all this will get us into advanced representation theory and free probability, that we will explore later.

In order to understand what the variables χ_t are about, let us first investigate the symmetric group S_N. The result here, which is something very classical, is as follows:

Theorem 5.4. *Consider the symmetric group S_N, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices.*

1. *The main character $\chi \in C(S_N)$, defined as usual as $\chi = \sum_i u_{ii}$, counts the number of fixed points, $\chi(\sigma) = \#\{i|\sigma(i) = i\}$.*
2. *The probability for a permutation $\sigma \in S_N$ to be a derangement, meaning to have no fixed points at all, becomes, with $N \to \infty$, equal to $1/e$.*
3. *The law of the main character $\chi \in C(S_N)$ becomes with $N \to \infty$ a Poisson law of parameter 1, with respect to the counting measure.*
4. *In fact, the law of any truncated character $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ becomes with $N \to \infty$ a Poisson law of parameter t, with respect to the counting measure.*

Proof. This is something very classical, the proof being as follows:

(1) We have indeed the following computation:

$$\chi(\sigma) = \sum_i u_{ii}(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i\middle|\sigma(i) = i\right\}$$

(2) We use the inclusion-exclusion principle. Consider the following sets:

$$S_N^i = \left\{\sigma \in S_N\middle|\sigma(i) = i\right\}$$

The set of permutations having no fixed points is then $X_N=(\bigcup_iS_N^i)^c$. In order to compute $|X_N|$, consider as well the following sets:

$$S_N^{i_1\ldots i_k}=\left\{\sigma\in S_N\Big|\sigma(i_1)=i_1,\ldots,\sigma(i_k)=i_k\right\}$$

The inclusion-exclusion principle tells us that we have:

$$\begin{aligned}&\left|\left(\bigcup_iS_N^i\right)^c\right|\\
=\ &|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^i\cap S_N^j|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1}\cup\ldots\cup S_N^{i_N}|\\
=\ &|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^{ij}|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1\ldots i_N}|\end{aligned}$$

Thus, the probability that we are interested in is given by:

$$\mathbb P(\chi=0)=\frac{1}{N!}\left(|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^{ij}|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1\ldots i_N}|\right)$$

Now observe that for any $i_1<\ldots<i_k$ we have $|S_N^{i_1\ldots i_k}|=(N-k)!$. We obtain:

$$\begin{aligned}\mathbb P(\chi=0)&=\frac{1}{N!}\sum_{k=0}^N(-1)^k\sum_{i_1<\ldots<i_k}|S_N^{i_1\ldots i_k}|\\
&=\frac{1}{N!}\sum_{k=0}^N(-1)^k\sum_{i_1<\ldots<i_k}(N-k)!\\
&=\frac{1}{N!}\sum_{k=0}^N(-1)^k\binom{N}{k}(N-k)!\\
&=\sum_{k=0}^N\frac{(-1)^k}{k!}\\
&=1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^{N-1}\frac{1}{(N-1)!}+(-1)^N\frac{1}{N!}\end{aligned}$$

Since we have here the expansion of $\frac{1}{e}$, we conclude that we have, as desired:

$$\lim_{N\to\infty}\mathbb P(\chi=0)=\frac{1}{e}$$

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N\to\infty}\mathbb{P}(\chi=k)=\frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law of parameter 1, as stated.

(4) As a first observation, and in analogy with the formula in (1) above, the truncated characters count as well certain fixed points, as follows:

$$\chi(\sigma)=\sum_{i=1}^{[tN]}u_{ii}(\sigma)=\sum_{i=1}^{[tN]}\delta_{\sigma(i)i}=\#\left\{i\in\{1,\ldots,[tN]\}\Big|\sigma(i)=i\right\}$$

Regarding now the computation of the law of χ_t, this follows by generalizing the computation in (3). Indeed, an application of the inclusion-exclusion principle gives:

$$\lim_{N\to\infty}\mathbb{P}(\chi_t=k)=\frac{t^k}{k!e^t}$$

Thus, we obtain in the limit a Poisson law of parameter t, as stated. $\square$

The above result will be something quite fundamental for us, and is worth a second proof, with the remark that in what concerns the case $t=1$, that we already discussed in Theorem 5.2, using easiness, this will be actually a third proof of it. We can use indeed the following integration formula over S_N, which has its own interest:

Theorem 5.5. *Consider the symmetric group S_N, with its standard coordinates:*

$$g_{ij}=\chi\left(\sigma\in S_N\Big|\sigma(j)=i\right)$$

The integrals over S_N are given, modulo linearity, by the formula

$$\int_{S_N}g_{i_1j_1}\ldots g_{i_kj_k}=\begin{cases}\frac{(N-|\ker i|)!}{N!}&\text{if }\ker i=\ker j\\0&\text{otherwise}\end{cases}$$

where $\ker i$ is the partition of $\{1,\ldots,k\}$ whose blocks collect the equal indices of i.

Proof. According to the definition of g_{ij}, the integrals in the statement are given by:

$$\int_{S_N}g_{i_1j_1}\ldots g_{i_kj_k}=\frac{1}{N!}\#\left\{\sigma\in S_N\Big|\sigma(j_1)=i_1,\ldots,\sigma(j_k)=i_k\right\}$$

Now observe that the existence of $\sigma\in S_N$ as above requires:

$$i_m=i_n\iff j_m=j_n$$

Thus, the above integral vanishes when $\ker i\neq\ker j$. Regarding now the case $\ker i=\ker j$, if we denote by $b\in\{1,\ldots,k\}$ the number of blocks of this partition $\ker i=\ker j$, we have $N-b$ points to be sent bijectively to $N-b$ points, and so $(N-b)!$ solutions, and the integral in the statement follows to be $\frac{(N-b)!}{N!}$, as claimed. $\square$

As an illustration for the above formula, we can recover the computation of the asymptotic laws of the truncated characters χ_t. We have indeed:

Theorem 5.6. *For the symmetric group $S_N \subset O_N$, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices, the truncated character*

$$\chi_t(g) = \sum_{i=1}^{[tN]} g_{ii}$$

counts the number of fixed points among $\{1, \ldots, [tN]\}$, and its law with respect to the counting measure becomes, with $N \to \infty$, a Poisson law of parameter t.

Proof. The first assertion is something that we already know. For the second assertion, we use the formula in Theorem 5.5. With S_{kb} being the Stirling numbers, we have:

$$\begin{aligned}
\int_{S_N} \chi_t^k &= \sum_{i_1 \ldots i_k=1}^{[tN]} \int_{S_N} g_{i_1 i_1} \ldots g_{i_k i_k} \\
&= \sum_{\pi \in P(k)} \frac{[tN]!}{([tN] - |\pi|!)} \cdot \frac{(N - |\pi|!)}{N!} \\
&= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N-b)!}{N!} \cdot S_{kb}
\end{aligned}$$

In particular with $N \to \infty$ we obtain the following formula:

$$\lim_{N \to \infty} \int_{S_N} \chi_t^k = \sum_{b=1}^{k} S_{kb} t^b$$

But this is the k-th moment of the Poisson law p_t, and so we are done. □

Summarizing, the truncated characters for S_N are definitely interesting objects. However, in what regards O_N, S_N^+, O_N^+, things are quite tricky, and we need a good motivation, coming on top of what we know about S_N, for getting into computations here.

For this purpose, recall from our comments preceding Definition 5.3 that the need for a parameter $t > 0$ basically comes from theoretical probability, and more precisely from the classical/free bijection there, at the semigroup level. So, let us explain this now.

In order to get started, recall that the Gaussian laws g_t and Poisson laws p_t appear via the Central Limit Theorem (CLT) and the Poisson Limit Theorem (PLT). Our first task will be that of explaining these results. The first of them is as follows:

Theorem 5.7 (CLT)**.** *Given a sequence of real random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim g_t$$

where g_t is the Gaussian law of parameter t, given by:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

Proof. We use the well-known fact that the log of the Fourier transform $F_f(x) = \mathbb{E}(e^{ixf})$ linearizes the convolution. The Fourier transform of the variable in the statement is:

$$\begin{aligned} F(x) &= \left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n \\ &= \left[1 - \frac{tx^2}{2n} + O(n^{-2})\right]^n \\ &\simeq e^{-tx^2/2} \end{aligned}$$

On the other hand, the Fourier transform of g_t is given by:

$$\begin{aligned} F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2/2t + ixy} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(y/\sqrt{2t} - \sqrt{t/2}ix)^2 - tx^2/2} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2 - tx^2/2} dz \\ &= e^{-tx^2/2} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Poisson Limit Theorem (PLT), this is as follows:

Theorem 5.8 (PLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\,\delta_1\right)^{*n} \to p_t$$

where p_t is the Poisson law of parameter $t > 0$, given by:

$$p_t = e^{-t} \sum_k \frac{t^k \delta_k}{k!}$$

Proof. Once again, we use the fact the log of the Fourier transform $F_f(z) = \mathbb{E}(e^{izf})$ linearizes the convolution. The Fourier transform of the variable in the statement is:

$$\begin{aligned} F(x) &= \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n \\ &= \lim_{n\to\infty}\left(1+\frac{(e^{ix}-1)t}{n}\right)^n \\ &= \exp\left((e^{ix}-1)t\right) \end{aligned}$$

On the other hand, the Fourier transform of p_t is given by:

$$\begin{aligned} F_{p_t}(x) &= e^{-t}\sum_k \frac{t^k}{k!}\,e^{ikx} \\ &= e^{-t}\sum_k \frac{(e^{ix}t)^k}{k!} \\ &= \exp(-t)\exp(e^{ix}t) \\ &= \exp\left((e^{ix}-1)t\right) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In order to discuss now the free version of the above results, we first need to talk about moments, laws and freeness. Let us start with the following definition:

Definition 5.9. *Let A be a C^*-algebra, given with a trace tr.*

1. *The elements $a \in A$ are called random variables.*
2. *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
3. *The law of such a variable is the functional $\mu_a : P \to tr(P(a))$.*

Here $k = \circ\bullet\bullet\circ\ldots$ is as usual a colored integer, and the powers a^k are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial P, this is a noncommuting $*$-polynomial in one variable:

$$P \in \mathbb{C} < X, X^* >$$

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

Let us discuss now the independence, and its noncommutative versions. As a starting point here, we have the following straightforward definition:

Definition 5.10. *We call two subalgebras $B, C \subset A$ independent when the following condition is satisfied, for any $x \in B$ and $y \in C$:*

$$tr(x) = tr(y) = 0 \implies tr(xy) = 0$$

Also, two variables $b, c \in A$ are called independent when the algebras that they generate

$$B = < b >, C = < c >$$

are independent, in the above sense.

It is possible to develop some theory here, but this is ultimately not very interesting, being just an abstract generalization of usual probability theory. As a much more interesting notion now, coming from [135], and that we will study next, we have:

Definition 5.11. *We call two subalgebras $B, C \subset A$ free when the following condition is satisfied, for any $x_i \in B$ and $y_i \in C$:*

$$tr(x_i) = tr(y_i) = 0 \implies tr(x_1 y_1 x_2 y_2 \ldots) = 0$$

Also, two variables $b, c \in A$ are called free when the algebras that they generate

$$B = < b >, C = < c >$$

are free, in the above sense.

Thus, freeness appears by definition as a kind of "free analogue" of independence. As a basic result now regarding these notions, and providing us with examples, we have:

Proposition 5.12. *We have the following results, valid for group algebras:*

(1) *$C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.*
(2) *$C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.*

Proof. In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, and this is in turn trivial. □

In short, we have now a notion of freeness, dealing with noncommutativity itself, in its most pure form, where there are no algebraic relations at all. This is very nice, but in practice now, we need a free analogue of the Fourier transform, or rather of the log of the Fourier transform. The result here, due to Voiculescu [135], is as follows:

Theorem 5.13. *Given a real probability measure μ, consider its Cauchy transform*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$$

and then define its R-transform as the solution of the following equation:

$$G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

The free convolution operation is then linearized by the R-transform.

Proof. The proof here, which is quite tricky, is in four steps, as follows:

(1) In order to model the free convolution, we can use the algebra of creation operators on the free Fock space $F(\mathbb{R}^2)$. This is the same as the semigroup algebra $C^*(\mathbb{N} * \mathbb{N})$, and we have some freeness here, a bit in the same way as for group algebras.

(2) In what concerns single distributions, the point here is that the variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and with $f \in \mathbb{C}[X]$ being a polynomial, are easily seen to model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$.

(3) Now let $f, g \in \mathbb{C}[X]$ and consider the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^* + (f + g)(S)$.

(4) Thus the operation $\mu \to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, which is standard, and whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as in the statement. □

We refer to [135] or [138] for full details on the above. Now with the above technology in hand, we are ready to state and prove the free CLT, once again following [135]:

Theorem 5.14 (FCLT). *Given noncommutative self-adjoint variables $x_1, x_2, x_3, \ldots \in A$ which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner semicircle law of parameter t, given by:

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t^2 - x^2}dx$$

Proof. We follow the same idea as in the proof of the CLT, explained before:

(1) At $t = 1$, the R-transform of the variable in the statement can be computed by using the linearization property from Theorem 5.13, and is given by:

$$R(\xi) = nR_x\left(\frac{\xi}{\sqrt{n}}\right) \simeq \xi$$

(2) On the other hand, standard computations show that the Cauchy transform of the Wigner semicircle law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi + \frac{1}{\xi}\right) = \xi$$

Thus we have the following formula, which by the way follows as well from $S + S^* \sim \gamma_1$, which is clear from the proof of Theorem 5.13 above:

$$R_{\gamma_1}(\xi) = \xi$$

(3) But this gives the result, and so we are done with the case $t = 1$. The passage to the general case, $t > 0$, is routine, by dilation. □

We can state and prove as well a free PLT, as follows:

Theorem 5.15 (FPLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}\to\pi_t$$

the limiting measure being the Marchenko-Pastur law of parameter $t > 0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t > 0$.

Proof. Consider the meaure in the statement, under the convolution sign:

$$\mu=\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1$$

The Cauchy transform of this measure is given by:

$$G_\mu(\xi)=\left(1-\frac{t}{n}\right)\frac{1}{\xi}+\frac{t}{n}\cdot\frac{1}{\xi-1}$$

We want to compute the following R-transform:

$$R=R_{\mu^{\boxplus n}}(y)=nR_\mu(y)$$

By Theorem 5.13, the equation satisfied by R is as follows:

$$\left(1-\frac{t}{n}\right)\frac{1}{y^{-1}+R/n}+\frac{t}{n}\cdot\frac{1}{y^{-1}+R/n-1}=y$$

By multiplying by n/y, and rearranging terms, this equation can be written as:

$$\frac{t+yR}{1+yR/n}=\frac{t}{1+yR/n-y}$$

With $n\to\infty$ the equation simplifies, and we obtain the following formula:

$$t+yR=\frac{t}{1-y}$$

Thus we have $R=\frac{t}{1-y}$, which equals R_{π_t}, and we obtain the result. □

As a conclusion to this, let us formulate the following statement:

Theorem 5.16. *The main limiting results in classical and free probability are*

$$\begin{matrix} \mathrm{FPLT} & - & \mathrm{FCLT} & & \pi_t & - & \gamma_t \\ | & & | & \to & | & & | \\ \mathrm{PLT} & - & \mathrm{CLT} & & p_t & - & g_t \end{matrix}$$

the limiting measures being Gaussian, Poisson, Wigner and Marchenko-Pastur.

Proof. This follows indeed by putting together all the above results, classical and free, and with $g_t, p_t, \gamma_t, \pi_t$ being respectively the measures in the statement. ☐

Now back to our permutation and rotation questions, the above result makes a clear connection with our quantum group scheme, from Theorem 5.2 above, namely:

$$\begin{matrix} S_N^+ & \to & O_N^+ & & \pi_1 & - & \gamma_1 \\ \uparrow & & \uparrow & : & | & & | \\ S_N & \to & O_N & & p_1 & - & g_1 \end{matrix}$$

In order to get beyond this, and reach to the parameter $t > 0$, we must do some further probability. Following [130], given a noncommutative random variable a, we can define its classical cumulants $k_n(a)$ and its free cumulants $\kappa_n(a)$ by the following formulae:

$$\log F_a(\xi) = \sum_n k_n(a)\xi^n$$

$$R_a(\xi) = \sum_n \kappa_n(a)\xi^n$$

With this notion in hand, we can define then more general quantities $k_\pi(a), \kappa_\pi(a)$, depending on partitions $\pi \in P(k)$, by multiplicativity over blocks, and we have:

Theorem 5.17. *We have the classical and free moment-cumulant formulae*

$$M_k(a) = \sum_{\pi \in P(k)} k_\pi(a)$$

$$M_k(a) = \sum_{\pi \in NC(k)} \kappa_\pi(a)$$

where $k_\pi(a), \kappa_\pi(a)$ are the generalized cumulants and free cumulants of a.

Proof. This is standard, by using the formulae of F_a, R_a, or by doing some direct combinatorics, based on the Möbius inversion formula from section 2 above. See [130]. ☐

Now with this classical and free cumulant technology in hand, we can reformulate Theorem 5.16 above in a more conceptual way, as follows:

Theorem 5.18. *The main central limiting results are as follows, with $g_t, p_t, \gamma_t, \pi_t$ being the Gaussian and Poisson semigroups, and Wigner and Marchenko-Pastur free semigroups,*

$$\begin{array}{ccc} \mathrm{FPLT} & - & \mathrm{FCLT} \\ | & & | \\ \mathrm{PLT} & - & \mathrm{CLT} \end{array} \quad \to \quad \begin{array}{ccc} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{array}$$

which are related by the Bercovici-Pata bijection, in the sense that "the classical cumulants of the classical measures are equal to the free cumulants of the free measures".

Proof. We already know the main assertion, from Theorem 5.16, so we just have to discuss the assertions regarding the semigroup properties, cumulants and the Bercovici-Pata bijection [46]. For this purpose, recall that at $t=1$ the moments of the limiting measures in the statement appear by counting diagrams, according to the following scheme:

$$\begin{array}{ccc} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{array} \quad : \quad \begin{array}{ccc} NC & \leftarrow & NC_2 \\ \downarrow & & \downarrow \\ P & \leftarrow & P_2 \end{array}$$

The point now is that at $t>0$ the moments of the measures in the statement can be recaptured as well from the above diagrams, accroding to the following formula:

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

Now by putting this into the classical and free cumulant machinery from Theorem 5.17, we obtain the conclusions in the statement, in relation with [46]. See [122]. □

Summarizing, we have now a good understanding of the limiting character measures appearing from O_N, O_N^+, S_N, S_N^+, and with the remark that the presence of a parameter $t>0$ would be desirable. But we already have our parameter $t>0$ in the quantum group setting, coming from truncated characters, introduced in Definition 5.3 above:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

So, following [26], [27], let us discuss now the computation of the law of χ_t. In general, and in particular in what regards O_N, S_N^+, O_N^+, there is no simple trick as for S_N, and we must use general integration methods, from [72], [144]. First, we have:

Theorem 5.19. *Assuming that $A = C(G)$ has Tannakian category $C = (C(k,l))$, the Haar integration over G is given by the Weingarten type formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in D_k}\delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

for any colored integer $k = e_1 \ldots e_k$ and indices i, j, where D_k is a linear basis of $C(\emptyset, k)$,

$$\delta_\pi(i)=<\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>$$

and $W_k = G_k^{-1}$, with $G_k(\pi,\sigma) =< \pi,\sigma >$.

Proof. We know from section 1 above that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k})=span(D_k)$$

Consider now the following linear map, with $D_k = \{\xi_k\}$ being as in the statement:

$$E(x)=\sum_{\pi\in D_k}<x,\xi_\pi>\xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $span(T_\pi|\pi \in D_k)$ of the restriction of E. But this restriction is the linear map given by G_k, and so W is the linear map given by W_k, and this gives the result. □

In the easy quantum group case, the above formula simplifies, as follows:

Theorem 5.20. *For an easy quantum group $G \subset O_N^+$, coming from a category of partitions $D = (D(k,l))$, we have the Weingarten integration formula*

$$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$, with

$$G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$$

where $|.|$ is the number of blocks.

Proof. With notations from Theorem 5.19, the Kronecker symbols are given by:

$$\delta_{\xi_\pi}(i)=<\xi_\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>=\delta_\pi(i_1,\ldots,i_k)$$

The Gram matrix being as well the correct one, we obtain the result. See [26]. □

With the above formula in hand, we can go back now to the question of computing the laws of truncated characters. First, we have the following moment formula, from [26]:

Theorem 5.21. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11}+\ldots+u_{ss})^k = Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN}=G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

Proof. We have indeed the following computation:

$$\begin{aligned}
\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\
&= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\
&= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\
&= Tr(W_{kN}G_{ks})
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to process now the above formula, things are quite technical, and won't work well in general. We must impose here a uniformity condition, as follows:

Theorem 5.22. *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, the following conditions are equivalent:*

1. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the embedding $U_{N-1}^+\subset U_N^+$ given by $u\to diag(u,1)$.*
2. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+\subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

If these conditions are satisfied, we say that $G=(G_N)$ is "uniform".

Proof. We use the general easiness theory from section 1 above.

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N\subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given $K\subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N\times N$ matrix $v=diag(u,1)$. Our claim is that for any $\pi\in P(k)$ we have:

$$\xi_\pi\in Fix(v^{\otimes k})\iff \xi_{\pi'}\in Fix(v^{\otimes k'}),\ \forall\pi'\in P(k'),\pi'\subset\pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}
& \xi_\pi \in Fix(v^{\otimes k}) \\
\iff & (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k} = (\xi_\pi)_{i_1\ldots i_k}, \forall i \\
\iff & \sum_j (v^{\otimes k})_{i_1\ldots i_k, j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k} = (\xi_\pi)_{i_1\ldots i_k}, \forall i \\
\iff & \sum_j \delta_\pi(j_1,\ldots,j_k) v_{i_1j_1}\ldots v_{i_kj_k} = \delta_\pi(i_1,\ldots,i_k), \forall i
\end{aligned}$$

Now let us recall that our corepresentation has the special form $v = diag(u,1)$. We conclude from this that for any index $a \in \{1,\ldots,k\}$, we must have:

$$i_a = N \implies j_a = N$$

With this observation in hand, if we denote by i', j' the multi-indices obtained from i, j obtained by erasing all the above $i_a = j_a = N$ values, and by $k' \leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'} \delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'} = \delta_\pi(i_1,\ldots,i_k), \forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

Case 1. Assume that the index set $\{a|i_a = N\}$ corresponds to a certain subpartition $\pi' \subset \pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'} \delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'} = \delta_\pi(i'_1,\ldots,i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a|i_a = N\}$ does not correspond to a subpartition $\pi' \subset \pi$. In this case the indices mix, and our formula reads:

$$0 = 0$$

Thus, we are led to $\xi_{\pi'} \in Fix(v^{\otimes k'})$, for any subpartition $\pi' \subset \pi$, as claimed. Now with this claim in hand, the result follows from Tannakian duality. $\square$

By getting back now to the truncated characters, we have the following result:

Theorem 5.23. *For a uniform easy quantum group $G = (G_N)$, we have the formula*

$$\lim_{N\to\infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with $D \subset P$ being the associated category of partitions.

Proof. We use the general moment formula from Theorem 5.21 above. With $s = [tN]$, this formula becomes:

$$\int_{G_N} \chi_t^k = Tr(W_{kN} G_{k[tN]})$$

The point now is that in the uniform case the Gram and Weingarten matrices are asymptotically diagonal, and this gives the result. See [26], [33], [43]. □

We can now improve our quantum group results, as follows:

Theorem 5.24. *The main truncated character laws for the basic quantum groups are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws $p_t, g_t, \pi_t, \gamma_t$*

$$\begin{array}{ccc} S_N^+ & \rightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \rightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{array}$$

in the $N \to \infty$ limit. Also, the convergences are not stationary at generic $t \in (0,1]$.

Proof. This follows indeed from the easiness property of our quantym groups, and from Theorem 5.23, which produces the moments in Theorem 5.18. As for the last assertion, this is something valid at any $t \in (0,1)$, which follows from standard computations. □

As an umbrella result now, summarizing all our knowledge, we have:

Theorem 5.25. *The asymptotic truncated character laws for the basic quantum permutation and rotation groups, which are all easy, as follows,*

$$\begin{array}{ccc} S_N^+ & \rightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \rightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} NC & \leftarrow & NC_2 \\ \downarrow & & \downarrow \\ P & \leftarrow & P_2 \end{array}$$

are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws $p_t, g_t, \pi_t, \gamma_t$, which appear from the main limiting laws in classical and free probability,

$$\begin{array}{ccc} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{array} \quad : \quad \begin{array}{ccc} \text{FPLT} & - & \text{FCLT} \\ | & & | \\ \text{PLT} & - & \text{CLT} \end{array}$$

and which form semigroups related by the Bercovici-Pata bijection, "the classical cumulants of the classical measures are equal to the free cumulants of the free measures".

Proof. This follows from Theorem 5.18 and Theorem 5.24, and from the various results leading to them. In fact, the present result summarizes our probabilistic knowledge in the $N \to \infty$ limit, with the only things left being the technical stationarity results for S_N^+, O_N^+, which from the present $N \to \infty$ perspective look rather "accidental". □

There are many ways of further extending the above results, and for a basic computation here, in the spirit of [86], we refer to [33]. Also, we will be back to this later, in sections 9-12 below, when doing reflection groups, with the result that the various square diagrams in Theorem 5.25 can be suitably modified, and then completed into cubes.

6. Partial permutations

We discuss in this section an extension of some of the results that we have seen so far, both of algebraic and analytic nature, from the case of the basic quantum permutation and rotation groups, to their "partial semigroup" analogues:

$$\begin{matrix}S_N^+&\to&O_N^+\\ \uparrow&&\uparrow\\ S_N&\to&O_N\end{matrix}\quad\to\quad\begin{matrix}\widetilde{S}_N^+&\to&\widetilde{O}_N^+\\ \uparrow&&\uparrow\\ \widetilde{S}_N&\to&\widetilde{O}_N\end{matrix}$$

Let us start with the semigroup $\widetilde{S}_N$ of partial permutations. This is a quite familiar object in combinatorics, defined as follows:

Definition 6.1. *$\widetilde{S}_N$ is the semigroup of partial permutations of $\{1\ldots,N\}$,*

$$\widetilde{S}_N=\left\{\sigma:X\simeq Y\Big|X,Y\subset\{1,\ldots,N\}\right\}$$

with the usual composition operation, $\sigma'\sigma:\sigma^{-1}(X'\cap Y)\to\sigma'(X'\cap Y)$.

Observe that $\widetilde{S}_N$ is not simplifiable, because the null permutation $\emptyset\in\widetilde{S}_N$, having the empty set as domain/range, satisfies $\emptyset\sigma=\sigma\emptyset=\emptyset$, for any $\sigma\in\widetilde{S}_N$. Observe also that $\widetilde{S}_N$ has a "subinverse" map, sending $\sigma:X\to Y$ to its usual inverse $\sigma^{-1}:Y\simeq X$.

A first interesting result about this semigroup $\widetilde{S}_N$, which shows that we are dealing here with some non-trivial combinatorics, is as follows:

Proposition 6.2. *The number of partial permutations is given by*

$$|\widetilde{S}_N|=\sum_{k=0}^N k!\binom{N}{k}^2$$

that is, $1,2,7,34,209,\ldots$, and with $N\to\infty$ we have:

$$|\widetilde{S}_N|\simeq N!\sqrt{\frac{\exp(4\sqrt{N}-1)}{4\pi\sqrt{N}}}$$

Proof. The first assertion is clear, because in order to construct a partial permutation $\sigma:X\to Y$ we must choose an integer $k=|X|=|Y|$, then we must pick two subsets $X,Y\subset\{1,\ldots,N\}$ having cardinality k, and there are $\binom{N}{k}$ choices for each, and finally we must construct a bijection $\sigma:X\to Y$, and there are $k!$ choices here. As for the estimate, which is non-trivial, this is however something standard, and well-known. $\square$

Another result, which is trivial, but quite fundamental, is as follows:

Proposition 6.3. *We have a semigroup embedding $u:\widetilde{S}_N\subset M_N(0,1)$, defined by*

$$u_{ij}(\sigma)=\begin{cases}1&\text{if }\sigma(j)=i\\ 0&\text{otherwise}\end{cases}$$

whose image are the matrices having at most one nonzero entry, on each row and column.

Proof. This is trivial from definitions, with $u:\widetilde{S}_N\subset M_N(0,1)$ extending the standard embedding $u:S_N\subset M_N(0,1)$, that we have been heavily using, so far. □

Finally, a third basic result about $\widetilde{S}_N$ is as follows:

Proposition 6.4. *We have an embedding $\widetilde{S}_N\subset S_{2N}$, mapping $\sigma:X\simeq Y$ to*

$$\sigma'(i)=\begin{cases}\sigma(i)&if\ i\in X\\ N+r&if\ i=x_r\\ y_r&if\ i=N+r\\ i&if\ i>N+L\end{cases}$$

where $X^c=\{x_1,\ldots,x_L\}$ and $Y^c=\{y_1,\ldots,y_L\}$, with $x_1<\ldots<x_L$ and $y_1<\ldots<y_L$.

Proof. This is something a bit more technical, which is clear from definitions, too. □

Let us discuss now some probabilistic aspects, related to the Poisson computations in section 5. We denote by $\kappa:\widetilde{S}_N\to\mathbb N$ the cardinality of the domain/range, and by $\chi:\widetilde{S}_N\to\mathbb N$ be the number of fixed points. These variables are given by:

$$\kappa=\sum_{ij}u_{ij}\quad,\quad\chi=\sum_iu_{ii}$$

These two quantities are in fact of similar nature:

Proposition 6.5. *The embedding $\widetilde{S}_N\subset S_{2N}$ in Proposition 6.4 makes correspond the variables $\chi,\kappa:\widetilde{S}_N\to\mathbb N$ to the variables*

$$\chi_{left},\chi_{right}:S_{2N}\to\mathbb N$$

counting the fixed points among $\{1,\ldots,N\}$ and among $\{N+1,\ldots,2N\}$, respectively.

Proof. By using the formula of $\sigma\to\sigma'$ from Proposition 6.4, we obtain:

$$\begin{aligned}\chi_{left}(\sigma')&=\#\{i\leq N|\sigma'(i)=i\}\\ &=\{i\in X|\sigma(i)=i\}\\ &=\chi(\sigma)\end{aligned}$$

We have as well the following formula:

$$\begin{aligned}\chi_{right}(\sigma') &= \#\{i > N|\sigma'(i) = i\}\\ &= \{i > N + L\}\\ &= N - 2\\ &= \kappa(\sigma)\end{aligned}$$

Thus we have obtained the formulae in the statement, and we are done. □

More generally, given a number $l \leq N$, we denote by $\chi_l : \widetilde{S}_N \to \mathbb{N}$ the number of fixed points among $\{1, \ldots, l\}$. Observe that $\widetilde{S}_N \subset S_{2N}$ maps in fact $\chi_l \to \chi_l$, for any l. Generally speaking, we are interested in the joint law of (χ_l, κ). There are many interesting questions here, and as a main result on this subject, we have:

Theorem 6.6. *The measures*

$$\mu_k^l = law\left(\chi_l\Big|\kappa = k\right)$$

are given by the formula

$$\mu_k^l = \sum_{q\geq 0}\binom{k}{q}\binom{l}{q}\binom{N}{q}^{-2}\frac{(\delta_1 - \delta_0)^{*q}}{q!}$$

and become Poisson (st) in the $k = sN, l = tN, N \to \infty$ limit.

Proof. Observe first that at $k = l = N$ this corresponds to the well-known fact that the number of fixed points $\chi : S_N \to \mathbb{N}$ becomes Poisson (1), in the $N \to \infty$ limit. More generally, at $k = N$ this corresponds to the fact that the truncated character $\chi_l : S_N \to \mathbb{N}$ becomes Poisson (t), in the $l = tN \to \infty$ limit. In general, we can use the same method, namely the inclusion-exclusion principle. Let us set indeed:

$$\widetilde{S}_N^{(k)} = \left\{\sigma \in \widetilde{S}_N\Big|\kappa(\sigma) = k\right\}$$

By inclusion-exclusion, we obtain the following formula:

$$\begin{aligned}&P\left(\chi_l = p\Big|\kappa = k\right)\\ = \;&\frac{1}{|\widetilde{S}_N^{(k)}|}\binom{l}{p}\#\left\{\sigma \in \widetilde{S}_{N-p}^{(k-p)}\Big|\sigma(i) \neq i, \forall i \leq l - p\right\}\\ = \;&\frac{1}{|\widetilde{S}_N^{(k)}|}\binom{l}{p}\sum_{r\geq 0}(-1)^r\binom{l-p}{r}\left|\widetilde{S}_{N-p-r}^{(k-p-r)}\right|\end{aligned}$$

Here the index r, which counts the fixed points among $\{1, \ldots, l - p\}$, runs a priori up to $\min(k, l) - p$. However, since the binomial coefficient or the cardinality of the set on the right vanishes by definition at $r > \min(k, l) - p$, we can sum over $r \geq 0$.

We have the following formula:

$$|\widetilde{S}_N^{(k)}|=k!\binom{N}{k}^2$$

By using this and then by cancelling various factorials, and grouping back into binomial coeffiecients, we obtain the following formula:

$$\begin{aligned}
&P\left(\chi_l=p\Big|\kappa=k\right)\\
=&\frac{1}{k!\binom{N}{k}^2}\binom{l}{p}\sum_{r\geq0}(-1)^r\binom{l-p}{r}(k-p-r)!\binom{N-p-r}{k-p-r}^2\\
=&\sum_{r\geq0}\frac{(-1)^r}{p!r!}\binom{k}{p+r}\binom{l}{p+r}\binom{N}{p+r}^{-2}
\end{aligned}$$

We can now compute the measure itself. With $p=q-r$, we obtain:

$$\begin{aligned}
&law\left(\chi_l\Big|\kappa=k\right)\\
=&\sum_{p\geq0}\sum_{r\geq0}\frac{(-1)^r}{p!r!}\binom{k}{p+r}\binom{l}{p+r}\binom{N}{p+r}^{-2}\delta_p\\
=&\sum_{q\geq0}\sum_{r\geq0}\frac{(-1)^r}{(q-r)!r!}\binom{k}{q}\binom{l}{q}\binom{N}{q}^{-2}\delta_{q-r}\\
=&\sum_{q\geq0}\binom{k}{q}\binom{l}{q}\binom{N}{q}^{-2}\cdot\frac{1}{q!}\sum_{r\geq0}(-1)^r\binom{q}{r}\delta_{q-r}
\end{aligned}$$

The sum at right being $(\delta_1-\delta_0)^{*q}$, this gives the formula in the statement.

Regarding now the asymptotics, in the regime $k=sN,l=tN,N\to\infty$ from the statement, the coefficient of $(\delta_1-\delta_0)^{*q}/q!$ in the formula of μ_k^l is:

$$\begin{aligned}
c_q&=\binom{k}{q}\binom{l}{q}\binom{N}{q}^{-2}\\
&=\frac{\binom{k}{q}}{\binom{N}{q}}\cdot\frac{\binom{l}{q}}{\binom{N}{q}}\\
&\simeq\left(\frac{k}{N}\right)^q\left(\frac{l}{N}\right)^q\\
&=(st)^q
\end{aligned}$$

We deduce that the Fourier transform of μ_k^l is given by:

$$\begin{aligned}\mathcal{F}(\mu_k^l)(y) &\simeq \sum_{q\geq 0}(st)^q\frac{(e^y-1)^q}{q!}\\ &= e^{st(e^y-1)}\end{aligned}$$

But this is the Fourier transform of Poisson (st), and we are done. □

Observe that the formula in Theorem 6.6 shows that we have $\mu_k^l = \mu_l^k$. This is an interesting equality, which seems to be quite unobvious to prove, with bare hands.

Let us discuss now the construction and main properties of the quantum semigroup of quantum partial permutations $\widetilde{S}_N^+$, in analogy with the above. For this purpose, let us go back to the embedding $u:\widetilde{S}_N\subset M_N(0,1)$ in Proposition 6.3.

Due to the formula $u_{ij}(\sigma)=\delta_{i\sigma(j)}$, the matrix $u=(u_{ij})$ is "submagic", in the sense that its entries are projections, which are pairwise orthogonal on each row and column. This suggests the following definition, given in [42]:

Definition 6.7. *$C(\widetilde{S}_N^+)$ is the universal C^*-algebra generated by the entries of a $N\times N$ submagic matrix u, with comultiplication and counit maps given by*

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$$

$$\varepsilon(u_{ij})=\delta_{ij}$$

where submagic means formed of projections, which are pairwise orthogonal on rows and columns. We call $\widetilde{S}_N^+$ semigroup of quantum partial permutations of $\{1,\ldots,N\}$.

Here the fact that the morphisms of algebras Δ,ε as above exist indeed follows from the universality property of $C(\widetilde{S}_N^+)$, with the needed submagic checks being nearly identical to the magic checks for $C(S_N^+)$, from section 2 above.

Observe that the morphisms Δ,ε satisfy the usual axioms for a comultiplication and antipode, namely:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

Thus, we have a bialgebra structure of $C(\widetilde{S}_N^+)$, which tells us that the underlying noncommutative space $\widetilde{S}_N^+$ is a compact quantum semigroup. This semigroup is of quite special type, because $C(\widetilde{S}_N^+)$ has as well a subantipode map, defined by:

$$S(u_{ij})=u_{ji}$$

To be more precise here, this map exists because the transpose of a submagic matrix is submagic too. As for the subantipode axiom satisfied by it, this is as follows, where $m^{(3)}$ is the triple multiplication, and $\Delta^{(2)}$ is the double comultiplication:

$$m^{(3)}(S\otimes id\otimes S)\Delta^{(2)}=S$$

Observe also that Δ,ε,S restrict to $C(\widetilde{S}_N)$, and correspond there, via Gelfand duality, to the usual multiplication, unit element, and subinversion map of $\widetilde{S}_N$.

As a conclusion to this discussion, the basic properties of the quantum semigroup $\widetilde{S}_N^+$ that we constructed in Definition 6.7 can be summarized as follows:

Proposition 6.8. *We have maps as follows*

$$\begin{matrix}C(\widetilde{S}_N^+)&\to&C(S_N^+)\\ \\ \downarrow&&\downarrow\\ \\ C(\widetilde{S}_N)&\to&C(S_N)\end{matrix}\qquad : \qquad\begin{matrix}\widetilde{S}_N^+&\supset&S_N^+\\ \\ \cup&&\cup\\ \\ \widetilde{S}_N&\supset&S_N\end{matrix}$$

with the bialgebras at left corresponding to the quantum semigroups at right.

Proof. This is clear from the above discussion, and from the well-known fact that projections which sum up to 1 are pairwise orthogonal. See [42]. □

As a first example, we have $\widetilde{S}_1^+=\widetilde{S}_1$. At $N=2$ now, recall that the algebra generated by two free projections p,q is isomorphic to the group algebra of $D_\infty=\mathbb{Z}_2*\mathbb{Z}_2$. We denote by $\varepsilon:C^*(D_\infty)\to\mathbb{C}1$ the counit map, given by the following formulae:

$$\varepsilon(1)=1$$

$$\varepsilon(\ldots pqpq\ldots)=0$$

With these conventions, we have the following result, from [42]:

Proposition 6.9. *We have an isomorphism*

$$C(\widetilde{S}_2^+)\simeq\left\{(x,y)\in C^*(D_\infty)\oplus C^*(D_\infty)\Big|\varepsilon(x)=\varepsilon(y)\right\}$$

which is given by the formula

$$u=\begin{pmatrix}p\oplus 0&0\oplus r\\0\oplus s&q\oplus 0\end{pmatrix}$$

where p,q and r,s are the standard generators of the two copies of $C^*(D_\infty)$.

Proof. Consider an arbitrary 2×2 matrix formed by projections:

$$u=\begin{pmatrix}P&R\\S&Q\end{pmatrix}$$

This matrix is submagic when the following conditions are satisfied:

$$PR = PS = QR = QS = 0$$

But these conditions mean that the non-unital algebras $X = < P, Q >$ and $Y = < R, S >$ must commute, and must satisfy $xy = 0$, for any $x \in X, y \in Y$.

Thus, if we denote by Z the universal non-unital algebra generated by two projections, we have an isomorphism as follows:

$$C(\widetilde{S}_2^+) \simeq \mathbb{C}1 \oplus Z \oplus Z$$

Now since $C^*(D_\infty) = \mathbb{C}1 \oplus Z$, we obtain an isomorphism as follows:

$$C(\widetilde{S}_2^+) \simeq \left\{ (\lambda + a, \lambda + b) \Big| \lambda \in \mathbb{C}, a, b \in Z \right\}$$

Thus, we are led to the conclusion in the statement. See [42]. $\square$

Let us extend now to our free setting the classical results. Proposition 6.2 has no free analogue, because $\widetilde{S}_N^+$ is infinite. Proposition 6.3 was already extended, as being part of Definition 6.7. Regarding now Proposition 6.4 and Proposition 6.5, we first have:

Proposition 6.10. *The following two elements of $C(\widetilde{S}_N^+)$ are self-adjoint,*

$$\chi = \sum_i u_{ii}, \qquad \kappa = \sum_{ij} u_{ij}$$

satisfy $0 \leq \chi, \kappa \leq N$, and coincide with the usual χ, κ on the quotient $C(\widetilde{S}_N)$.

Proof. All the assertions are clear from definitions, with the inequalities $0 \leq \chi, \kappa \leq N$ being taken of course in an operator-theoretic sense. $\square$

With this observation in hand, if we denote by $v = (v_{ij})$ the magic unitary for S_{2N}, the formulae in Proposition 6.5 tell us that the surjection $C(S_{2N}) \to C(\widetilde{S}_N)$ maps:

$$\begin{aligned} v_{11} + \ldots + v_{NN} &\to \chi \\ v_{N+1,N+1} + \ldots + v_{2N,2N} &\to \kappa \end{aligned}$$

Let us look now at Theorem 6.6. Since $C(\widetilde{S}_N^+)$ has no integration functional, we cannot talk about the joint law of (χ, κ). Thus, we need an alternative approach to μ_k^l.

For this purpose, we use the following simple observation:

Proposition 6.11. *Any partial permutation $\sigma : X \simeq Y$ can be factorized as*

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, "\gamma"'] & Y \\
\{1,\ldots,k\} \arrow[r, "\beta"'] & \{1,\ldots,k\} \arrow[u, "\alpha"']
\end{tikzcd}$$

with $\alpha, \beta, \gamma \in S_k$ being certain non-unique permutations, where $k = \kappa(\sigma)$.

Proof. Since we have $|X| = |Y| = k$, we can choose any two bijections $X \simeq \{1,\ldots,k\}$ and $\{1,\ldots,k\} \simeq Y$, and then complete them up to permutations $\gamma, \alpha \in S_N$. The remaining permutation $\beta \in S_k$ is then uniquely determined by the formula $\sigma = \alpha\beta\gamma$. $\square$

We can now formulate an alternative definition for the measures μ_k^l. We fix $k \leq N$, and we denote by p, q, r the magic matrices for S_N, S_k, S_N. We have:

Proposition 6.12. *Consider the map $\varphi : S_N \times S_k \times S_N \to \widetilde{S}_N$, sending (α, β, γ) to the partial permutation $\sigma : \gamma^{-1}\{1,\ldots,k\} \simeq \alpha\{1,\ldots,k\}$ given by $\sigma(\gamma^{-1}(t)) = \alpha(\beta(t))$.*

1. *The image of φ is the set $\widetilde{S}_N^{(k)} = \{\sigma \in \widetilde{S}_N | \kappa(\sigma) = k\}$.*
2. *The transpose of φ is given by $\varphi^*(u_{ij}) = \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}$.*
3. *μ_k^l equals the law of the variable $\varphi^*(\chi_l) \in C(S_N \times S_k \times S_N)$.*

Proof. This is an elementary statement, whose proof goes as follows:

(1) Since $\alpha, \gamma \in S_N$, the domain and range of the associated element $\sigma \in \widetilde{S}_N$ have indeed cardinality k. The surjectivity follows from Proposition 6.11 above.

(2) For the element $\sigma \in \widetilde{S}_N$ in the statement, we have:

$$\begin{aligned}
u_{ij}(\sigma) = 1 \quad &\iff \quad \sigma(j) = i \\
&\iff \quad \exists t \leq k, \gamma^{-1}(t) = j, \alpha(\beta(t)) = i \\
&\iff \quad \exists s, t \leq k, \gamma^{-1}(t) = j, \beta(t) = s, \alpha(s) = i \\
&\iff \quad \exists s, t \leq k, r_{tj}(\gamma) = 1, q_{st}(\beta) = 1, p_{is}(\alpha) = 1 \\
&\iff \quad \exists s, t \leq k, (p_{is} \otimes q_{st} \otimes r_{tj})(\alpha, \beta, \gamma) = 1
\end{aligned}$$

Now since the numbers $s, t \leq k$ are uniquely determined by $\alpha, \beta, \gamma, i, j$, if they exist, we conclude that we have the following formula:

$$u_{ij}(\sigma) = \sum_{s,t \leq k} (p_{is} \otimes q_{st} \otimes r_{tj})(\alpha, \beta, \gamma)$$

But this gives the formula in the statement, and we are done.

(3) This comes from the fact that the map $\varphi_k : S_N \times S_k \times S_N \to \widetilde{S}_N^{(k)}$ obtained by restricting the target of φ commutes with the normalized (mass one) counting measures.

At $k=N$ this follows from the well-known fact that given $(\alpha,\beta,\gamma)\in S_N\times S_N\times S_N$ random, the product $\alpha\beta\gamma\in S_N$ is random, and the general case is clear as well. $\square$

The point now is that we can use the same trick, "$\sigma=\alpha\beta\gamma$", in the free case. The precise preliminary statement that we will need is as follows:

Proposition 6.13. *Let p,q,r be the magic matrices for S_N^+,S_k^+,S_N^+.*

1. *The matrix $U_{ij}=\sum_{s,t\leq k}p_{is}\otimes q_{st}\otimes r_{tj}$ is submagic.*
2. *We have a representation $\pi:C(\widetilde{S}_N^+)\to C(S_N^+\times S_k^+\times S_N^+)$, $\pi(u_{ij})=U_{ij}$.*
3. *π factorizes through the algebra $C(\widetilde{S}_N^{+(k)})=C(\widetilde{S}_N^+)/<\kappa=k>$.*
4. *At $k=N$, this factorization π_k commutes with the Haar functionals.*

Proof. Once again, this is an elementary statement, whose proof goes at follows:

(1) By using the fact that p,q,r are magic, we obtain:

$$\begin{aligned}
U_{ij}U_{il} &= \sum_{s,t\leq k}\sum_{v,w\leq k}p_{is}p_{iv}\otimes q_{st}q_{vw}\otimes r_{tj}r_{wl}\\
&= \sum_{s,t\leq k}\sum_{w\leq k}p_{is}\otimes q_{st}q_{sw}\otimes r_{tj}r_{wl}\\
&= \sum_{s,t\leq k}p_{is}\otimes q_{st}\otimes r_{tj}r_{tl}\\
&= \delta_{jl}U_{ij}
\end{aligned}$$

The proof of $U_{ij}U_{lj}=\delta_{il}U_{ij}$ is similar, and we conclude that U is submagic.

(2) This follows from (1), and from the definition of $C(\widetilde{S}_N^+)$.

(3) By using the fact that p,q,r are magic, we obtain indeed:

$$\sum_{ij}U_{ij}=\sum_{ij}\sum_{s,t\leq k}p_{is}\otimes q_{st}\otimes r_{tj}=\sum_{s,t\leq k}1\otimes q_{st}\otimes 1=k$$

Thus the representation π factorizes indeed through the algebra in the statement.

(4) This is a well-known analogue of the fact that "the product of random permutations is a random permutation", that we already used before. Here is a representation theory

proof, using Peter-Weyl theory. With $P=Proj(Fix(u^{\otimes n}))$, we have:

$$\begin{aligned}
&\int_{S_N^+\times S_N^+\times S_N^+}U_{i_1j_1}\ldots U_{i_nj_n}\\
=&\sum_{st}\int_{S_N^+}p_{i_1s_1}\ldots p_{i_ns_n}\int_{S_N^+}q_{s_1t_1}\ldots q_{s_nt_n}\int_{S_N^+}r_{t_1j_1}\ldots r_{t_nj_n}\\
=&\sum_{st}P_{i_1\ldots i_n,s_1\ldots s_n}P_{s_1\ldots s_n,t_1\ldots t_n}P_{t_1\ldots t_n,j_1\ldots j_n}\\
=&(P^3)_{i_1\ldots i_n,j_1\ldots j_n}\\
=&P_{i_1\ldots i_n,j_1\ldots j_n}\\
=&\int_{S_N^+}u_{i_1j_1}\ldots u_{i_nj_n}
\end{aligned}$$

Thus π_N commutes indeed with the Haar functionals, and we are done. □

Observe that, since κ is now continuous, $0\leq\kappa\leq N$, the algebras $C(\widetilde{S}_N^{+(k)})$ constructed above don't sum any longer up to the algebra $C(\widetilde{S}_N^+)$ itself. Thus, in a certain sense, the above measures μ_k^l encode only a part of the "probabilistic theory" of $\widetilde{S}_N^+$.

We can however formulate a free analogue of Theorem 6.6, as follows:

Theorem 6.14. *The measures $\mu_k^l=law(\pi_k(\chi_l))$, where π_k is defined as*

$$\pi_k:C(\widetilde{S}_N^+)\to C(S_N^+\times S_k^+\times S_N^+)$$

$$u_{ij}\to\sum_{s,t\leq k}p_{is}\otimes q_{st}\otimes r_{tj}$$

become free Poisson (st) in the $k=sN,l=tN,N\to\infty$ limit.

Proof. Observe first that at $k=l=N$ this corresponds to the fact that the law of the main character $\chi:S_N^+\to\mathbb{N}$ becomes free Poisson (1), in the $N\to\infty$ limit. Unlike in the classical case, the convergence here is stationary, starting from $N=4$.

More generally, at $k=N$ this corresponds to the fact that the truncated character $\chi_l:S_N^+\to\mathbb{N}$ becomes free Poisson (t), in the $l=tN\to\infty$ limit.

In general, we can use the same technique, namely the moment method, and the Weingarten formula. The variable that we are interested in, $\chi_k^l=\pi_k(\chi_l)$, is given by:

$$\chi_k^l=\sum_{i\leq l}\sum_{s,t\leq k}p_{is}\otimes q_{st}\otimes r_{ti}$$

By raising to the power n and integrating, we obtain the following formula:

$$\begin{aligned}&\int_{S_N^+\times S_k^+\times S_N^+}(\chi_k^l)^n\\ = &\sum_{i_a\leq l}\sum_{s_a,t_a\leq k}\int_{S_N^+}p_{i_1s_1}\ldots p_{i_ns_n}\int_{S_k^+}q_{s_1t_1}\ldots q_{s_nt_n}\int_{S_N^+}r_{t_1i_1}\ldots r_{t_ni_n}\end{aligned}$$

By using now the Weingarten formula, the above moment is:

$$\begin{aligned}c_n &= \sum_{ist}\sum_{\alpha\ldots\rho\in NC(n)}\delta_\alpha(i)\delta_\beta(s)W_{nN}(\alpha,\beta)\cdot\delta_\gamma(s)\delta_\delta(t)W_{nk}(\gamma,\delta)\cdot\delta_\varepsilon(t)\delta_\rho(i)W_{nN}(\varepsilon,\rho)\\
&= \sum_{\alpha\ldots\rho\in NC(n)}W_{nN}(\alpha,\beta)W_{nk}(\gamma,\delta)W_{nN}(\varepsilon,\rho)\sum_{ist}\delta_\alpha(i)\delta_\beta(s)\delta_\gamma(s)\delta_\delta(t)\delta_\varepsilon(t)\delta_\rho(i)\\
&= \sum_{\alpha\ldots\rho\in NC(n)}W_{nN}(\alpha,\beta)W_{nk}(\gamma,\delta)W_{nN}(\varepsilon,\rho)\sum_{ist}\delta_{\alpha\vee\rho}(i)\delta_{\beta\vee\gamma}(s)\delta_{\delta\vee\varepsilon}(t)\\
&= \sum_{\alpha\ldots\rho\in NC(n)}W_{nN}(\alpha,\beta)W_{nk}(\gamma,\delta)W_{nN}(\varepsilon,\rho)\cdot l^{|\alpha\vee\rho|}k^{|\beta\vee\gamma|}k^{|\delta\vee\varepsilon|}\end{aligned}$$

Let us examine now the asymptotic regime $k=sN,l=tN,N\to\infty$ in the statement. We use here two standard facts from [27], namely the fact that in the $N\to\infty$ limit the Gram and Weingarten matrices are concentrated on the diagonal, and the fact that we have $|\pi\vee\sigma|\leq\frac{|\pi|+|\sigma|}{2}$, with equality when $\pi=\sigma$. We obtain, as in [27]:

$$\begin{aligned}c_n &\simeq \sum_{\alpha,\gamma,\varepsilon\in NC(n)}N^{-|\alpha|}k^{-|\gamma|}N^{-|\varepsilon|}\cdot l^{|\alpha\vee\varepsilon|}k^{|\alpha\vee\gamma|}k^{|\gamma\vee\varepsilon|}\\
&\simeq \sum_{\alpha,\gamma,\varepsilon\in NC(n)}N^{-|\alpha|-|\gamma|-|\varepsilon|+|\alpha\vee\varepsilon|+|\alpha\vee\gamma|+|\gamma\vee\varepsilon|}\cdot s^{-|\gamma|+|\alpha\vee\gamma|+|\gamma\vee\varepsilon|}\cdot t^{|\alpha\vee\varepsilon|}\\
&\simeq \sum_{\alpha\in NC(n)}(st)^{|\alpha|}\end{aligned}$$

We recognize at right the well-known formula for the moments of the free Poisson law of parameter st, and this finishes the proof. □

As a conclusion, with Theorem 6.6 and Theorem 6.14 in hand, and by using the well-known fact that Poisson $(st)\to$ free Poisson (st) is indeed a liberation, in the sense of free probability [46], [138], we can now state that $\widetilde{S}_N\to\widetilde{S}_N^+$ is a "correct" liberation.

More generally now, we can include as well rotations and quantum rotations in our picture. Our starting point will be the following definition:

Definition 6.15. *$\widetilde{O}_N$ is the semigroup of partial linear isometries of $\mathbb{R}^N$,*

$$\widetilde{O}_N = \left\{ T : A \to B \text{ isometry} \Big| A, B \subset \mathbb{R}^N \right\}$$

with the usual composition operation for such maps, namely:

$$T'T : T^{-1}(A' \cap B) \to T'(A' \cap B)$$

As a first remark, $\widetilde{O}_N$ is indeed a semigroup, with respect to the operation in the statement, and this is best seen in the matrix model picture, as follows:

Proposition 6.16. *We have an embedding $\widetilde{O}_N \subset M_N(\mathbb{R})$, obtained by completing maps $T : A \to B$ into linear maps $U : \mathbb{R}^N \to \mathbb{R}^N$, by setting $U_{|A^\perp} = 0$. Moreover:*

1. *This embedding makes $\widetilde{O}_N$ correspond to the set of matrix-theoretic partial isometries, i.e. to the matrices $U \in M_N(\mathbb{R})$ satisfying $UU^tU = U$.*
2. *The semigroup operation on $\widetilde{O}_N$ corresponds in this way to the semigroup operation for matrix-theoretic partial isometries, $U \circ V = U(U^tU \wedge VV^t)V$.*

Proof. All the assertions are elementary. For $C = A, B$ let $I_C : C \subset \mathbb{R}^N$ be the inclusion, and $P_C : \mathbb{R}^N \to C$ be the projection. The correspondence $T \leftrightarrow U$ is then given by:

$$\begin{tikzcd} A \arrow[r, "T"] & B \arrow[d, "I_B"] \\ \mathbb{R}^N \arrow[u, "P_A"] \arrow[r, "U"'] & \mathbb{R}^N \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "T"] \arrow[d, "I_A"'] & B \\ \mathbb{R}^N \arrow[r, "U"'] & \mathbb{R}^N \arrow[u, "P_B"'] \end{tikzcd}$$

The fact that the composition $U \circ V$ is indeed a partial isometry comes from the fact that the projections U^tU and VV^t are absorbed when performing the product:

$$U(U^tU \wedge VV^t)V \cdot V^t(U^tU \wedge VV^t)U^t \cdot U(U^tU \wedge VV^t)V = U(U^tU \wedge VV^t)V$$

Thus, we are led to the conclusions in the statement. $\square$

Observe also that we have a set-theoretic embedding $\widetilde{O}_N \subset O_{2N}$, that can be obtained by suitably adapting the formula in Proposition 6.4 above.

In general, the multiplication formula $U \circ V = U(U^tU \wedge VV^t)V$ in Proposition 6.16 (2), while being quite complicated, is quite unavoidable.

In view of some future liberation purposes, we would need a functional analytic interpretation of it. We have here the following result:

Proposition 6.17. *$C(\widetilde{O}_N)$ is the universal commutative C^*-algebra generated by the entries of a $N\times N$ matrix $u=(u_{ij})$ satisfying the relations*

$$u=\bar{u}\quad,\quad uu^tu=u$$

with comultiplication given by the formula

$$(id\otimes\Delta)u=u_{12}(p_{13}\wedge q_{12})u_{13}=\lim_{n\to\infty}\underbrace{UU^t\ldots U^tU}_{2n+1\ \text{terms}}$$

where $p=uu^t,q=u^tu$ and $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$.

Proof. The presentation assertion is standard, by using the Gelfand and Stone-Weierstrass theorems. Let us find now the comultiplication map of $C(\widetilde{O}_N)$. We know that this is the map given by $\Delta(u_{ij})=\Phi^{-1}(L_{ij})$, where $\Phi:C(\widetilde{O}_N)\otimes C(\widetilde{O}_N)\to C(\widetilde{O}_N\times\widetilde{O}_N)$ is the canonical isomorphism, and where $L_{ij}(U,V)=(U\circ V)_{ij}$.

In order to write now this map L_{ij} in tensor product form, we can use the following well-known formula:

$$P\wedge Q=\lim_{n\to\infty}(PQ)^n$$

More precisely, with $P=VV^t$ and $Q=U^tU$, we obtain the following formula:

$$(U\circ V)_{ij}=\sum_{kl}U_{ik}(P\wedge Q)_{kl}V_{lj}=\lim_{n\to\infty}\sum_{kl}U_{kl}(PQ)^n_{kl}V_{lj}$$

With $a_0=k,a_{2n}=l$, and by expanding the product, we obtain:

$$\begin{aligned}(U\circ V)_{ij}&=\lim_{n\to\infty}\sum_{a_0\ldots a_{2n}}U_{ia_0}P_{a_0a_1}Q_{a_1a_2}\ldots P_{a_{2n-2}a_{2n-1}}Q_{a_{2n-1}a_{2n}}V_{a_{2n}j}\\&=\lim_{n\to\infty}\sum_{a_0\ldots a_{2n}}U_{ia_0}Q_{a_1a_2}\ldots Q_{a_{2n-1}a_{2n}}\cdot P_{a_0a_1}\ldots P_{a_{2n-2}a_{2n-1}}V_{a_{2n}j}\end{aligned}$$

Now by getting back to $\Delta(u_{ij})=\Phi^{-1}(L_{ij})$, with $L_{ij}(U,V)=(U\circ V)_{ij}$, we conclude that we have the following formula, with $p=uu^t$ and $q=u^tu$:

$$\Delta(u_{ij})=\lim_{n\to\infty}\sum_{a_0\ldots a_{2n}}u_{ia_0}q_{a_1a_2}\ldots q_{a_{2n-1}a_{2n}}\otimes p_{a_0a_1}\ldots p_{a_{2n-2}a_{2n-1}}u_{a_{2n}j}$$

Let us expand now both matrix products $p = uu^t$ and $q = u^tu$. In terms of the element $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ in the statement, the sum on the right, say $S_{ij}^{(n)}$, becomes:

$$\begin{aligned}
S_{ij}^{(n)} &= \sum_{a_s} u_{ia_0}(u^tu)_{a_1a_2}\ldots(u^tu)_{a_{2n-1}a_{2n}} \otimes (uu^t)_{a_0a_1}\ldots(uu^t)_{a_{2n-2}a_{2n-1}}u_{a_{2n}j}\\
&= \sum_{a_sb_sc_s} u_{ia_0}u_{b_1a_1}u_{b_1a_2}\ldots u_{b_na_{2n-1}}u_{b_na_{2n}} \otimes u_{a_0c_1}u_{a_1c_1}\ldots u_{a_{2n-2}c_n}u_{a_{2n-1}c_n}u_{a_{2n}j}\\
&= \sum_{b_sc_s} U_{ic_1}U_{b_1c_1}U_{b_1c_2}\ldots U_{b_nc_n}U_{b_nj}\\
&= (\underbrace{UU^t\ldots U^tU}_{2n+1\ \text{terms}})_{ij}
\end{aligned}$$

Thus we have obtained the second formula in the statement. Regarding now the first formula, observe that we have $U = u_{12}u_{13}$. This gives:

$$\begin{aligned}
\underbrace{UU^t\ldots U^tU}_{2n+1\ \text{terms}} &= (u_{12}u_{13})(u_{13}^tu_{12}^t)\ldots(u_{13}^tu_{12}^t)(u_{12}u_{13})\\
&= u_{12}(u_{13}u_{13}^t)(u_{12}^tu_{12})\ldots(u_{13}u_{13}^t)(u_{12}^tu_{12})u_{13}\\
&= u_{12}p_{13}q_{12}\ldots p_{13}q_{12}u_{13}
\end{aligned}$$

Now since the product on the right converges in the $n \to \infty$ limit to $u_{12}(p_{13} \wedge q_{12})u_{13}$, this gives the first formula in the statement as well, and we are done. $\square$

Observe that if we further assume that u is unitary, or that its entries satisfy the condition $u_{ij}^2 = p_{ij}$ (projection) with $p = (p_{ij})$ magic, then $UU^tU = U$, so the convergence in the formula of Δ is stationary, and we obtain $\Delta(u_{ij}) = U_{ij}$.

Thus, we can recover in this way the fact that both the inclusions $O_N \subset \widetilde{O}_N \subset M_N(\mathbb{R})$ are semigroup maps, with respect to the usual multiplication of the $N \times N$ matrices.

We will be back to this observation, with full details directly in the free case, in Proposition 6.20 below.

Let us construct now the liberations. We have here the following definition:

Definition 6.18. *To any $N \in \mathbb{N}$ we associate the following algebra,*

$$C(\widetilde{O}_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u_{ij} = u_{ij}^*, uu^t = p, \text{projection}\right)$$

and we call the underlying object $\widetilde{O}_N^+$ space of quantum partial isometries.

As a first observation, due to the presentation results in Proposition 6.17, we have an inclusions $\widetilde{O}_N \subset \widetilde{O}_N^+$. We have as well a liberated version of Proposition 6.16, or rather of the last assertion there, the rest being already known.

These functoriality statements are best summarized as follows:

Proposition 6.19. *We have embeddings of compact quantum spaces as follows,*

$$\begin{tikzcd} O_N^+ \arrow[r] & \widetilde{O}_N^+ \\ O_N \arrow[u] \arrow[r] & \widetilde{O}_N \arrow[u] \end{tikzcd}$$

and the compact quantum spaces on the right produce the compact quantum groups on the left by dividing, at the algebra level, by the relations

$$p = p' = q = q' = 1$$

where $q = u^*u$ and $q' = u^t\bar{u}$, as in Definition 6.18.

Proof. It follows from definitions that we have embeddings as in the statement. Regarding now the second assertion, in the case of $\widetilde{O}_N^+$, the relations $p = p' = q = q' = 1$ read:

$$uu^t = \bar{u}u^t = u^t u = u^t \bar{u} = 1$$

We deduce that both u, u^t are unitaries, and so when dividing by these relations we obtain the quantum group O_N^+. As for the result regarding the classical versions, this is clear too, by dividing by the commutation relations $ab = ba$. $\square$

Let us discuss now the multiplicative structure. We have here:

Proposition 6.20. $\widetilde{O}_N^+$ *has a non-associative multiplication given by*

$$(id \otimes \Delta)u = u_{12}(p_{13} \wedge q_{12})u_{13} = \lim_{n\to\infty} \underbrace{UU^t \ldots U^t U}_{2n+1 \text{ terms}}$$

where $p = uu^t, q = u^t u$ and $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$. The embeddings

$$\widetilde{O}_N, O_N^+ \subset \widetilde{O}_N^+$$

commute with the multiplications.

Proof. First of all, the equality between the two matrices on the right in the statement follows as in the proof of Proposition 6.17. Let us call $W = (W_{ij})$ this matrix.

In order to check that $\Delta(u_{ij}) = W_{ij}$ defines indeed a morphism, we must verify that $W = (W_{ij})$ satisfies the conditions in Definition 6.18. We have:

$$\begin{aligned} WW^tW &= u_{12}(p_{13} \wedge q_{12})u_{13} \cdot u_{13}^t(p_{13} \wedge q_{12})u_{12}^t \cdot u_{12}(p_{13} \wedge q_{12})u_{13} \\ &= u_{12}(p_{13} \wedge q_{12})p_{13}(p_{13} \wedge q_{12})p_{12}(p_{13} \wedge q_{12})u_{13} \\ &= u_{12}(p_{13} \wedge q_{12})u_{13} = W \end{aligned}$$

Regarding now the last assertion, for the inclusion $\widetilde{O}_N \subset \widetilde{O}_N^+$ this is clear. For the inclusion $O_N^+ \subset \widetilde{O}_N^+$ this is clear too, because with $p = q = 1$ we obtain $(id \otimes \Delta)u = U$, which is the usual comultiplication formula for $C(O_N^+)$. $\square$

Let us discuss now probabilistic aspects. We will see that, while our space $\widetilde{O}_N^+$ is not a semigroup, the Bercovici-Pata bijection criterion is satisfied for it.

We use the same method as for $\widetilde{S}_N^+$, namely a "$\sigma = \alpha\beta\gamma$" type trick. So, pick an exponent $\circ \in \{\emptyset, +\}$, set $\kappa = \sum_{ij} u_{ij} u_{ij}^t$, and consider the following algebra:

$$C(\widetilde{O}_N^{\circ(k)}) = C(\widetilde{O}_N^\circ)/ < \kappa = k >$$

With this convention, we have the following result:

Proposition 6.21. *For any $\circ \in \{\emptyset, +\}$ we have a representation*

$$\pi_k : C(\widetilde{O}_N^{\circ(k)}) \to C(O_N^\circ \times O_k^\circ \times O_N^\circ)$$

$$\pi_k(u_{ij}) = \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}$$

which commutes with the Haar functionals at $k = N$.

Proof. In the classical case, the first observation is that any partial isometry $T : A \to B$, with the linear spaces $A, B \subset \mathbb{R}^N$ having dimension $\dim(A) = \dim(B) = k$, decomposes as $T = UVW$, with $U, W \in O_N$ and $V \in O_k$:

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "W"'] & B \\ \mathbb{R}^k \arrow[r, "V"'] & \mathbb{R}^k \arrow[u, "U"'] \end{tikzcd}$$

We conclude that we have a surjection $\varphi : O_N \times O_k \times O_N \to \widetilde{O}_N^{(k)}$ mapping (U, V, W) to the partial isometry $T : W^{-1}(\mathbb{R}^k) \to U(\mathbb{R}^k)$ given by $T(W^{-1}x) = U(Vx)$. By proceeding now as in the proof of Proposition 6.12 (2) above, we see that the transpose map $\pi = \varphi^*$ is the representation in the statement, and we are done with the classical case.

In the free case, this is a straightforward extension of Proposition 6.13 above. Let us first check that the matrix $U = (U_{ij})$ formed by the elements appearing on the right in the statement satisfies the partial isometry condition. We have:

$$\begin{aligned}(UU^tU)_{ij} &= \sum_{kl} U_{ik}U_{lk}U_{lj}\\ &= \sum_{kl}\sum_{s,t\leq k}\sum_{v,w\leq k}\sum_{y,z\leq k} p_{is}p_{lv}p_{ly}\otimes q_{st}q_{vw}q_{yz}\otimes r_{tk}r_{wk}r_{zj}\\ &= \sum_{s,t\leq k}\sum_{y,z\leq k} p_{is}\otimes q_{st}q_{yt}q_{yz}\otimes r_{zj}\\ &= \sum_{s,z\leq k} p_{is}\otimes q_{sz}\otimes r_{zj}\\ &= U_{ij}\end{aligned}$$

Since we have as well that $u_{ij}=u_{ij}^*$ implies $U_{ij}=U_{ij}^*$, this proves the partial isometry condition.

Let us ckeck now that the representation that we have just constructed vanishes on the ideal $<\kappa=k>$. We have:

$$\begin{aligned}\sum_{ij} U_{ij}U_{ij}^t &= \sum_{ij}\sum_{s,t\leq k}\sum_{v,w\leq k} p_{is}p_{iv}\otimes q_{st}q_{vw}\otimes r_{tj}r_{wj}\\ &= \sum_{s,t\leq k} 1\otimes q_{st}q_{st}\otimes 1\\ &= k\end{aligned}$$

Thus we have a representation π_k as in the statement. Finally, the last assertion is already known, from the proof of Proposition 6.13 (3). $\square$

With the above result in hand, we can construct measures μ_k^l as in the discrete case, $\mu_k^l=law(\chi_k^l)$ with $\chi_k^l=\pi_k(\chi_l)$, and we have the following result:

Theorem 6.22. *$\widetilde{O}_N\to\widetilde{O}_N^+$ is a liberation, in the sense that we have the Bercovici-Pata bijection for μ_k^l, in the $k=sN,l=tN,N\to\infty$ limit.*

Proof. This follows by using standard integration technology, from [26], [43], [72]. More precisely, the Weingarten computation in the proof of Theorem 6.14 above gives the following formula, in the $k=sN$, $l=tN$, $N\to\infty$ limit, where $D(n)\subset P(n)$ denotes the set of partitions associated to the quantum group O_N° under consideration:

$$\lim_{N\to\infty}\int_{O_N^\circ\times O_k^\circ\times O_N^\circ}(\chi_k^l)^n=\sum_{\alpha\in D(n)}(st)^{|\alpha|}$$

On the other hand, we know from [26], [43], [72] that the law of the truncated character χ_l is given by the following formula, in the $l = tN$, $N \to \infty$ limit:

$$\lim_{N\to\infty}\int_{O_N^\circ}(\chi_l)^n = \sum_{\alpha\in D(n)} t^{|\alpha|}$$

We conclude that in the $k = sN$, $l = tN$, $N \to \infty$ limit, we have the following equality of distributions:

$$\lim_{N\to\infty}\mu_k^l = \lim_{N\to\infty}\mu_N^{sl}$$

Thus, we are led to the conclusion in the statement. □

7. De Finetti theorems

We discuss in this section probabilistic invariance questions with respect to the basic quantum permutation and rotation groups, namely:

$$\begin{tikzcd} S_N^+ \arrow[r] & O_N^+ \\ S_N \arrow[u] \arrow[r] & O_N \arrow[u] \end{tikzcd}$$

In general, given an easy quantum group $G=(G_N)$, there is a natural notion of G-invariance for a sequence of noncommutative random variables $(x_i)_{i\in\mathbb N}$, which agrees with the usual definition when G is a classical group.

Following the classical theory of the De Finetti theorem, then [112], then [76], [77], and then [34], we will discuss here De Finetti type theorems for the above quantum permutation and rotation groups that we are interested in.

Let us start by fixing some notations. We first have:

Definition 7.1. *Given an easy quantum group $G_N\subset O_N^+$, we consider the free complex algebra on N variables*

$$\mathcal P_N=\mathbb C<t_1,\ldots,t_N>$$

and we construct a coaction of $C(G_N)$ on it, as follows:

$$\alpha_N:\mathcal P_N\to\mathcal P_N\otimes C(G_N)$$

$$t_j\to\sum t_i\otimes u_{ij}$$

Observe that α_N is indeed a coaction, in the sense that we have:

$$(id\otimes\Delta)\alpha_N=(\alpha_N\otimes id)\alpha_N$$

$$(id\otimes\varepsilon)\alpha_N=id$$

With this notion in hand, we can talk about invariant sequences, as follows:

Definition 7.2. *Let $(x_1,\ldots,x_N)$ be a sequence of random variables in a noncommutative probability space (B,φ). We say that the sequence is G_N-invariant if the distribution functional $\varphi_x:\mathcal P_N\to\mathbb C$ is invariant under the coaction α_N,*

$$(\varphi_x\otimes id)\alpha_N(p)=\varphi_x(p)$$

for all $p\in\mathcal P_N$. More explicitly, the sequence $(x_1,\ldots,x_N)$ is G_N-invariant if

$$\varphi(x_{j_1}\ldots x_{j_k})=\sum_{i_1\ldots i_k}\varphi(x_{i_1}\ldots x_{i_k})u_{i_1j_1}\ldots u_{i_kj_k}$$

as an equality in $C(G_N)$, for any $k\in\mathbb N$ and any $1\leq j_1,\ldots,j_k\leq N$.

In the classical case we recover in this way the usual invariance notion from probability, as shown by the following result:

Proposition 7.3. *In the classical group case, where $G_N\subset O_N$, a sequence $(x_1,\ldots,x_N)$ is G_N-invariant in the above sense if and only if*

$$\varphi(x_{j_1}\ldots x_{j_k})=\sum_{i_1\ldots i_k}g_{i_1j_1}\ldots g_{i_kj_k}\varphi(x_{i_1}\ldots x_{i_k})$$

for each $k\in\mathbb N$, $1\leq j_1,\ldots,j_k\leq N$ and $g=(g_{ij})\in G_N$, and this coincides with the usual notion of G_N-invariance for a sequence of classical random variables.

Proof. This follows indeed by evaluating both sides of the equation in Definition 7.2 at a given group element $g\in G_N$. □

In the classical De Finetti theorem, the independence occurs after conditioning. Likewise the free De Finetti theorem is a statement about freeness with amalgamation.

Both these concepts may be expressed in terms of operator-valued probability theory, that we will recall now. First, we have the following definition:

Definition 7.4. *An operator-valued probability space consists of:*

1. *A unital algebra A.*
2. *A unital subalgebra $B\subset A$.*
3. *An expectation $E:A\to B$, satisfying $E(1)=1$ and $E(b_1ab_2)=b_1E(a)b_2$.*

Given such an operator-valued probability space, the joint distribution of a family of variables $(x_i)_{i\in I}$ in the algebra A is by definition the following functional:

$$E_x:B<(t_i)_{i\in I}>\to B$$
$$P\to E(P(x))$$

We refer to [130] and related papers for more on all this, general theory and examples. Next, we have the following key definition:

Definition 7.5. *Let $(x_i)_{i\in I}$ be a family of variables.*

1. *These variables are called independent if the algebra $<B,(x_i)_{i\in I}>$ is commutative, and if $i_1,\ldots,i_k\in I$ are distinct and $p_1,\ldots,p_k\in B<t>$ then:*
$$E(p_1(x_{i_1})\ldots p_k(x_{i_k}))=E(p_1(x_{i_1}))\ldots E(p_k(x_{i_k}))$$
2. *These variables are called free if for any $i_1,\ldots,i_k\in I$ such that $i_l\neq i_{l+1}$, and any $p_1,\ldots,p_k\in B<t>$ such that $E(p_l(x_{i_l}))=0$, we have:*
$$E(p_1(x_{i_1})\ldots p_k(x_{i_k}))=0$$

In order to deal with invariance questions, we will need the theory of classical and free cumulants, in this setting. Let us start with the following definition:

Definition 7.6. *Let (A, B, E) be an operator-valued probability space.*

(1) *A B-functional is a N-linear map $\rho : A^N \to B$ such that:*
$$\rho(b_0 a_1 b_1, a_2 b_2 \ldots, a_N b_N) = b_0 \rho(a_1, b_1 a_2, \ldots, b_{N-1} a_N) b_N$$
Equivalently, ρ is a linear map $A^{\otimes_B N} \to B$, where the tensor product is taken with respect to the natural $B - B$ bimodule structure on A.

(2) *Suppose that B is commutative. For $k \in \mathbb{N}$ let $\rho^{(k)}$ be a B-functional. Given $\pi \in P(n)$, we define a B-functional $\rho^{(\pi)} : A^N \to B$ by the formula*
$$\rho^{(\pi)}(a_1, \ldots, a_N) = \prod_{V \in \pi} \rho(V)(a_1, \ldots, a_N)$$
where if $V = (i_1 < \ldots < i_s)$ is a block of π then:
$$\rho(V)(a_1, \ldots, a_N) = \rho_s(a_{i_1}, \ldots, a_{i_s})$$

If B is noncommutative, there is no natural order in which to compute the product appearing in the above formula for $\rho^{(\pi)}$.

However, the nesting property of the noncrossing partitions allows for a natural definition of $\rho^{(\pi)}$ for $\pi \in NC(N)$, which we now recall:

Definition 7.7. *For $k \in \mathbb{N}$ let $\rho^{(k)} : A^k \to B$ be a B-functional. Given $\pi \in NC(N)$, define a B-functional $\rho^{(N)} : A^N \to B$ recursively as follows:*

(1) *If $\pi = 1_N$ is the partition having one block, define $\rho^{(\pi)} = \rho^{(N)}$.*

(2) *Otherwise, let $V = \{l + 1, \ldots, l + s\}$ be an interval of π and define:*
$$\rho^{(\pi)}(a_1, \ldots, a_N) = \rho^{(\pi - V)}(a_1, \ldots, a_l \rho^{(s)}(a_{l+1}, \ldots, a_{l+s}), a_{l+s+1}, \ldots, a_N)$$

Finally, we have the following definition:

Definition 7.8. *Let $(x_i)_{i \in I}$ be a family of random variables in A.*

(1) *The operator-valued classical cumulants $c_E^{(k)} : A^k \to B$ are the B-functionals defined by the following classical moment-cumulant formula:*
$$E(a_1 \ldots a_N) = \sum_{\pi \in P(N)} c_E^{(\pi)}(a_1, \ldots, a_N)$$

(2) *The operator-valued free cumulants $\kappa_E^{(k)} : A^k \to B$ are the B-functionals defined by the following free moment-cumulant formula:*
$$E(a_1, \ldots, a_N) = \sum_{\pi \in NC(N)} \kappa_E^{(\pi)}(a_1, \ldots, a_N)$$

We refer to [130] for more on the above notions.

We have the following result, which is well-known in the classical case, and which in the free case is due to Speicher [130]:

Theorem 7.9. *Let $(x_i)_{i\in I}$ a family of random variables in A.*

(1) *If the algebra $< B,(x_i)_{i\in I} >$ is commutative, then $(x_i)_{i\in I}$ are conditionally independent given B if and only if when there are $1\leq k,l\leq N$ such that $i_k\neq i_l$:*
$$c_E^{(N)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

(2) *The variables $(x_i)_{i\in I}$ are free with amalgamation over B if and only if when there are $1\leq k,l\leq N$ such that $i_k\neq i_l$:*
$$\kappa_E^{(N)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

Note that the condition in (1) is equivalent to the statement that if $\pi\in P(N)$, then the following happens, unless $\pi\leq\ker i$:
$$c_E^{(\pi)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

Similarly, the condition (2) is equivalent to the statement that if $\pi\in NC(N)$, then the following happens, unless $\pi\leq\ker i$:
$$\kappa_E^{(\pi)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

Observe also that in the case $B=\mathbb C$ we obtain the usual notions of independence and freeness. As before, we refer to [122], [130] for more on all this.

Stronger characterizations of the joint distribution of $(x_i)_{i\in I}$ can be given by specifying what types of partitions may contribute nonzero cumulants.

To be more precise, we have here the following result:

Theorem 7.10. *Let $(x_i)_{i\in I}$ be a family of random variables in A.*

(1) *Suppose that $< B,(x_i)_{i\in I} >$ is commutative. The B-valued joint distribution of $(x_i)_{i\in I}$ is independent for $D=P$ and independent centered Gaussian for $D=P_2$ if and only if, for any $\pi\in P(N)$, unless $\pi\in D(N)$ and $\pi\leq\ker i$:*
$$c_E^{(\pi)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

(2) *The B-valued joint distribution of $(x_i)_{i\in I}$ is freely independent for $D=NC$ and freely independent centered semicircular for $D=NC_2$ if and only if, for any $\pi\in NC(N)$, unless $\pi\in D(N)$ and $\pi\leq\ker i$:*
$$\kappa_E^{(\pi)}(b_0x_{i_1}b_1,\ldots,x_{i_N}b_N)=0$$

Proof. These results are well-known, coming from the definition of the classical and free cumulants, in the present setting, via some combinatorics. For the detailed proofs, examples and comments on all this, we refer to [122], [130]. □

Finally, here is one more basic result that we will need:

Theorem 7.11. *Let $(x_i)_{i\in I}$ be a family of random variables. Define the B-valued moment functionals $E^{(N)}$ by the following formula:*

$$E^{(N)}(a_1,\ldots,a_N)=E(a_1\ldots a_N)$$

(1) *If B is commutative, then for any $\sigma\in P(N)$ and $a_1,\ldots,a_N\in A$ we have:*

$$c_E^{(\sigma)}(a_1,\ldots,a_N)=\sum_{\pi\in P(N),\pi\leq\sigma}\mu_{P(N)}(\pi,\sigma)E^{(\pi)}(a_1,\ldots,a_N)$$

(2) *For any $\sigma\in NC(N)$ and $a_1,\ldots,a_N\in A$ we have:*

$$\kappa_E^{(\sigma)}(a_1,\ldots,a_N)=\sum_{\pi\in NC(N),\pi\leq\sigma}\mu_{NC(N)}(\pi,\sigma)E^{(\pi)}(a_1,\ldots,a_N)$$

Proof. This follows indeed from the Möbius inversion formula. □

We can now prove a reverse De Finetti theorem, as follows:

Theorem 7.12. *Let $(x_1,\ldots,x_N)$ be a sequence in A.*

(1) *If $x_1,\ldots,x_N$ are freely independent and identically distributed with amalgamation over B, then the sequence is S_N^+-invariant.*
(2) *If $x_1,\ldots,x_N$ are freely independent and identically distributed with amalgamation over B, and have centered semicircular distributions with respect to E, then the sequence is O_N^+-invariant.*
(3) *If $<B,x_1,\ldots,x_N>$ is commutative and $x_1,\ldots,x_N$ are conditionally independent and identically distributed given B, then the sequence is S_N-invariant.*
(4) *If $<x_1,\ldots,x_N>$ is commutative and $x_1,\ldots,x_N$ are conditionally independent and identically distributed given B, and have centered Gaussian distributions with respect to E, then the sequence is O_N-invariant.*

Proof. Suppose that the joint distribution of $(x_1,\ldots,x_N)$ satisfies one of the conditions in the statement, and let D be the partition family associated to the corresponding easy quantum group. We have then the following computation:

$$\begin{aligned}
&\sum_{i_1\ldots i_k}\varphi(x_{i_1}\ldots x_{i_k})u_{i_1j_1}\ldots u_{i_kj_k}\\
=&\sum_{i_1\ldots i_k}\varphi(E(x_{j_1}\ldots x_{j_k}))u_{i_1j_1}\ldots u_{i_kj_k}\\
=&\sum_{i_1\ldots i_k}\sum_{\pi\leq\ker i}\varphi(\xi_E^{(\pi)}(x_1,\ldots,x_1))u_{i_1j_1}\ldots u_{i_kj_k}\\
=&\sum_{\pi\in D(k)}\varphi(\xi_E^{(\pi)}(x_1,\ldots,x_1))\sum_{i,\pi\leq\ker i}u_{i_1j_1}\ldots u_{i_kj_k}
\end{aligned}$$

Here ξ denotes the free and classical cumulants in the cases (1,2) and (3,4) respectively. It follows from a direct computation that if $\pi\in D(k)$ then:

$$\sum_{i,\pi\leq\ker i} u_{i_1j_1}\ldots u_{i_kj_k}=\begin{cases}1 & \text{if } \pi\leq\ker j\\ 0 & \text{otherwise}\end{cases}$$

Applying this above, we find:

$$\begin{aligned}\sum_{i_1,\ldots,i_k}\varphi(x_{i_1}\ldots x_{i_k})u_{i_1j_1}\ldots u_{i_kj_k} &= \sum_{\pi\leq\ker j}\varphi(\xi_E^{(\pi)}(x_1,\ldots,x_1))\\ &= \varphi(x_{j_1}\ldots x_{j_k})\end{aligned}$$

This completes the proof. □

We now begin the technical preparations for our approximation result.

We will use the following simple fact:

Proposition 7.13. *Suppose that a sequence $(x_1,\ldots,x_N)$ is G_N-invariant. Then there is a right coaction*

$$\widetilde{\alpha}_N:M_N(\mathbb{C})\to M_N(\mathbb{C})\otimes L^\infty(G_N)$$

determined by the following formula:

$$\widetilde{\alpha}_N(p(x))=(ev_x\otimes\pi_N)\alpha_N(p)$$

Moreover, the fixed point algebra of $\widetilde{\alpha}_N$ is the G_N-invariant subalgebra B_N.

Proof. This follows indeed after identifying the GNS representation of $\mathcal{P}_N$ for the state φ_x with the morphism $ev_x:\mathcal{P}_N\to M_n(\mathbb{C})$. □

There is a natural conditional expectation given by integrating the coaction $\widetilde{\alpha}_n$ with respect to the Haar state:

$$E_N:M_N(\mathbb{C})\to B_N$$

$$E_N(m)=\left(id\otimes\int\right)\widetilde{\alpha}_N(m)$$

The point now is that by using the Weingarten calculus, we can give a simple combinatorial formula for the moment functionals with respect to E_N, in the case where G_N is one of the easy quantum groups under consideration. To be more precise, we have:

Proposition 7.14. *Suppose that $(x_1,\ldots,x_N)$ is G_N-invariant, and that either $G_N=O_N^+,S_N^+$, or that $G_N=O_N,S_N$ and $(x_1,\ldots,x_N)$ commute. We have then*

$$E_N^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)=\frac{1}{N^{|\pi|}}\sum_{\pi\leq\ker i}b_0x_{i_1}\ldots bx_{i_k}b_k$$

for any π in the partition category $D(k)$ for G_N, and any $b_0,\ldots,b_k\in B_N$.

Proof. We prove this by induction on the number of blocks of π. First suppose that $\pi = 1_k$ is the partition with only one block. Then:

$$\begin{aligned} E_N^{(1_k)}(b_0x_1b_1, \ldots, x_1b_k) &= E_N(b_0x_1 \ldots x_1b_k) \\ &= \sum_{i_1 \ldots i_k} b_0x_{i_1} \ldots x_{i_k}b_k \int u_{i_11} \ldots u_{i_k1} \end{aligned}$$

Here we have used the fact that $b_0, \ldots, b_k$ are fixed by the coaction $\widetilde{\alpha}_n$. Applying the Weingarten integration formula, we have:

$$\begin{aligned} E_N(b_0x_1 \ldots x_1b_k) &= \sum_{i_1 \ldots i_k} b_0x_{i_1} \ldots x_{i_k}b_k \sum_{\pi \leq \ker i} \sum_{\sigma} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi \in D(k)} \left(\sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) \right) \sum_{i, \pi \leq \ker i} b_0x_{i_1} \ldots x_{i_k}b_k \end{aligned}$$

Now observe that for any $\sigma \in D(k)$ we have:

$$G_{kN}(\sigma, 1_k) = N^{|\sigma \vee 1_k|} = N$$

It follows that for any $\pi \in D(k)$, we have:

$$\begin{aligned} N \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) &= \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) G_{kN}(\sigma, 1_k) \\ &= \delta_{\pi 1_k} \end{aligned}$$

Applying this above, we find, as desired:

$$\begin{aligned} E_N(b_0x_1 \ldots x_1b_k) &= \sum_{\pi \in D(k)} \frac{1}{N} \delta_{\pi 1_k} \sum_{i, \pi \leq \ker i} b_0x_{i_1} \ldots x_{i_k}b_k \\ &= \frac{1}{N} \sum_{i=1}^{N} b_0x_i \ldots x_ib_k \end{aligned}$$

If the condition (3) or (4) is satisfied, then the general case follows from:

$$E_N^{(\pi)}(b_0x_1b_1, \ldots, x_1b_k) = b_1 \ldots b_k \prod_{V \in \pi} E_N(V)(x_1, \ldots, x_1)$$

The one thing we must check here is that if $\pi \in D(k)$ and V is a block of π with s elements, then $1_s \in D(s)$. This is easily verified, in each case.

Suppose now that the condition (1) or (2) is satisfied. Let $\pi \in D(k)$. Since π is noncrossing, π contains an interval $V = \{l+1, \ldots, l+s+1\}$. We then have:

$$E_N^{(\pi)}(b_0x_1b_1, \ldots, x_1b_k) = E_N^{(\pi - V)}(b_0x_1b_1, \ldots, E_n(x_1b_{l+1} \ldots x_1b_{l+s})x_1, \ldots, x_1b_k)$$

To apply induction, we must check that $\pi - V \in D(k-s)$ and $1_s \in D(s)$. Indeed, this is easily verified for NC, NC_2. Applying induction, we have:

$$\begin{aligned}
& E_N^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)\\
= & \frac{1}{N^{|\pi|-1}}\sum_{i,\pi-V\leq\ker i} b_0x_{i_1}\ldots b_l\left(E_n(x_1b_{l+1}\ldots x_1b_{l+s})\right)x_{i_{l+s}}\ldots x_{i_k}b_k\\
= & \frac{1}{N^{|\pi|-1}}\sum_{i,\pi-V\leq\ker i} b_0x_{i_1}\ldots b_l\left(\frac{1}{N}\sum_{i=1}^n x_ib_{l+1}\ldots bx_ib_{l+s}\right)x_{i_{l+s}}\ldots x_{i_k}b_k\\
= & \frac{1}{N^{|\pi|}}\sum_{i,\pi\leq\ker i} b_0x_{i_1}\ldots x_{i_k}b_k
\end{aligned}$$

This completes the proof. $\square$

In order to advance, we will need some standard Weingarten estimates for our quantum groups, which have their own interest, and that we will discuss now.

Regarding the symmetric group S_N, the situation here is very simple, as follows:

Proposition 7.15. *For S_N the Weingarten function is given by*

$$W_{kN}(\pi,\sigma)=\sum_{\tau\leq\pi\wedge\sigma}\mu(\tau,\pi)\mu(\tau,\sigma)\frac{(N-|\tau|)!}{N!}$$

and satisfies the folowing estimate,

$$W_{kN}(\pi,\sigma)=N^{-|\pi\wedge\sigma|}(\mu(\pi\wedge\sigma,\pi)\mu(\pi\wedge\sigma,\sigma)+O(N^{-1}))$$

with μ being the Möbius function of $P(k)$.

Proof. The first assertion follows from the Weingarten formula, namely:

$$\int_{S_N}u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in P(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

Indeed, in this formula the integrals on the left are known, from the explicit integration formula over S_N that we established in section 5, namely:

$$\int_{S_N}g_{i_1j_1}\ldots g_{i_kj_k}=\begin{cases}\frac{(N-|\ker i|)!}{N!} & \text{if }\ \ker i=\ker j\\ 0 & \text{otherwise}\end{cases}$$

But this allows the computation of the right term, via the Möbius inversion formula, explained in section 2. As for the second assertion, this follows from the first one. $\square$

The above result is of course something very special, coming from the fact that the integration over S_N is something very simple.

Regarding now the quantum group S_N^+, that we are particularly interested in, let us begin with some explicit computations.

We first have the following simple and final result at $k=2,3$, directly in terms of the quantum group integrals:

Proposition 7.16. *At $k=2,3$ we have the following estimate:*
$$\int_{S_N^+} u_{i_1j_1}\ldots u_{i_kj_k}=\begin{cases}0 & (\ker i\neq\ker j)\\ \simeq N^{-|\ker i|} & (\ker i=\ker j)\end{cases}$$

Proof. Since at $k\leq 3$ we have $NC(k)=P(k)$, the Weingarten integration formulae for S_N^+ and S_N coincide, and we obtain, by using the above formula for S_N:
$$\begin{aligned}\int_{S_N^+} u_{i_1j_1}\ldots u_{i_kj_k} &= \int_{S_N} u_{i_1j_1}\ldots u_{i_kj_k}\\ &= \delta_{\ker i,\ker j}\frac{(N-|\ker i|)!}{N!}\end{aligned}$$

But this gives the formula in the statement. $\square$

In general now, the idea will be that of working out a "master estimate" for the Weingarten function, as above. Before starting here, let us record the formulae at $k=2,3$, which will be useful, as illustrations. At $k=2$, with indices $||,\sqcap$, and with the convention that $\approx$ means componentwise dominant term, we have:
$$W_{2N}\approx\begin{pmatrix}N^{-2}&-N^{-2}\\-N^{-2}&N^{-1}\end{pmatrix}$$

At $k=3$ now, with indices $|||,|\sqcap,\sqcap|,\sqcap\!\!\!\!|\ ,\sqcap\!\sqcap$, and same meaning for $\approx$, we have:
$$W_{3N}\approx\begin{pmatrix}N^{-3}&-N^{-3}&-N^{-3}&-N^{-3}&2N^{-3}\\-N^{-3}&N^{-2}&N^{-3}&N^{-3}&-N^{-2}\\-N^{-3}&N^{-3}&N^{-2}&N^{-3}&-N^{-2}\\-N^{-3}&N^{-3}&N^{-3}&N^{-2}&-N^{-2}\\2N^{-3}&-N^{-2}&-N^{-2}&-N^{-2}&N^{-1}\end{pmatrix}$$

These formulae follow indeed from the plain formulae for W_{kN} at $k=2,3$ from [26], after rearranging the matrix indices as above. Observe in particular that we have the following formula, which will be of interest in what follows:
$$W_{3N}(|\sqcap,\sqcap|)\simeq N^{-3}$$

In order to deal now with the general case, let us start with:

Proposition 7.17. *The following happen, regarding the partitions in $P(k)$:*

(1) $|\pi|+|\sigma|\leq|\pi\vee\sigma|+|\pi\wedge\sigma|$.
(2) $|\pi\vee\tau|+|\tau\vee\sigma|\leq|\pi\vee\sigma|+|\tau|$.
(3) $d(\pi,\sigma)=\frac{|\pi|+|\sigma|}{2}-|\pi\vee\sigma|$ *is a distance.*

Proof. All this is well-known, the idea being as follows:

(1)This is well-known, coming from the fact that $P(k)$ is a semi-modular lattice.

(2) This follows from (1), as explained for instance in [34].

(3) This follows from (2), which says that the following holds:

$$\begin{aligned}&\frac{|\pi|+|\tau|}{2}-d(\pi,\tau)+\frac{|\tau|+|\sigma|}{2}-d(\tau,\sigma)\\ \leq\ &\frac{|\pi|+|\sigma|}{2}-d(\pi,\sigma)+|\tau|\end{aligned}$$

Thus, we obtain the triangle inequality:

$$d(\pi,\tau)+d(\tau,\sigma)\geq d(\pi,\sigma)$$

As for the other distance conditions, these are all clear. □

Actually in what follows we will only need (3) in the above statement. For more on this, and on the geometry and combinatorics of partitions, see [122].

As a main result now regarding the Weingarten function, we have:

Theorem 7.18. *The Weingarten function W_{kN} has a series expansion in N^{-1},*

$$W_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|-|\pi|-|\sigma|}\sum_{g=0}^\infty K_g(\pi,\sigma)N^{-g}$$

where the objects on the right are defined as follows:

(1) *A path from π to σ is a sequence $p=[\pi=\tau_0\neq\tau_1\neq\ldots\neq\tau_r=\sigma]$.*
(2) *The signature of such a path is $+$ when r is even, and $-$ when r is odd.*
(3) *The geodesicity defect of such a path is $g(p)=\sum_{i=1}^rd(\tau_{i-1},\tau_i)-d(\pi,\sigma)$.*
(4) *K_g counts the signed paths from π to σ, with geodesicity defect g.*

Proof. The Gram matrix $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$ can be written as follows:

$$\begin{aligned}G_{kN}(\pi,\sigma)&=N^{\frac{|\pi|}{2}}N^{|\pi\vee\sigma|-\frac{|\pi|+|\sigma|}{2}}N^{\frac{|\sigma|}{2}}\\&=N^{\frac{|\pi|}{2}}N^{-d(\pi,\sigma)}N^{\frac{|\sigma|}{2}}\end{aligned}$$

Consider now the diagonal matrix $\Delta=diag(N^{\frac{|\pi|}{2}})$, and let us set as well:

$$H(\pi,\sigma)=\begin{cases}0&(\pi=\sigma)\\N^{-d(\pi,\sigma)}&(\pi\neq\sigma)\end{cases}$$

In terms of these two matrices, the above formula simply reads:

$$G_{kN}=\Delta(1+H)\Delta$$

Thus, the Weingarten matrix is given by the following formula:

$$W_{kN}=\Delta^{-1}(1+H)^{-1}\Delta^{-1}$$

In order to compute the inverse of $1+H$, consider the set $P_r(\pi,\sigma)$ of length r paths between π and σ. The powers of H are then given by:

$$\begin{aligned}H^r(\pi,\sigma)&=\sum_{p\in P_r(\pi,\sigma)}H(\tau_0,\tau_1)\ldots H(\tau_{r-1},\tau_r)\\&=\sum_{p\in P_r(\pi,\sigma)}N^{-d(\pi,\sigma)-g(p)}\end{aligned}$$

Thus by using the formula $(1+H)^{-1}=1-H+H^2-H^3+\ldots$ we obtain:

$$\begin{aligned}(1+H)^{-1}(\pi,\sigma)&=\sum_{r=0}^\infty(-1)^rH^r(\pi,\sigma)\\&=N^{-d(\pi,\sigma)}\sum_{r=0}^\infty\sum_{p\in P_r(\pi,\sigma)}(-1)^rN^{-g(p)}\end{aligned}$$

It follows that the Weingarten matrix is given by:

$$\begin{aligned}W_{kN}(\pi,\sigma)&=\Delta^{-1}(\pi)(1+H)^{-1}(\pi,\sigma)\Delta^{-1}(\sigma)\\&=N^{-\frac{|\pi|}{2}-\frac{|\sigma|}{2}-d(\pi,\sigma)}\sum_{r=0}^\infty\sum_{p\in P_r(\pi,\sigma)}(-1)^rN^{-g(p)}\\&=N^{|\pi\vee\sigma|-|\pi|-|\sigma|}\sum_{r=0}^\infty\sum_{p\in P_r(\pi,\sigma)}(-1)^rN^{-g(p)}\end{aligned}$$

Now by rearranging the various terms of the double sum according to their geodesicity defect $g=g(p)$, this gives the formula in the statement. □

As an illustration, we have the following explicit estimates:

Theorem 7.19. *Consider an easy quantum group $G=(G_N)$, coming from a category of partitions $D=(D(k))$. For any $\pi\leq\sigma$ we have the estimate*

$$W_{kN}(\pi,\sigma)=N^{-|\pi|}(\mu(\pi,\sigma)+O(N^{-1}))$$

and for π,σ arbitrary we have

$$W_{kN}(\pi,\sigma)=O(N^{|\pi\vee\sigma|-|\pi|-|\sigma|})$$

with μ being the Möbius function of $D(k)$.

Proof. We have two assertions here, the idea being as follows:

(1) The first estimate is clear from Theorem 7.18.

(2) In the case $\pi \leq \sigma$ it is known that K_0 coincides with the Möbius function of $NC(k)$, as explained for instance in [34], so we obtain once again from Theorem 7.18 the fine estimate as well, namely:

$$W_{kN}(\pi,\sigma) = N^{-|\pi|}(\mu(\pi,\sigma) + O(N^{-1})) \qquad \forall \pi \leq \sigma$$

Observe that, by symmetry of W_{kN}, we obtain as well the following estimate:

$$W_{kN}(\pi,\sigma) = N^{-|\sigma|}(\mu(\sigma,\pi) + O(N^{-1})) \qquad \forall \pi \geq \sigma$$

Thus, we are led to the conclusions in the statement. $\square$

When π,σ are not comparable by $\leq$, the situation is quite unclear. The simplest example appears at $k=3$, where we have the following formula, which is elementary:

$$W_{3N}(|\sqcap,\sqcap|) \simeq N^{-3}$$

Observe that the exponent -3 is precisely the dominant one, because:

$$\Big|\,|\sqcap \vee \sqcap|\,\Big| - \Big|\,|\sqcap\,\Big| - \Big|\,\sqcap|\,\Big| = 1 - 2 - 2 = -3$$

As for the corresponding coefficient, $K_0(|\sqcap,\sqcap|) = 1$, this is definitely not the Möbius function, which vanishes for partitions which are not comparable by $\leq$. According to Theorem 7.18, this is rather the number of signed geodesic paths from $|\sqcap$ to $\sqcap|$.

In relation to this, observe that geometrically, $NC(5)$ consists of the partitions $|\sqcap,\sqcap|,\sqcap\!\!\!\!\!\;\;|$, which form an equilateral triangle with edges worth 1, and then the partitions $|||,\sqcap\!\sqcap$, which are at distance 1 apart, and each at distance $1/2$ from each of the vertices of the triangle.

It is not exactly obvious how to recover the formula $K_0(|\sqcap,\sqcap|) = 1$ from this.

Finally, we will need as well the following result:

Proposition 7.20. *We have the following results:*

(1) *If $D = NC, NC_2$, then $\mu_{D(k)}(\pi,\sigma) = \mu_{NC(k)}(\pi,\sigma)$.*
(2) *If $D = P, P_2$ then $\mu_{D(k)}(\pi,\sigma) = \mu_{P(k)}(\pi,\sigma)$.*

Proof. Let $Q = NC, P$ according to the cases (1,2). It is easy to see in each case that $D(k)$ is closed under taking intervals in $Q(k)$, i.e., if $\pi_1,\pi_2 \in D(k)$, $\sigma \in Q(k)$ and $\pi_1 < \sigma < \pi_2$ then $\sigma \in D(k)$. The result now follows from the definition of the Möbius function. $\square$

With all these combinatorial ingredients in hand, we are now prepared to prove an approximation result for finite sequences, from [34], as follows:

Theorem 7.21. *Suppose that $(x_1,\dots,x_N)$ is G_N-invariant, and that $G_N = O_N^+, S_N^+$, or that $G_N = O_N, S_N$ and $(x_1,\dots,x_N)$ commute. Let $(y_1,\dots,y_N)$ be a sequence of B_N-valued random variables with B_N-valued joint distribution determined as follows:*

(1) *$G = O^+$: Free semicircular, centered with same variance as x_1.*
(2) *$G = S^+$: Freely independent, y_i has same distribution as x_1.*
(3) *$G = O$: Independent Gaussian, centered with same variance as x_1.*
(4) *$G = S$: Independent, y_i has same distribution as x_1.*

Then if $1 \le j_1,\dots,j_k \le N$ and $b_0,\dots,b_k \in B_N$, we have the following estimate,

$$||E_N(b_0x_{j_1}\dots x_{j_k}b_k) - E(b_0y_{j_1}\dots y_{j_k}b_k)|| \le \frac{C_k(G)}{N}||x_1||^k||b_0||\dots||b_k||$$

with $C_k(G)$ being a constant depending only on k and G.

Proof. First we note that it suffices to prove the result for N sufficiently large. We will assume that N is sufficiently large as for the Gram matrix G_{kN} to be invertible.

Let $1 \le j_1,\dots,j_k \le N$ and $b_0,\dots,b_k \in B_N$. We have:

$$\begin{aligned}
& E_N(b_0x_{j_1}\dots x_{j_k}b_k)\\
= & \sum_{i_1\dots i_k} b_0x_{i_1}\dots x_{i_k}b_k\int u_{i_1j_1}\dots u_{i_kj_k}\\
= & \sum_{i_1\dots i_k} b_0x_{i_1}\dots x_{i_k}b_k\sum_{\pi\le\ker i}\sum_{\sigma\le\ker j}W_{kN}(\pi,\sigma)\\
= & \sum_{\sigma\le\ker j}\sum_{\pi}W_{kN}(\pi,\sigma)\sum_{i,\pi\le\ker i}b_0x_{i_1}\dots x_{i_k}b_k
\end{aligned}$$

On the other hand, it follows from the assumptions on $(y_1,\dots,y_N)$ and the various moment-cumulant formulae, that we have:

$$E(b_0y_{j_1}\dots y_{j_k}b_k) = \sum_{\sigma\le\ker j}\xi_{E_N}^{(\sigma)}(b_0x_1b_1,\dots,x_1b_k)$$

Here, and in what follows, ξ are the relevant free or classical cumulants:

The right hand side can be expanded, via Möbius inversion, in terms of expectation functionals as follows, with π being a partition in NC, P according to the cases (1,2) or (3,4), and with $\pi \le \sigma$ for some $\sigma \in D(k)$:

$$E_N^{(\pi)}(b_0x_1b_1,\dots,x_1b_k)$$

Now if $\pi \notin D(k)$ then we claim that this expectation functional is zero.

Indeed this is only possible if $D = NC_2, P_2$ and π has a block with an odd number of legs. But it is easy to see that in these cases x_1 has an even distribution with respect to

E_N, and therefore we have, as claimed:

$$E_N^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)=0$$

This observation allows to to rewrite the above equation as:

$$E(b_0y_{j_1}\ldots y_{j_k}b_k)=\sum_{\sigma\le\ker j}\sum_{\pi\le\sigma}\mu_{D(k)}(\pi,\sigma)E_N^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)$$

We therefore obtain the following formula:

$$E(b_0y_{j_1}\ldots y_{j_k}b_k)=\sum_{\sigma\le\ker j}\sum_{\pi\le\sigma}\mu_{D(k)}(\pi,\sigma)N^{-|\pi|}\sum_{i,\pi\le\ker i}b_0x_{i_1}\ldots x_{i_k}b_k$$

Comparing these two equations, we find that:

$$\begin{aligned}&E_N(b_0x_{j_1}\ldots x_{j_k}b_k)-E(b_0y_{j_1}\ldots y_{j_k}b_k)\\ =\ &\sum_{\sigma\le\ker j}\sum_{\pi}\left(W_{kN}(\pi,\sigma)-\mu_{D(k)}(\pi,\sigma)N^{-|\pi|}\right)\sum_{i,\pi\le\ker i}b_0x_{i_1}\ldots x_{i_k}b_k\end{aligned}$$

Now since $x_1,\ldots,x_N$ are identically distributed with respect to the faithful state φ, it follows that these variables have the same norm. Therefore, for any $\pi\in D(k)$:

$$\left|\left|\sum_{i,\pi\le\ker i}b_0x_{i_1}\ldots x_{i_k}b_k\right|\right|\le N^{|\pi|}||x_1||^k||b_0||\ldots||b_k||$$

Combining this with the former equation, we obtain:

$$\begin{aligned}&||E_N(b_0x_{j_1}\ldots x_{j_k}b_k)-E(b_0y_{j_1}\ldots y_{j_k}b_k)||\\ \le\ &\sum_{\sigma\le\ker j}\sum_{\pi}\left|W_{kN}(\pi,\sigma)N^{|\pi|}-\mu_{D(k)}(\pi,\sigma)\right|||x_1||^k||b_0||\ldots||b_k||\end{aligned}$$

Let us set now:

$$C_k(G)=\sup_{N\in\mathbb{N}}N\times\sum_{\sigma,\pi\in D(k)}\left|W_{kN}(\pi,\sigma)N^{|\pi|}-\mu_{D(k)}(\pi,\sigma)\right|$$

But this is finite by our main estimate, which completes the proof. □

We will make use of the inclusions $G_N\subset G_M$ for $N<M$, which correspond to the Hopf algebra morphisms $\omega_{N,M}:C(G_M)\to C(G_N)$ determined by:

$$\omega_{N,M}(u_{ij})=\begin{cases}u_{ij}&\text{if }1\le i,j\le N\\ \delta_{ij}&\text{if }\max(i,j)>N\end{cases}$$

We begin by extending the notion of G_N-invariance to infinite sequences:

Definition 7.22. *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in a noncommutative probability space (A,φ). We say that $(x_i)_{i\in\mathbb{N}}$ is G-invariant if*
$$(x_1,\ldots,x_N)$$
is G_N-invariant for each $N\in\mathbb{N}$.

In other words, the condition is that the joint distribution functional of $(x_1,\ldots,x_N)$ should be invariant under the following action, for each $n\in\mathbb{N}$:
$$\alpha_N:\mathcal{P}_N\to\mathcal{P}_N\otimes C(G_N)$$
It will be convenient to extend these actions to $\mathcal{P}_\infty=\mathbb{C}<t_i|i\in\mathbb{N}>$, by defining $\beta_N:\mathcal{P}_\infty\to\mathcal{P}_\infty\otimes C(G_N)$ to be the unique unital morphism such that:
$$\beta_N(t_j)=\begin{cases}\sum_{i=1}^N t_i\otimes u_{ij} & \text{if } 1\le j\le N\\ t_j\otimes 1 & \text{if } j>N\end{cases}$$
It is clear that β_N is an action of G_N. Also, we have the following relations, where $\iota_N:\mathcal{P}_N\to\mathcal{P}_\infty$ is the natural inclusion:
$$(id\otimes\omega_{N,M})\beta_M=\beta_N$$
$$(\iota_N\otimes id)\alpha_N=\beta_N\iota_N$$
By using these compatibilities, we have the following result:

Proposition 7.23. *A a sequence $(x_i)_{i\in\mathbb{N}}$ is G-invariant if and only if the joint distribution functional*
$$\varphi_x:\mathcal{P}_\infty\to\mathbb{C}$$
is invariant under β_N for each $N\in\mathbb{N}$.

Proof. This is clear indeed from the above discussion. $\square$

In what follows $(x_i)_{i\in\mathbb{N}}$ will be a sequence of self-adjoint random variables in a von Neumann algebra (M,φ). We will assume that M is generated by $(x_i)_{i\in\mathbb{N}}$.

We denote by $L^2(M,\varphi)$ the corresponding GNS Hilbert space, with inner product which is by definition as follows:
$$<m_1,m_2>=\varphi(m_1m_2^*)$$
Also, the strong topology on M will be taken by definition with respect to the faithful representation on $L^2(M,\varphi)$.

We let $\mathcal{P}_\infty^{\beta_N}$ be the fixed point algebra of the action β_N, and we set:
$$B_N=\left\{p(x)\Big|p\in\mathcal{P}_\infty^{\beta_N}\right\}''$$
We have then the following formula:
$$(id\otimes\omega_{N,N+1})\beta_{N+1}=\beta_N$$

Thus we have an inclusion as follows, for any $n \geq 1$:

$$B_{N+1} \subset B_N$$

We then define the G-invariant subalgebra by:

$$B = \bigcap_{N \geq 1} B_N$$

With these conventions, we have the following result:

Proposition 7.24. *If an infinite sequence $(x_i)_{i \in \mathbb{N}}$ is G-invariant, then for each $N \in \mathbb{N}$ there is a right coaction*

$$\widetilde{\beta}_N : M \to M \otimes L^\infty(G_N)$$

determined by the following formula, for any $p \in \mathcal{P}_\infty$:

$$\widetilde{\beta}_N(p(x)) = (ev_x \otimes \pi_N)\beta_N(p)$$

The fixed point algebra of $\widetilde{\beta}_N$ is then B_N.

Proof. This is indeed clear from definitions. □

We have as well the following result, which is clear as well:

Proposition 7.25. *In the above context, for each $N \in \mathbb{N}$ there is then a φ-preserving conditional expectation $E_N : M \to B_N$ given by integrating the action $\widetilde{\beta}_N$:*

$$E_N(m) = \left(id \otimes \int \right) \widetilde{\beta}_N(m)$$

By taking the limit as $N \to \infty$, we obtain a φ-preserving conditional expectation onto the G-invariant subalgebra.

Proof. Once again, this is clear from definitions. □

Next, we have the following result:

Proposition 7.26. *Suppose that $(x_i)_{i \in \mathbb{N}}$ is G-invariant. Then:*

1. *For any $m \in M$, the sequence $E_N(m)$ converges in 2-norm and with respect to the strong topology to a limit $E(m) \in B$.*
2. *E is a φ-preserving conditional expectation of M onto B.*
3. *For $\pi \in NC(k)$ and $m_1, \ldots, m_k \in M$ we have, with strong convergence:*

$$E^{(\pi)}(m_1 \otimes \ldots \otimes m_k) = \lim_{n \to \infty} E_N^{(\pi)}(m_1 \otimes \ldots \otimes m_k)$$

Proof. This is again clear from definitions. Note that (1) is just a simple noncommutative reversed martingale convergence theorem. □

We are now prepared to state and prove the main theorem, from [34]:

Theorem 7.27. *Let $(x_i)_{i\in\mathbb{N}}$ be a G-invariant sequence of self-adjoint random variables in (M,φ), and assume that $M=<(x_i)_{i\in\mathbb{N}}>$. Then there is a subalgebra $B\subset M$ and a φ-preserving conditional expectation $E:M\to B$ such that:*

1. *If $G=(S_N)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and identically distributed given B.*
2. *If $G=(S_N^+)$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with amalgamation over B.*
3. *If $G=(O_N)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have Gaussian distributions with mean zero and common variance, given B.*
4. *If $G=(O_N^+)$, then $(x_i)_{i\in\mathbb{N}}$ form a B-valued free semicircular family with mean zero and common variance.*

Proof. Let $j_1,\ldots,j_k\in\mathbb{N}$ and $b_0,\ldots,b_k\in B$. We have:

$$\begin{aligned}
E(b_0x_{j_1}\ldots x_{j_k}b_k) &= \lim_{N\to\infty}E_n(b_0x_{j_1}\ldots x_{j_k}b_k)\\
&= \lim_{N\to\infty}\sum_{\sigma\leq\ker j}\sum_{\pi}W_{kN}(\pi,\sigma)\sum_{i,\pi\leq\ker i}b_0x_{i_1}\ldots x_{i_k}b_k\\
&= \lim_{N\to\infty}\sum_{\sigma\leq\ker j}\sum_{\pi\leq\sigma}\mu_{D(k)}(\pi,\sigma)N^{-|\pi|}\sum_{i,\pi\leq\ker i}b_0x_{i_1}\ldots x_{i_k}b_k
\end{aligned}$$

Let us recall now from the above that we have the following compatibility formula, where $\widetilde{\iota}_N:W^*(x_1,\ldots,x_N)\to M$ is the obvious inclusion, and $\widetilde{\alpha}_N$ is as before:

$$(\widetilde{\iota}_N\otimes id)\widetilde{\alpha}_N=\widetilde{\beta}_N\widetilde{\iota}_N$$

By using this, and the above cumulant results, we have:

$$E(b_0x_{j_1}\ldots x_{j_k}b_k)=\lim_{N\to\infty}\sum_{\sigma\leq\ker j}\sum_{\pi\leq\sigma}\mu_{D(k)}(\pi,\sigma)E_N^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)$$

We therefore obtain the following formula:

$$E(b_0x_{j_1}\ldots x_{j_k}b_k)=\sum_{\sigma\leq\ker j}\sum_{\pi\leq\sigma}\mu_{D(k)}(\pi,\sigma)E^{(\pi)}(b_0x_1b_1,\ldots,x_1b_k)$$

We can replace the sum of expectation functionals by cumulants to obtain:

$$E(b_0x_{j_1}\ldots x_{j_k}b_k)=\sum_{\sigma\leq\ker j}\xi_E^{(\sigma)}(b_0x_1b_1,\ldots,x_1b_k)$$

Here and in what follows ξ denotes the relevant free or classical cumulants, depending on the quantum group that we are dealing with, free or classical.

Now since the cumulants are determined by the moment-cumulant formulae, we find that we have the following formula:

$$\xi_E^{(\sigma)}(b_0x_{j_1}b_1,\ldots,x_{j_k}b_k)=\begin{cases}\xi_E^{(\sigma)}(b_0x_1b_1,\ldots,x_1b_k) & \text{if } \sigma\in D(k) \text{ and } \sigma\leq\ker j\\ 0 & \text{otherwise}\end{cases}$$

The result then follows from the characterizations of these joint distributions in terms of cumulants. □

We refer to [34] and related papers for more on the above.

8. Hypergeometric laws

We have seen so far that, in what concerns the probability theory on classical or quantum groups, the very first problem which appears, and which is of key importance, is that of computing the laws of characters, and more generally of truncated characters:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

For the quantum rotation and permutation groups, this problem can be investigated by using easiness and combinatorics, and satisfactory results in this sense, which are in tune with free probability theory, can be obtained in the $N \to \infty$ limit.

That was for the basic theory. In this section we discuss more advanced aspects, regarding the case where $N \in \mathbb{N}$ is fixed, or variables which are more general than the truncated characters χ_t, or regarding both, more advanced variables at fixed $N \in \mathbb{N}$. Let us first discuss the case of O_N. In certain situations, we can use:

Proposition 8.1. *Each row of coordinates on O_N has the same joint distribution as the sequence of coordinates on the real sphere $S^{N-1}_\mathbb{R}$,*

$$(u_{i1}, \ldots, u_{iN}) \sim (x_1, \ldots, x_N)$$

and the same happens for the columns.

Proof. Given an index $i \in \{1, \ldots, N\}$, our claim is that we have an embedding as follows, which commutes with the corresponding uniform integration functionals:

$$C(S^{N-1}_\mathbb{R}) \subset C(O_N) \quad , \quad x_j \to u_{ij}$$

In order to prove this claim, consider the subalgebra $C(S) \subset C(O_N)$ generated by the variables u_{ij}, with i being fixed, and with $j = 1, \ldots, N$. Since these N variables are real, and their squares sum up to 1, we have a quotient map, as follows:

$$C(S^{N-1}_\mathbb{R}) \to C(S) \subset C(O_N) \quad , \quad x_j \to u_{ij}$$

Now observe that $S \subset S^{N-1}_\mathbb{R}$ must be an isomorphism, because by Gram-Schmidt we can complete any vector of $S^{N-1}_\mathbb{R}$ into an orthogonal matrix. Thus, the above composition of morphisms is an embedding. As for the commutation with the uniform integration functionals, this follows from the fact that we have an action $O_N \curvearrowright S$. $\square$

Motivated by the above, let us compute now the hyperspherical laws at fixed values of $N \in \mathbb{N}$. Let us begin with a full discussion in the classical case. At $N = 2$ the sphere is the unit circle $\mathbb{T}$, and with $z = e^{it}$ the coordinates are $\cos t, \sin t$. The integrals of the arbitrary products of such coordinates can be computed as follows:

Theorem 8.2. *We have the following formula,*

$$\int_0^{\pi/2}\cos^pt\sin^qt\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}$$

where $\varepsilon(p)=1$ if p is even, and $\varepsilon(p)=0$ if p is odd, and where

$$m!!=(m-1)(m-3)(m-5)\ldots$$

with the product ending at 2 if m is odd, and ending at 1 if m is even.

Proof. This is something known to everyone loving and teaching calculus. We compute the integral in the statement I_p by partial integration. For this purpose, we use:

$$\begin{aligned}(\cos^pt\sin t)'&=p\cos^{p-1}t(-\sin t)\sin t+\cos^pt\cos t\\&=p\cos^{p+1}t-p\cos^{p-1}t+\cos^{p+1}t\\&=(p+1)\cos^{p+1}t-p\cos^{p-1}t\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(p+1)I_{p+1}=pI_{p-1}$$

Thus we can compute I_p by recurrence, and we obtain:

$$\begin{aligned}I_p&=\frac{p-1}{p}\,I_{p-2}\\&=\frac{p-1}{p}\cdot\frac{p-3}{p-2}\,I_{p-4}\\&=\frac{p-1}{p}\cdot\frac{p-3}{p-2}\cdot\frac{p-5}{p-4}\,I_{p-6}\\&\ \vdots\\&=\frac{p!!}{(p+1)!!}\,I_{1-\varepsilon(p)}\end{aligned}$$

Together with $I_0=\frac{\pi}{2}$ and $I_1=1$, which are both clear, we obtain:

$$I_p=\left(\frac{\pi}{2}\right)^{\varepsilon(p)}\frac{p!!}{(p+1)!!}$$

Summarizing, we have proved the following formula, with one equality coming from the above computation, and with the other equality coming from this, via $t=\frac{\pi}{2}-s$:

$$\int_0^{\pi/2}\cos^pt\,dt=\int_0^{\pi/2}\sin^pt\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)}\frac{p!!}{(p+1)!!}$$

In relation with the formula in the statement, we are therefore done with the case $p=0$ or $q=0$. Let us investigate now the general case. We must compute:

$$I_{pq}=\int_0^{\pi/2}\cos^pt\sin^qt\,dt$$

In order to do the partial integration, observe that we have:

$$\begin{aligned}(\cos^pt\sin^qt)' &= p\cos^{p-1}t(-\sin t)\sin^qt\\ &+ \cos^pt\cdot q\sin^{q-1}t\cos t\\ &= -p\cos^{p-1}t\sin^{q+1}t+q\cos^{p+1}t\sin^{q-1}t\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain, for $p,q>0$:

$$pI_{p-1,q+1}=qI_{p+1,q-1}$$

Thus, we can compute I_{pq} by recurrence. When q is even we have:

$$\begin{aligned}I_{pq} &= \frac{q-1}{p+1}\,I_{p+2,q-2}\\ &= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\,I_{p+4,q-4}\\ &= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\cdot\frac{q-5}{p+5}\,I_{p+6,q-6}\\ &= \vdots\\ &= \frac{p!!q!!}{(p+q)!!}\,I_{p+q}\end{aligned}$$

But the last term was already computed above, and we obtain the result:

$$\begin{aligned}I_{pq} &= \frac{p!!q!!}{(p+q)!!}\,I_{p+q}\\ &= \frac{p!!q!!}{(p+q)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(p+q)}\frac{(p+q)!!}{(p+q+1)!!}\\ &= \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}\end{aligned}$$

Observe that this gives the result for p even as well, by symmetry. Indeed, we have $I_{pq}=I_{qp}$, by using the following change of variables:

$$t=\frac{\pi}{2}-s$$

In the remaining case now, where both p,q are odd, we can use once again the formula $pI_{p-1,q+1}=qI_{p+1,q-1}$ established above, and the recurrence goes as follows:

$$\begin{aligned}
I_{pq} &= \frac{q-1}{p+1}\,I_{p+2,q-2}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\,I_{p+4,q-4}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\cdot\frac{q-5}{p+5}\,I_{p+6,q-6}\\
&= \vdots\\
&= \frac{p!!q!!}{(p+q-1)!!}\,I_{p+q-1,1}
\end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned}
I_{p1} &= \int_0^{\pi/2}\cos^p t\sin t\,dt\\
&= -\frac{1}{p+1}\int_0^{\pi/2}(\cos^{p+1}t)'\,dt\\
&= \frac{1}{p+1}
\end{aligned}$$

Thus, we can finish our computation in the case p,q odd, as follows:

$$\begin{aligned}
I_{pq} &= \frac{p!!q!!}{(p+q-1)!!}\,I_{p+q-1,1}\\
&= \frac{p!!q!!}{(p+q-1)!!}\cdot\frac{1}{p+q}\\
&= \frac{p!!q!!}{(p+q+1)!!}
\end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of $\pi/2$ appearing there being $\varepsilon(p)\varepsilon(q)=0\cdot 0=0$ in the present case, and this finishes the proof. $\square$

More generally, we can compute arbitrary polynomial integrals, over the spheres of arbitrary dimension, the result being is as follows:

Theorem 8.3. *The spherical integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each $a\in\{1,\ldots,N\}$ appears an even number of times in the sequence $i_1,\ldots,i_k$. We have*

$$\int_{S^{N-1}_\mathbb{R}} x_{i_1}\ldots x_{i_k}\,dx=\frac{(N-1)!!l_1!!\ldots l_N!!}{(N+\Sigma l_i-1)!!}$$

with l_a being this number of occurrences.

Proof. We can restrict attention to the case $l_a \in 2\mathbb{N}$, since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I = \frac{2^N}{V}\int_0^{\pi/2}\ldots\int_0^{\pi/2} x_1^{l_1}\ldots x_N^{l_N}\, J\, dt_1\ldots dt_{N-1}$$

In this formula V is the volume of the sphere, J is the Jacobian, and the 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive. The normalization constant in front of the integral is:

$$\frac{2^N}{V} = \frac{2^N}{N\pi^{N/2}}\cdot\Gamma\left(\frac{N}{2}+1\right) = \left(\frac{2}{\pi}\right)^{[N/2]}(N-1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned} I' = \int_0^{\pi/2}\ldots\int_0^{\pi/2} \quad & (\cos t_1)^{l_1}(\sin t_1\cos t_2)^{l_2} \\ & \vdots \\ & (\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1})^{l_{N-1}} \\ & (\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1})^{l_N} \\ & \sin^{N-2}t_1\sin^{N-3}t_2\ldots\sin^2 t_{N-3}\sin t_{N-2} \\ & dt_1\ldots dt_{N-1} \end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned} I' \quad = \quad & \int_0^{\pi/2}\cos^{l_1}t_1\sin^{l_2+\ldots+l_N+N-2}t_1\,dt_1 \\ & \int_0^{\pi/2}\cos^{l_2}t_2\sin^{l_3+\ldots+l_N+N-3}t_2\,dt_2 \\ & \vdots \\ & \int_0^{\pi/2}\cos^{l_{N-2}}t_{N-2}\sin^{l_{N-1}+l_N+1}t_{N-2}\,dt_{N-2} \\ & \int_0^{\pi/2}\cos^{l_{N-1}}t_{N-1}\sin^{l_N}t_{N-1}\,dt_{N-1} \end{aligned}$$

Now by using the formula at $N=2$, from Theorem 8.2, this gives:

$$\begin{aligned}
I' &= \frac{l_1!!(l_2+\ldots+l_N+N-2)!!}{(l_1+\ldots+l_N+N-1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-2)}\\
&\quad \frac{l_2!!(l_3+\ldots+l_N+N-3)!!}{(l_2+\ldots+l_N+N-2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-3)}\\
&\quad \vdots\\
&\quad \frac{l_{N-2}!!(l_{N-1}+l_N+1)!!}{(l_{N-2}+l_{N-1}+l_N+2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(1)}\\
&\quad \frac{l_{N-1}!!l_N!!}{(l_{N-1}+l_N+1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(0)}
\end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a $(N-1)!!$ factor, and that the $\frac{\pi}{2}$ factors multiply up to $F=\left(\frac{\pi}{2}\right)^{[N/2]}$. Thus by multiplying with the normalization constant, we obtain the result. □

In connection now with our probabilistic questions, we have:

Theorem 8.4. *The even moments of the hyperspherical variables are*

$$\int_{S^{N-1}_\mathbb{R}} x_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $y_i = x_i/\sqrt{N}$ become normal and independent with $N\to\infty$.

Proof. The moment formula in the statement follows from Theorem 8.3. Now observe that with $N\to\infty$ we have the following estimate:

$$\begin{aligned}
\int_{S^{N-1}_\mathbb{R}} x_i^k dx &= \frac{(N-1)!!}{(N+k-1)!!}\times k!!\\
&\simeq N^{k/2}\times k!!\\
&= N^{k/2}M_k(g_1)
\end{aligned}$$

Thus, we have $x_i/\sqrt{N}\sim g_1$, as claimed. Finally, the independence assertion follows as well from the formula in Theorem 8.3, via standard probability theory. □

In the case of the free sphere now, from [37], the computations are substantially more complicated. Let us start with the following result, that we know from section 5:

Theorem 8.5. *For the free sphere $S^{N-1}_{\mathbb{R},+}$, the rescaled coordinates*

$$y_i = \sqrt{N}x_i$$

become semicircular and free, in the $N\to\infty$ limit.

Proof. As explained in section 5 above, the Weingarten formula for the free sphere, together with the standard fact that the Gram matrix, and hence the Weingarten matrix too, is asymptotically diagonal, gives the following estimate:

$$\int_{S^{N-1}_{\mathbb{R},+}} x_{i_1}\dots x_{i_k}\,dx \simeq N^{-k/2}\sum_{\sigma\in NC_2(k)}\delta_\sigma(i_1,\dots,i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i. Indeed, by setting $i_1=\dots=i_k=i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S^{N-1}_{\mathbb{R},+}} x_i^k\,dx \simeq N^{-k/2}|NC_2(k)|$$

Thus the rescaled coordinates $y_i=\sqrt{N}x_i$ become semicircular in the $N\to\infty$ limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [26]. □

The problem now, which is highly non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of $N\in\mathbb{N}$. The answer here, from [30], based on advanced quantum group and calculus techniques, is as follows:

Theorem 8.6. *The moments of the free hyperspherical law are given by*

$$\int_{S^{N-1}_{\mathbb{R},+}} x_1^{2l}\,dx=\frac{1}{(N+1)^l}\cdot\frac{q+1}{q-1}\cdot\frac{1}{l+1}\sum_{r=-l-1}^{l+1}(-1)^r\binom{2l+2}{l+r+1}\frac{r}{1+q^r}$$

where $q\in[-1,0)$ is such that $q+q^{-1}=-N$.

Proof. The idea is that $x_1\in C(S^{N-1}_{\mathbb{R},+})$ has the same law as $u_{11}\in C(O_N^+)$, which has the same law as a certain variable $w\in C(SU_2^q)$, which can be in turn modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F\in GL_N(\mathbb{R})$ satisfying $F^2=1$ we associate the following universal algebra:

$$C(O_F^+)=C^*\left((u_{ij})_{i,j=1,\dots,N}\Big|u=F\bar{u}F=\text{unitary}\right)$$

Observe that $O_{I_N}^+=O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [147], which do not include the strong antipode axiom $S^2=id$.

At $N=2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q\in[-1,0)$:

$$F=\begin{pmatrix}0&\sqrt{-q}\\1/\sqrt{-q}&0\end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+=SU_2^q$$

Indeed, the relations $u=F\bar{u}F$ tell us that u must be of the following special form:

$$u=\begin{pmatrix}\alpha&-q\gamma^*\\ \gamma&\alpha^*\end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α,γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma=q\gamma\alpha$$

$$\alpha\gamma^*=q\gamma^*\alpha$$

$$\gamma\gamma^*=\gamma^*\gamma$$

$$\alpha^*\alpha+\gamma^*\gamma=1$$

$$\alpha\alpha^*+q^2\gamma\gamma^*=1$$

We recognize here the relations in [147] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+)\simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi\in NC_2(2k)$ and $i=(i_1,\ldots,i_{2k})$, we set:

$$\delta_\pi^F(i)=\prod_{s\in\pi}F_{i_{s_l}i_{s_r}}$$

Here the product is over all strings $s=\{s_l\curvearrowright s_r\}$ of π. Our claim is that the following family of vectors, with $\pi\in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi=\sum_i\delta_\pi^F(i)e_{i_1}\otimes\ldots\otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u=F\bar{u}F$ is unitary.

By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+}u_{i_1j_1}\ldots u_{i_{2k}j_{2k}}=\sum_{\pi\sigma}\delta_\pi^F(i)\delta_\sigma^F(j)W_{kN}(\pi,\sigma)$$

With these preliminaries in hand, let us start the computation. Let $N\in\mathbb{N}$, and consider the number $q\in[-1,0)$ satisfying:

$$q+q^{-1}=-N$$

Our claim is that we have:

$$\int_{O_N^+}\varphi(\sqrt{N+2}\,u_{ij})=\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+} u_{ij}^{2k} = \sum_{\pi\sigma} W_{kN}(\pi,\sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $v=(v_{ij})$ is given by $w=\sum_{ij} v_{ij}$, are given by:

$$\int_{SU_2^q} w^{2k} = \sum_{ij}\sum_{\pi\sigma} \delta_\pi^F(i)\delta_\sigma^F(j) W_{kN}(\pi,\sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as u_{ij}, which proves our claim.

In order to do now the computation over SU_2^q, we can use a matrix model due to Woronowicz [146], where the standard generators α,γ are mapped as follows:

$$\begin{aligned}\pi_u(\alpha)e_k &= \sqrt{1-q^{2k}}e_{k-1}\\ \pi_u(\gamma)e_k &= uq^k e_k\end{aligned}$$

Here $u\in\mathbb{T}$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb{N})$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D is the diagonal operator given by $D(e_k)=q^{2k}e_k$, then the formula is as follows:

$$\int_{SU_2^q} x = (1-q^2)\int_{\mathbb{T}} tr(D\pi_u(x))\frac{du}{2\pi iu}$$

With the above model in hand, the law of the variable that we are interested in is of the following form:

$$\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*) = (1-q^2)\int_{\mathbb{T}} tr(D\varphi(M))\frac{du}{2\pi iu}$$

To be more precise, this formula holds indeed, with:

$$M(e_k) = e_{k+1} + q^k(u-qu^{-1})e_k + (1-q^{2k})e_{k-1}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. We refer here to [30]. □

Following now [22], let us discuss the free hypergeometric laws. We will use here a twisting result established in section 4 above, which is also from [22]. We know from that twisting result that we have, at the probabilistic level:

Theorem 8.7. *The following two algebras are isomorphic, via $u_{ij}^2\to X_{ij}$,*

(1) *The algebra generated by the variables $u_{ij}^2\in C(O_n^+)$,*
(2) *The algebra generated by $X_{ij}=\frac{1}{n}\sum_{a,b=1}^n p_{ia,jb}\in C(S_{n^2}^+)$,*

and this isomorphism commutes with the respective Haar integration functionals.

Proof. This follows indeed from the general twisting result from section 4. □

As pointed out in [22], it is possible to derive as well this result directly, by using the Weingarten formula, and manipulations on the partitions:

Theorem 8.8. *The following families of variables have the same joint law,*

(1) $\{u_{ij}^2\}\in C(O_n^+)$,
(2) $\{X_{ij}=\frac{1}{n}\sum_{ab}p_{ia,jb}\}\in C(S_{n^2}^+)$,

where $u=(u_{ij})$ and $p=(p_{ia,jb})$ are the corresponding fundamental corepresentations.

Proof. As already mentioned, this result can be obtained via twisting methods. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation $\pi\to\pi'$. Indeed, we obtain the following moment formulae:

$$\begin{aligned}\int_{O_n^+}u_{ij}^{2k}&=\sum_{\pi,\sigma\in NC_2(2k)}W_{2k,n}(\pi,\sigma)\\ \int_{S_{n^2}^+}X_{ij}^k&=\sum_{\pi,\sigma\in NC_2(2k)}n^{|\pi'|+|\sigma'|-k}W_{k,n^2}(\pi',\sigma')\end{aligned}$$

According to the fattening results in section 2 the summands coincide, and so the moments are equal, as desired. The proof for joint moments is similar. □

In what follows we will be interested in single variables. We have here:

Definition 8.9. *The noncommutative random variable*

$$X(n,m,N)=\sum_{i=1}^n\sum_{j=1}^m u_{ij}\in C(S_N^+)$$

is called free hypergeometric, of parameters (n,m,N).

The terminology comes from the fact that the variable $X'(n,m,N)$, defined as above, but over the algebra $C(S_N)$, follows a hypergeometric law of parameters (n,m,N). Following [22], here is an exploration of the basic asymptotic properties of these laws:

Theorem 8.10. *The free hypergeometric laws have the following properties:*

(1) *Let $n,m,N\to\infty$, with $\frac{nm}{N}\to t\in(0,\infty)$. Then the law of $X(n,m,N)$ converges to Marchenko-Pastur law π_t.*
(2) *Let $n,m,N\to\infty$, with $\frac{n}{N}\to\nu\in(0,1)$ and $\frac{m}{N}\to 0$. Then the law of $S(n,m,N)=(X(n,m,N)-m\nu)/\sqrt{m\nu(1-\nu)}$ converges to the semicircle law γ_1.*

Proof. This is standard, by using the Weingarten formula, as follows:

(1) From the Weingarten formula, we have:

$$\int X(n,m,N)^k=\sum_{\pi,\sigma\in NC(k)}W_{kN}(\pi,\sigma)n^{|\pi|}m^{|\sigma|}$$

The point now is that we have the following estimate:

$$W_{kN}(\pi,\sigma)=\begin{cases}N^{-|\pi|}+O(N^{-|\pi|-1}) & \text{if } \pi=\sigma\\ O(N^{|\pi\vee\sigma|-|\pi|-|\sigma|}) & \text{if } \pi\neq\sigma\end{cases}$$

It follows that we have:

$$W_{kN}(\pi,\sigma)n^{|\pi|}m^{|\sigma|}\to\begin{cases}t^{|\pi|} & \text{if } \pi=\sigma\\ 0 & \text{if } \pi\neq\sigma\end{cases}$$

Thus the k-th moment of $X(n,m,N)$ converges to $\sum_{\pi\in NC(k)}\lambda^{|\pi|}$, which is the k-th moment of the Marchenko-Pastur law π_t, and we are done.

(2) We need to show that the free cumulants satisfy:

$$\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\to\begin{cases}1 & \text{if } p=2\\ 0 & \text{if } p\neq 2\end{cases}$$

The case $p=1$ is trivial, so suppose $p\geq 2$. We have:

$$\begin{aligned}&\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\\ =\ &(m\nu(1-\nu))^{-p/2}\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]\end{aligned}$$

On the other hand, from the Weingarten formula, we have:

$$\begin{aligned}&\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]\\ =\ &\sum_{w\in NC(p)}\mu_p(w,1_p)\prod_{V\in w}\sum_{\pi_V,\sigma_V\in NC(V)}W_{NC(V),N}(\pi_V,\sigma_V)n^{|\pi_V|}m^{|\sigma_V|}\\ =\ &\sum_{w\in NC(p)}\mu_p(w,1_p)\prod_{V\in w}\sum_{\pi_V,\sigma_V\in NC(V)}(N^{-|\pi_V|}\mu_{|V|}(\pi_V,\sigma_V)+O(N^{-|\pi_V|-1}))n^{|\pi_V|}m^{|\sigma_V|}\\ =\ &\sum_{\substack{\pi,\sigma\in NC(p)\\ \pi\leq\sigma}}(N^{-|\pi|}\mu_p(\pi,\sigma)+O(N^{-|\pi|-1}))n^{|\pi|}m^{|\sigma|}\sum_{\substack{w\in NC(p)\\ \sigma\leq w}}\mu_p(w,1_p)\end{aligned}$$

We use now the following standard identity:

$$\sum_{\substack{w\in NC(p)\\ \sigma\leq w}}\mu_p(w,1_p)=\begin{cases}1 & \text{if } \sigma=1_p\\ 0 & \text{if } \sigma\neq 1_p\end{cases}$$

This gives the following formula for the cumulants:

$$\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]=m\sum_{\pi\in NC(p)}(N^{-|\pi|}\mu_p(\pi,1_p)+O(N^{-|\pi|-1}))n^{|\pi|}$$

It follows that for $p\geq 3$ we have, as desired:

$$\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\to 0$$

As for the remaining case $p=2$, here we have:

$$\begin{aligned}\kappa^{(2)}[S(n,m,N),S(n,m,N)] &\to \frac{1}{\nu(1-\nu)}\sum_{\pi\in NC(2)}\nu^{|\pi|}\mu_2(\pi,1_2)\\ &= \frac{1}{\nu(1-\nu)}(\nu-\nu^2)\\ &= 1\end{aligned}$$

This gives the result. □

As a final analytic topic, let us discuss now, following [33], the computation of the asymptotic laws of powers $Tr(u^k)$ with $k\in\mathbb N$, called Diaconis-Shahshahani variables, following the paper [86], generalizing the usual characters $\chi=Tr(u)$. In order to do our computation, let us start with the following standard definition:

Definition 8.11. *Associated to $k_1,\ldots,k_s\in\mathbb N$ is the trace permutation $\gamma\in S_k$, with $k=\sum k_i$, having cycles*

$$\begin{gathered}(1,\ldots,k_1)\\ (k_1+1,\ldots,k_1+k_2)\\ \vdots\\ (k-k_s+1,\ldots,k)\end{gathered}$$

called trace permutation associated to $k_1,\ldots,k_s\in\mathbb N$.

We denote by $\gamma(\sigma)$ the partition given by:

$$i\sim_\sigma j\iff \gamma(i)\sim_{\gamma(\sigma)}\gamma(j)$$

With these conventions, we have the following result:

Theorem 8.12. *Given an easy quantum group G, we have:*

$$\int_G Tr(u^{k_1})\ldots Tr(u^{k_s})\,du=\#\left\{\pi\in D_k\Big|\pi=\gamma(\pi)\right\}+O(N^{-1})$$

If G is classical, this estimate is exact, without any lower order corrections.

Proof. We have two assertions to be proved, the idea being as follows:

(1) Let I be the integral to be computed. According to the definition of γ, we have:

$$\begin{aligned}I &= \int_G Tr(u^{k_1})\ldots Tr(u^{k_s})\,du\\ &= \sum_{i_1\ldots i_k}\int_G (u_{i_1i_2}\ldots u_{i_ki_1})\ldots(u_{i_{k-k_s+1}i_{k-k_s+2}}\ldots u_{i_ki_{k-k_s+1}})\\ &= \sum_{i_1\ldots i_k}\int_G u_{i_1i_{\gamma(1)}}\ldots u_{i_ki_{\gamma(k)}}\end{aligned}$$

We use now the Weingarten formula. We obtain:

$$\begin{aligned} I &= \sum_{i_1\ldots i_k}\sum_{\pi\leq\ker i}\sum_{\sigma\leq\ker i\gamma} W_{kN}(\pi,\sigma) \\ &= \sum_{i_1\ldots i_k}\sum_{\pi\leq\ker i}\sum_{\gamma(\sigma)\leq\ker i} W_{kN}(\pi,\sigma) \\ &= \sum_{\pi,\sigma\in D_k} N^{|\pi\vee\gamma(\sigma)|}W_{kN}(\pi,\sigma) \\ &= \sum_{\pi,\sigma\in D_k} N^{|\pi\vee\gamma(\sigma)|}N^{|\pi\vee\sigma|-|\pi|-|\sigma|}(1+O(N^{-1})) \end{aligned}$$

The leading order of $N^{|\pi\vee\gamma(\sigma)|+|\pi\vee\sigma|-|\pi|-|\sigma|}$ is N^0, which is achieved if and only if $\sigma\geq\pi$ and $\pi\geq\gamma(\sigma)$, or equivalently when $\pi=\sigma=\gamma(\sigma)$. But this gives the formula.

(2) In the classical case, instead of using the approximation for $W_{Nk}(\pi,\sigma)$, we can write $N^{|\pi\vee\gamma(\sigma)|}=G_{kN}(\gamma(\sigma),\pi)$, and we can continue as follows:

$$\begin{aligned} I &= \sum_{\pi,\sigma\in D_k} G_{kN}(\gamma(\sigma),\pi)W_{kN}(\pi,\sigma) \\ &= \sum_{\sigma\in D_k}\delta(\gamma(\sigma),\sigma) \\ &= \#\{\sigma\in D_k|\sigma=\gamma(\pi)\} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

If c is a cycle we use the notation $c^1=c$, and c^*= cycle opposite to c. We have the following definition, generalizing Definition 8.11 above:

Definition 8.13. *Associated to any $k_1,\ldots,k_s\in\mathbb N$ and any $e_1,\ldots,e_s\in\{1,*\}$ is the trace permutation $\gamma\in S_k$, with $k=\sum k_i$, having as cycles*

$$(1,\ldots,k_1)^{e_1}$$
$$(k_1+1,\ldots,k_1+k_2)^{e_2}$$
$$\vdots$$
$$(k-k_s+1,\ldots,k)^{e_s}$$

called trace permutation associated to $k_1,\ldots,k_s\in\mathbb N$ and $e_1,\ldots,e_s\in\{1,*\}$.

With this convention, Theorem 8.12 extends to this setting, as follows:

Theorem 8.14. *Given an easy quantum group G, we have:*

$$\int_G Tr(u^{k_1})^{e_1}\ldots Tr(u^{k_s})^{e_s}\,du=\#\left\{\pi\in D_k\Big|\pi=\gamma(\pi)\right\}+O(N^{-1})$$

If G is classical, this estimate is exact, without any lower order corrections.

Proof. This is similar to the proof of Theorem 8.12. □

In terms of cumulants, we have the following result, also from [33]:

Theorem 8.15. *For $G = O_N, S_N$ we have the following cumulant formula:*
$$c_r(Tr(u^{k_1})^{e_1},\ldots,Tr(u^{k_r})^{e_r}) = \#\left\{\pi\in D_k\Big|\pi\vee\gamma=1_k,\,\pi=\gamma(\pi)\right\}$$
Also, for $G = O_N^+, S_N^+$ we have the following free cumulant formula:
$$\kappa_r(Tr(u^{k_1})^{e_1},\ldots,Tr(u^{k_r})^{e_r}) = \#\left\{\pi\in D_k\Big|\pi\vee\gamma=1_k,\,\pi=\gamma(\pi)\right\}+O(N^{-1})$$

Proof. We have two assertions to be proved, the idea being as follows:

(1) Let c_r be the considered cumulant. We write, for those partitions $\pi\in P_k$ such that the restriction of π to a block of σ is an element in the corresponding set $D_{|v|}$:
$$D_\sigma = \left\{\pi\in P_k\Big|p_{|v}\in D_{|v|}\,\forall v\in\sigma\right\}$$

We have then the following equivalent formula:
$$D_\sigma = \left\{\pi\in D_k\Big|\pi\leq\sigma^\gamma\right\}$$

Then, by the definition of the classical cumulants via Möbius inversion, we get:
$$\begin{aligned}
c_r &= \sum_{\sigma\in P(r)}\mu(\sigma,1_r)\cdot\#\{\pi\in D_\sigma|\pi=\gamma(\pi)\}\\
&= \sum_{\sigma\in P(r)}\mu(\sigma,1_r)\cdot\#\{\pi\in D_k|\pi\leq\sigma^\gamma,\pi=\gamma(\pi)\}\\
&= \sum_{\sigma\in P(r)}\mu(\sigma,1_r)\sum_{\pi\leq\sigma^\gamma,\pi=\gamma(\pi)}1
\end{aligned}$$

In order to exchange the two summations, we first have to replace the summation over $\sigma\in P(r)$ by a summation over $\tau=\sigma^\gamma\in P(k)$. Note that the condition on the latter is exactly $\tau\geq\gamma$ and that we have $\mu(\sigma,1_r)=\mu(\sigma^\gamma,1_k)$. Thus:
$$\begin{aligned}
c_r &= \sum_{\tau\geq\gamma}\mu(\tau,1_k)\sum_{\pi\leq\tau,\pi=\gamma(\pi)}1\\
&= \sum_{\pi=\gamma(\pi)}\sum_{\pi\vee\gamma\leq\tau}\mu(\tau,1_k)
\end{aligned}$$

The definition of the Möbius function gives for the second summation:
$$\sum_{\pi\vee\gamma\leq\tau}\mu(\tau,1_k)=\begin{cases}1&\text{if }\pi\vee\gamma=1_k\\0&\text{otherwise}\end{cases}$$

With this formula in hand, the assertion follows.

(2) In the free case, the proof runs in the same way, by using free cumulants and the corresponding Möbius function on noncrossing partitions. Note that we have the analogue of our equation in this case only for noncrossing σ. $\square$

We can now recover the theorem of Diaconis and Shahshahani in [86]:

Theorem 8.16. *The variables $u_k = \lim_{N\to\infty} Tr(u^k)$ are as follows:*

(1) *For O_N, the u_k are real Gaussian variables, with variance k and mean 0 or 1, depending on whether k is odd or even. The u_k are independent.*
(2) *For O_N^+, at $k = 1, 2$ we get semicircular variables of variance 1 and mean 0 for u_1 and mean 1 for u_2, and at $k \geq 3$ we get circular variables of mean 0 and covariance 1. The u_k are $*$-free.*

Proof. This follows by using the formula in Theorem 8.15, as follows:

(1) In this case D_k consists of all pairings of k elements. We have to count all pairings π with the properties that $\pi \vee \gamma = 1_k$ and $\pi = \gamma(\pi)$.

Note that if π connects two different cycles of γ, say c_i and c_j, then the property $\pi = \gamma(p)$ implies that each element from c_i must be paired with an element from c_j. Thus those cycles cannot be connected to other cycles and they must contain the same number of elements. This means that for $s \geq 3$ there is no such π. Thus all cumulants of order 3 and higher vanish asymptotically and all traces are asymptotically Gaussian.

Since in the case $s = 2$ we only have permissible pairings if the two cycles have the same number of elements, we also see that the covariance between traces of different powers vanishes and thus different powers are asymptotically independent. The variance of u_k is given by the number of matchings between $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ which are invariant under rotations. Since such a matching is determined by the partner of the first element 1, for which we have k possibilities, the variance of u_k is k. For the mean, if k is odd there is clearly no pairing at all, and if $k = 2p$ is even then the only pairing of $\{1, \ldots, 2p\}$ which is invariant under rotations is $(1, p+1), (2, p+2), \ldots, (p, 2p)$. Thus the mean of u_k is zero if k is odd and 1 if k is even.

(2) In the quantum case D_k consists of noncrossing pairings. We can essentially repeat the arguments from above but have to take care that only noncrossing pairings are counted. We also have to realize that for $k \geq 3$, the u_k are not selfadjoint any longer, thus we have to consider also u_k^* in these cases. This means that in our arguments we have to allow cycles which are rotated "backwards" under γ.

By the same reasoning as before we see that free cumulants of order three and higher vanish. The pairing which gave mean 1 in the classical case is only in the case $k = 2$ a noncrossing one, thus the mean of u_2 is 1, all other means are zero. For the variances, one has again that different powers allow no pairings at all and are asymptotically $*$-free. For the matchings between $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ one has to observe that there is only one non-crossing possibility, namely $(1,2k),(2,2k-1),\ldots,(k,k+1)$ and this satisfies $\pi=\gamma(\pi)$ only if γ rotates both cycles in different directions.

For $k=1$ and $k=2$ there is no difference between both directions, but for $k\geq 3$ this implies that we get only a non-vanishing covariance between u_k and u_k^*, with value 1. This shows that u_1 and u_2 are semicircular, whereas the higher u_k are circular. □

In order to discuss permutations and quantum permutations, let us start with:

Proposition 8.17. *The cumulants of $u_k=\lim_{N\to\infty}Tr(u^k)$ are as follows:*

(1) *For S_N, the classical cumulants are given by:*

$$c_r(u_{k_1},\ldots,u_{k_r})=\sum_{q|k_i\forall i=1,\ldots,r}q^{r-1}$$

(2) *For S_N^+, the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1},\ldots,u_{k_r}^{e_r})=\begin{cases}2&\text{if } r=1,\ k_1\geq 2\\ 2&\text{if } r=2,\ k_1=k_2,\ e_1=e_2^*\\ 2&\text{if } r=2,\ k_1=k_2=2\\ 1&\text{otherwise}\end{cases}$$

Proof. We have two assertions to be proved, the idea being as follows:

(1) Here D_k consists of all partitions. We have to count partitions π which have the properties that $\pi\vee\gamma=1_k$ and $\pi=\gamma(\pi)$.

Consider a partition π which connects different cycles of γ. Consider the restriction of π to one cycle. Let k be the number of elements in this cycle and t be the number of the points in the restriction. Then the orbit of those t points under γ must give a partition of that cycle, which means that t is a divisor of k and that the t points are equally spaced. The same must be true for all cycles of γ which are connected via π, and the ratio between t and k is the same for all those cycles. This means that if one block of π connects some cycles then the orbit under γ of this block connects exactly those cycles and exhausts all points of those cycles. So if we want to connect all cycles of γ then this can only happen in the way that we have one block of π intersecting each of the cycles of γ.

To be more precise, let us consider $c_r(u_{k_1},\ldots,u_{k_r})$. We have then to look for a common divisor q of all $k_1,\ldots,k_r$, and a contributing π is then one the blocks of which are of the following form: k_1/q points in the first cycle, equally spaced, and so on up to k_r/q points in the last cycle, equally spaced. We can specify this by saying to which points in the other cycles the first point in the first cycle is connected. There are q^{r-1} possibilities for such choices, and this gives the formula in the statement.

(2) For S_N^+ we have to consider noncrossing partitions instead of all partitions. Most of the contributing partitions from the classical case are crossing, so do not count for

the quantum case. Actually, whenever a restriction of a block to one cycle has two or more elements then the corresponding partition is crossing, unless the restriction exhausts the whole group. This is the case $q = 1$ from the considerations above, corresponding to the partition which has only one block, giving a contribution 1 to each cumulant $c_r(u_{k_1}, \ldots, u_{k_r})$. For cumulants of order 3 or higher there are no other contributions. For cumulants of second order one might also have contributions coming from pairings, where each restriction of a block to a cycle has one element. But this is the same problem as in the O_N^+ case, and we only get an additional contribution for the second order cumulants $c_2(u_k, u_k^*)$. For first order cumulants, singletons can also appear and make an additional contribution. Taking this all together gives the formula in the statement. □

We can now formulate a result about permutations and quantum permutations:

Theorem 8.18. *The variables $u_k = \lim_{N\to\infty} Tr(u^k)$ are as follows:*

(1) *For S_N we have a decomposition of type*

$$u_k = \sum_{l|k} lC_l$$

with the variables C_k being Poisson of parameter $1/k$, and independent.

(2) *For S_N^+ we have a decomposition of the type*

$$u_1 = C_1 \quad , \quad u_k = C_1 + C_k \quad (k \geq 2)$$

where the variables C_l are $*$-free, C_1 is free Poisson, C_2 is semicircular, and C_k with $k \geq 3$ are circular.

Proof. We have several assertions to be proved, the idea being as follows:

(1) Let C_k be the number of cycles of length k. We have $u_k = \sum_{l|k} lC_l$. We are claiming now that the C_k are independent and each is a Poisson variable of parameter $1/k$, i.e., that $c_r(C_{l_1}, \ldots, C_{l_r})$ is zero unless all the l_i are the same, say $= l$, in which case it is $1/l$, independently of r. This is compatible with the cumulants for the u_k, according to:

$$\begin{aligned} c_r(u_{k_1}, \ldots, u_{k_r}) &= \sum_{l_1|k_1} \ldots \sum_{l_r|k_r} l_1 \ldots l_r c_r(C_{l_1}, \ldots, C_{l_r}) \\ &= \sum_{l|k_i \forall i} l^r \, \frac{1}{l} \end{aligned}$$

Since the C_k are uniquely determined by the u_k, via some kind of Möbius inversion, this shows that the C_k are independent, and that C_k is Poisson with parameter $1/k$.

(2) In the classical case the random variable C_l can be defined by:

$$C_l = \frac{1}{l} \sum_{i_1 \ldots i_l \text{ distinct}} u_{i_1 i_2} u_{i_2 i_3} \ldots u_{i_l i_1}$$

Note that we divide by l because each term appears actually l times, in cyclically permuted versions, which are all the same because our variables commute.

Note that, by using commutativity and the monomial condition, in general the expression $u_{i_1i_2}u_{i_2i_3}\ldots u_{i_ki_1}$ has to be zero unless the indices $(i_1,\ldots,i_k)$ are of the form $(i_1,\ldots,i_l,i_1,\ldots,i_l,\ldots)$, where l divides k and $i_1,\ldots,i_l$ are distinct. This yields then the following relation, which we used before to define C_l:

$$\begin{aligned}
Tr(u^k) &= \sum_{i_1\ldots i_l} u_{i_1i_2}u_{i_2i_3}\ldots u_{i_li_1}\\
&= \sum_{l|k}\sum_{i_1\ldots i_l\ \text{distinct}} (u_{i_1i_2}u_{i_2i_3}\ldots u_{i_li_1})^{k/l}\\
&= \sum_{l|k} lC_l
\end{aligned}$$

This explicit form of C_l in terms of u_{ij} can be used to give a direct proof, by using the Weingarten formula, of the fact that the C_l are independent and Poisson.

(3) In the free case we define the "cycle" C_l by requiring neighboring indices to be different, as follows:

$$C_l = \sum_{i_1\neq i_2\neq\ldots\neq i_l\neq i_1} u_{i_1i_2}u_{i_2i_3}\ldots u_{i_li_1}$$

Note that if two adjacent indices are the same in $u_{i_1i_2}u_{i_2i_3}\ldots u_{i_li_1}$ then, because of the relation $u_{ij}u_{ik}=0$ for $j\neq k$, all must be the same or the term vanishes. For the case where all indices are the same we have:

$$\sum_i u_{ii}u_{ii}\ldots u_{ii} = \sum_i u_{ii} = C_1$$

But this gives then the following relation:

$$Tr(u^k) = C_k + C_1$$

Again, the C_l are uniquely determined by the $Tr(u^k)$ and thus our calculations also show that the C_l defined by our equation are $*$-free and have the distributions as stated. □

9. **Finite graphs**

Many interesting examples of quantum permutation groups appear as particular cases of the following general construction from [6], [7], involving finite graphs:

Proposition 9.1. *Given a finite graph X, with adjacency matrix $d \in M_N(0,1)$, the following construction produces a quantum permutation group,*

$$C(G^+(X)) = C(S_N^+)\Big/\Big\langle du = ud\Big\rangle$$

whose classical version $G(X)$ is the usual automorphism group of X.

Proof. The fact that we have a quantum group comes from the fact that $du = ud$ reformulates as $d \in End(u)$, which makes it clear that we are dividing by a Hopf ideal. Regarding the second assertion, we must establish here the following equality:

$$C(G(X)) = C(S_N)\Big/\Big\langle du = ud\Big\rangle$$

For this purpose, recall that we have $u_{ij}(\sigma) = \delta_{\sigma(j)i}$. By using this formula, we have:

$$(du)_{ij}(\sigma) = \sum_k d_{ik}u_{kj}(\sigma) = \sum_k d_{ik}\delta_{\sigma(j)k} = d_{i\sigma(j)}$$

On the other hand, we have as well:

$$(ud)_{ij}(\sigma) = \sum_k u_{ik}(\sigma)d_{kj} = \sum_k \delta_{\sigma(k)i}d_{kj} = d_{\sigma^{-1}(i)j}$$

Thus the condition $du = ud$ reformulates as $d_{ij} = d_{\sigma(i)\sigma(j)}$, and we are led to the usual notion of an action of a permutation group on X, as claimed. □

Let us work out some basic examples. We have the following result:

Theorem 9.2. *The construction $X \to G^+(X)$ has the following properties:*

(1) *For the N-point graph, having no edges at all, we obtain S_N^+.*
(2) *For the N-simplex, having edges everywhere, we obtain as well S_N^+.*
(3) *We have $G^+(X) = G^+(X^c)$, where X^c is the complementary graph.*
(4) *For a disconnected union, we have $G^+(X) \,\hat{*}\, G^+(Y) \subset G^+(X \sqcup Y)$.*
(5) *For the square, we obtain a non-classical, proper subgroup of S_4^+.*

Proof. All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have $d = 0$.

(2) Here $d = \mathbb{I}$ is the all-one matrix, and the magic condition gives $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$. We conclude that $du = ud$ is automatic in this case, and so $G^+(X) = S_N^+$.

(3) The adjacency matrices of X, X^c being related by the formula $d_X + d_{X^c} = \mathbb{I}$. We can use here the above formula $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$, and we conclude that $d_Xu = ud_X$ is equivalent to $d_{X^c}u = ud_{X^c}$. Thus, we obtain, as claimed, $G^+(X) = G^+(X^c)$.

(4) The adjacency matrix of a disconnected union is given by $d_{X\sqcup Y}=diag(d_X,d_Y)$. Now let $w=diag(u,v)$ be the fundamental corepresentation of $G^+(X)\,\hat{*}\,G^+(Y)$. Then $d_Xu=ud_X$ and $d_Yv=vd_Y$ imply, as desired, $d_{X\sqcup Y}w=wd_{X\sqcup Y}$.

(5) We know from (3) that we have $G^+(\square)=G^+(|\ |)$. We know as well from (4) that we have $\mathbb{Z}_2\,\hat{*}\,\mathbb{Z}_2\subset G^+(|\ |)$. It follows that $G^+(\square)$ is non-classical. Finally, the inclusion $G^+(\square)\subset S_4^+$ is indeed proper, because $S_4\subset S_4^+$ does not act on the square. $\square$

In order to further advance, and to explicitely compute various quantum automorphism groups, we can use the spectral decomposition of d, as follows:

Proposition 9.3. *A closed subgroup $G\subset S_N^+$ acts on a graph X precisely when*
$$P_\lambda u=uP_\lambda\quad,\quad\forall\lambda\in\mathbb{R}$$
where $d=\sum_\lambda\lambda\cdot P_\lambda$ is the spectral decomposition of the adjacency matrix of X.

Proof. Since $d\in M_N(0,1)$ is a symmetric matrix, we can consider indeed its spectral decomposition, $d=\sum_\lambda\lambda\cdot P_\lambda$. We have then the following formula:
$$<d>=span\left\{P_\lambda\Big|\lambda\in\mathbb{R}\right\}$$

But this shows that we have the following equivalence:
$$d\in End(u)\iff P_\lambda\in End(u),\forall\lambda\in\mathbb{R}$$

Thus, we are led to the conclusion in the statement. $\square$

In order to exploit this, we will often combine it with the following standard fact:

Proposition 9.4. *Consider a closed subgroup $G\subset S_N^+$, with associated coaction map*
$$\Phi:\mathbb{C}^N\to\mathbb{C}^N\otimes C(G)$$
For a linear subspace $V\subset\mathbb{C}^N$, the following are equivalent:

1. *The magic matrix $u=(u_{ij})$ commutes with P_V.*
2. *V is invariant, in the sense that $\Phi(V)\subset V\otimes C(G)$.*

Proof. Let $P=P_V$. For any $i\in\{1,\ldots,N\}$ we have the following formula:
$$\Phi(P(e_i))=\Phi\left(\sum_kP_{ki}e_k\right)=\sum_{jk}P_{ki}e_j\otimes u_{jk}=\sum_je_j\otimes(uP)_{ji}$$

On the other hand the linear map $(P\otimes id)\Phi$ is given by a similar formula:
$$(P\otimes id)(\Phi(e_i))=\sum_kP(e_k)\otimes u_{ki}=\sum_{jk}P_{jk}e_j\otimes u_{ki}=\sum_je_j\otimes(Pu)_{ji}$$

Thus $uP=Pu$ is equivalent to $\Phi P=(P\otimes id)\Phi$, and the conclusion follows. $\square$

We have as well the following useful complementary result, from [6]:

Proposition 9.5. *Let $p \in M_N(\mathbb{C})$ be a matrix, and consider its "color" decomposition, obtained by setting $(p_c)_{ij} = 1$ if $p_{ij} = c$ and $(p_c)_{ij} = 0$ otherwise:*

$$p = \sum_{c \in \mathbb{C}} c \cdot p_c$$

Then $u = (u_{ij})$ commutes with p if and only if it commutes with all matrices p_c.

Proof. Consider the multiplication and counit maps of the algebra $\mathbb{C}^N$:

$$M : e_i \otimes e_j \to e_i e_j$$

$$C : e_i \to e_i \otimes e_i$$

Since M, C intertwine $u, u^{\otimes 2}$, their iterations $M^{(k)}, C^{(k)}$ intertwine $u, u^{\otimes k}$, and so:

$$p^{(k)} = M^{(k)} p^{\otimes k} C^{(k)} = \sum_{c \in \mathbb{C}} c^k p_c \in End(u)$$

Let $S = \{c \in \mathbb{C} | p_c \neq 0\}$, and $f(c) = c$. By Stone-Weierstrass we have $S =< f >$, and so for any $e \in S$ the Dirac mass at e is a linear combination of powers of f:

$$\delta_e = \sum_k \lambda_k f^k = \sum_k \lambda_k \left(\sum_{c \in S} c^k \delta_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) \delta_c$$

The corresponding linear combination of matrices $p^{(k)}$ is given by:

$$\sum_k \lambda_k p^{(k)} = \sum_k \lambda_k \left(\sum_{c \in S} c^k p_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) p_c$$

The Dirac masses being linearly independent, in the first formula all coefficients in the right term are 0, except for the coefficient of δ_e, which is 1. Thus the right term in the second formula is p_e, and it follows that we have $p_e \in End(u)$, as claimed. $\square$

The above results can be combined, and we are led to the following statement:

Theorem 9.6. *A closed subgroup $G \subset S_N^+$ acts on a graph X precisely when*

$$u = (u_{ij})$$

commutes with all the matrices coming from the color-spectral decomposition of d.

Proof. This follows by combining Proposition 9.3 and Proposition 9.5, with the "color-spectral" decomposition in the statement referring to what comes out by succesively doing the color and spectral decomposition, until the process stabilizes. $\square$

The above statement might seem in need of some further discussion, and axiomatization, in what regards the two operations used there. In answer to all this, the point is that we are in fact doing planar algebras. We have the following result, from [7]:

Theorem 9.7. *The planar algebra associated to $G^+(X)$ is equal to the planar algebra generated by d, viewed as a 2-box in the spin planar algebra $\mathcal{S}_N$, with $N = |X|$.*

Proof. We recall from section 3 above that any quantum permutation group $G \subset S_N^+$ produces a subalgebra $P \subset \mathcal{S}_N$ of the spin planar algebra, given by:

$$P_k = Fix(u^{\otimes k})$$

In our case, the idea is that $G = G^+(X)$ comes via the relation $d \in End(u)$, but we can view this relation, via Frobenius duality, as a relation of type:

$$\xi_d \in Fix(u^{\otimes 2})$$

Indeed, let us view the adjacency matrix $d \in M_N(0,1)$ as a 2-box in $\mathcal{S}_N$, by using the canonical identification between $M_N(\mathbb{C})$ and the algebra of 2-boxes $\mathcal{S}_N(2)$:

$$(d_{ij}) \leftrightarrow \sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix}$$

Let P be the planar algebra associated to $G^+(X)$ and let Q be the planar algebra generated by d. The action of $u^{\otimes 2}$ on d viewed as a 2-box is given by:

$$\begin{aligned} u^{\otimes 2}\left(\sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix}\right) &= \sum_{ijkl} d_{ij} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes u_{ki}u_{lj} \\ &= \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (udu^t)_{kl} \end{aligned}$$

Since v is a magic unitary commuting with d we have:

$$udu^t = duu^t = d$$

This means that d, viewed as a 2-box, is in the algebra P_2 of fixed points of $u^{\otimes 2}$. Thus we have $Q \subset P$. For $P \subset Q$ we use the duality found in section 3. Let indeed (B, v) be the pair whose associated planar algebra is Q. The same computation with v at the place of u shows that v commutes with d. Thus we have a morphism $A \to B$, given by $u_{ij} \to v_{ij}$, and it follows that we have $P \subset Q$, and we are done. $\square$

With the above results in hand, it is quite clear that our assumption that $d \in M_N(0,1)$ is the adjacency matrix of a usual graph X is somehow unnatural, and that we can look at more general objects. We can consider for instance general permutation quantum groups of the following type, depending on an arbitrary matrix $d \in M_N(\mathbb{C})$:

$$C(G^+(X)) = C(S_N^+) \Big/ \Big\langle du = ud \Big\rangle$$

Here X stands for the combinatorial object associated to d, namely the complete graph having as vertices $\{1, \ldots, N\}$, with each oriented edge $i \to j$ colored by $d_{ij} \in \mathbb{C}$. Generally speaking, the theory extends well to this setting, and we have analogues of the above results, some valid for any $d \in M_N(\mathbb{C})$, and some other valid under the asumption $d = d^*$. We refer to [7] and subsequent papers for a full discussion here.

With these issues discussed, so let us get back now to concrete things. As a basic application of the above results, following [6], we can further study $G^+(\square)$, as follows:

Theorem 9.8. *The quantum automorphism group of the N-cycle is as follows:*
(1) *At $N \neq 4$ we have $G^+(X) = D_N$.*
(2) *At $N = 4$ we have $D_4 \subset G^+(X) \subset S_4^+$, with proper inclusions.*

Proof. We aknow that the results hold at $N \leq 4$, so let us assume $N \geq 5$. Given a N-th root of unity, $w^N = 1$, the vector $\xi = (w^i)$ is an eigenvector of d, with eigenvalue:

$$\lambda = w + w^{N-1}$$

Now by taking $w = e^{2\pi i/N}$, it follows that the are eigenvectors of d are:

$$1, f, f^2, \ldots, f^{N-1}$$

More precisely, the invariant subspaces of d are as follows, with the last subspace having dimension 1 or 2 depending on the parity of N:

$$\mathbb{C}1,\ \mathbb{C}f \oplus \mathbb{C}f^{N-1},\ \mathbb{C}f^2 \oplus \mathbb{C}f^{N-2}, \ldots$$

Consider now the associated coaction $\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G)$, and write:

$$\Phi(f) = f \otimes a + f^{N-1} \otimes b$$

By taking the square of this equality we obtain:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2 + 1 \otimes (ab + ba)$$

It follows that $ab = -ba$, and that $\Phi(f^2)$ is given by the following formula:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2$$

By multiplying this with $\Phi(f)$ we obtain:

$$\Phi(f^3) = f^3 \otimes a^3 + f^{N-3} \otimes b^3 + f^{N-1} \otimes ab^2 + f \otimes ba^2$$

Now since $N \geq 5$ implies that $1, N-1$ are different from $3, N-3$, we must have $ab^2 = ba^2 = 0$. By using this and $ab = -ba$, we obtain by recurrence on k that:

$$\Phi(f^k) = f^k \otimes a^k + f^{N-k} \otimes b^k$$

In particular at $k = N-1$ we obtain:

$$\Phi(f^{N-1}) = f^{N-1} \otimes a^{N-1} + f \otimes b^{N-1}$$

On the other hand we have $f^* = f^{N-1}$, so by applying $*$ to $\Phi(f)$ we get:

$$\Phi(f^{N-1}) = f^{N-1} \otimes a^* + f \otimes b^*$$

Thus $a^*=a^{N-1}$ and $b^*=b^{N-1}$. Together with $ab^2=0$ this gives:

$$(ab)(ab)^*=abb^*a^*=ab^Na^{N-1}=(ab^2)b^{N-2}a^{N-1}=0$$

From positivity we get from this $ab=0$, and together with $ab=-ba$, this shows that a,b commute. On the other hand $C(G)$ is generated by the coefficients of Φ, which are powers of a,b, and so $C(G)$ must be commutative, and we obtain the result. $\square$

Summarizing, this was a bad attempt in understanding $G^+(\square)$, which appears to be "exceptional" among the quantum symmetry groups of the N-cycles. An alternative approach to $G^+(\square)$ comes by regarding the square as the $N=2$ particular case of the N-hypercube $\square_N$. Indeed, the usual symmetry group of $\square_N$ is the hyperoctahedral group H_N, so we should have a formula of type $G(\square)=H_2^+$. Quite surprisingly, we will see that $G^+(\square_N)$ is in fact a twist of O_N. In order to discuss this material, let us start with:

Theorem 9.9. *There is a signature map $\varepsilon: P_{even}\to\{-1,1\}$, given by*

$$\varepsilon(\tau)=(-1)^c$$

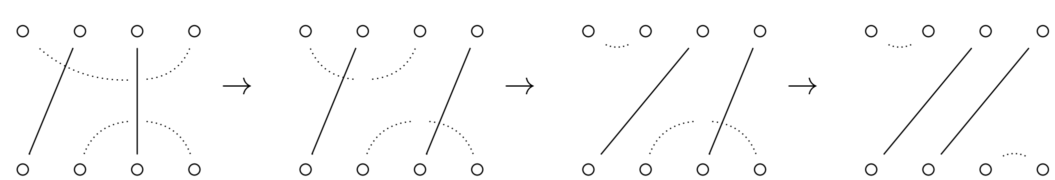
where c is the number of switches needed to make τ noncrossing. In addition:

1. *For $\tau\in S_k$, this is the usual signature.*
2. *For $\tau\in P_2$ we have $(-1)^c$, where c is the number of crossings.*
3. *For $\tau\leq\pi\in NC_{even}$, the signature is 1.*

Proof. The fact that ε is indeed well-defined comes from the fact that the number c in the statement is well-defined modulo 2, which is standard combinatorics.

In order to prove the remaining assertion, observe that any partition $\tau\in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



With this convention, the proof of the remaining assertions is as follows:

(1) For $\tau\in S_k$ the standard form is $\tau'=id$, and the passage $\tau\to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau\in P_2$, the standard form is of type $\tau'=|\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau\to\tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{even}$ comes from $\pi \in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

With the above result in hand, we can now formulate:

Definition 9.10. *Associated to a partition $\pi \in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where the signed Kronecker symbols

$$\bar{\delta}_\pi \in \{-1, 0, 1\}$$

are given by $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with $\tau = \ker\binom{i}{j}$.

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j:\ker\binom{i}{j} \geq \pi} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon : P_{even} \to \{\pm 1\}$, we are led to the following formula, which coincides with the one from Definition 9.10:

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{j:\ker\binom{i}{j} = \tau} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

We have the following key categorical result:

Proposition 9.11. *The assignement $\pi \to \bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi \otimes \bar{T}_\sigma = \bar{T}_{[\pi\sigma]} \quad , \quad \bar{T}_\pi \bar{T}_\sigma = N^{c(\pi,\sigma)} \bar{T}_{[^\sigma_\pi]} \quad , \quad \bar{T}_\pi^* = \bar{T}_{\pi^*}$$

where $c(\pi, \sigma)$ are certain positive integers.

Proof. In order to prove this result we can go back to the proof from the easy case, and insert signs, where needed. We have to check three conditions, as follows:

<u>1. Concatenation</u>. It is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_q \end{pmatrix}\right) \varepsilon\left(\ker\begin{pmatrix} k_1 \ldots k_r \\ l_1 \ldots l_s \end{pmatrix}\right) = \varepsilon\left(\ker\begin{pmatrix} i_1 \ldots i_p & k_1 \ldots k_r \\ j_1 \ldots j_q & l_1 \ldots l_s \end{pmatrix}\right)$$

Let us denote by τ, ν the partitions on the left, so that the partition on the right is of the form $\rho \leq [\tau\nu]$. Now by switching to the noncrossing form, $\tau \to \tau'$ and $\nu \to \nu'$, the partition on the right transforms into $\rho \to \rho' \leq [\tau'\nu']$. Now since $[\tau'\nu']$ is noncrossing, we can use Theorem 9.9 (3), and we obtain the result.

2. Composition. Here we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}\right)$$

Let τ,ν be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch τ,ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\ldots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing.

Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[^{\tau'}_{\nu'}]$. Now since $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Theorem 9.9 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\ldots j_q\\ i_1\ldots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality. So, we can now formulate:

Theorem 9.12. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset O_N^+$, for any $N\in\mathbb{N}$.

Proof. This follows indeed from the Tannakian results from section 1 above, exactly as in the easy case, by using this time Proposition 9.11 as technical ingredient. □

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 9.13. *A closed subgroup $G\subset O_N^+$ is called q-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

for a certain category of partitions $D\subset P_{even}$, where, for $q=-1,1$:

$$\dot{T}=\bar{T},T$$

The Schur-Weyl twist of G is the quizzy quantum group $\bar{G}\subset O_N^+$ obtained via $q\to-q$.

Let us compute now the twist of O_N. We recall that the Möbius function of any lattice, and in particular of P_{even}, is given by:

$$\mu(\sigma,\pi)=\begin{cases}1 & \text{if } \sigma=\pi\\ -\sum_{\sigma\leq\tau<\pi}\mu(\sigma,\tau) & \text{if } \sigma<\pi\\ 0 & \text{if } \sigma\not\leq\pi\end{cases}$$

With this notation, we have the following result:

Proposition 9.14. *For any partition $\pi\in P_{even}$ we have the formula*

$$\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$$

where $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$, with μ being the Möbius function of P_{even}.

Proof. The linear combinations $T=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$ acts on tensors as follows:

$$\begin{aligned}T(e_{i_1}\otimes\ldots\otimes e_{i_k}) &= \sum_{\tau\leq\pi}\alpha_\tau T_\tau(e_{i_1}\otimes\ldots\otimes e_{i_k})\\ &= \sum_{\tau\leq\pi}\alpha_\tau\sum_{\sigma\leq\tau}\sum_{j:\ker\binom{i}{j}=\sigma}e_{j_1}\otimes\ldots\otimes e_{j_l}\\ &= \sum_{\sigma\leq\pi}\left(\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau\right)\sum_{j:\ker\binom{i}{j}=\sigma}e_{j_1}\otimes\ldots\otimes e_{j_l}\end{aligned}$$

Thus, in order to have $\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$, we must have, for any $\sigma\leq\pi$:

$$\varepsilon(\sigma)=\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau$$

But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$ in the statement. $\square$

We can now twist the orthogonal group. The result here is as follows:

Theorem 9.15. *The twist of O_N is obtained by replacing the relations $ab=ba$ with*

$$ab=\pm ba$$

with anticommutation on rows and columns, and commutation otherwise.

Proof. The basic crossing, $\ker\binom{ij}{ji}$ with $i\neq j$, comes from the transposition $\tau\in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\slash\!\!\!\backslash}(e_i\otimes e_j)=\begin{cases}-e_j\otimes e_i & \text{for } i\neq j\\ e_j\otimes e_i & \text{otherwise}\end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from $\bar{T}_\chi$ correspond to the relations defining $\bar{O}_N$, we obtain the result. $\square$

Getting back now to graphs, we have the following result, from [20]:

Theorem 9.16. *The quantum symmetry group of the N-hypercube is*

$$G^+(\square_N)=\bar{O}_N$$

with the corresponding coaction map on the vertex set being the map

$$\Phi:C^*(\mathbb{Z}_2^N)\to C^*(\mathbb{Z}_2^N)\otimes C(\bar{O}_N)$$

$$g_i\to\sum_j g_j\otimes u_{ji}$$

via the standard identification $\square_N=\widehat{\mathbb{Z}_2^N}$.

Proof. We use here the fact that the cube $\square_N$, when regarded as a graph, is the Cayley graph of the group $\mathbb{Z}_2^N$. The eigenvectors and eigenvalues of $\square_N$ are as follows:

$$\begin{aligned}v_{i_1\ldots i_N}&=\sum_{j_1\ldots j_N}(-1)^{i_1j_1+\ldots+i_Nj_N}g_1^{j_1}\ldots g_N^{j_N}\\ \lambda_{i_1\ldots i_N}&=(-1)^{i_1}+\ldots+(-1)^{i_N}\end{aligned}$$

Modulo some standard computations, explained in [20], it is enough to construct a map Φ as in the statement. For this purpose, consider the following variables:

$$G_i=\sum_j g_j\otimes u_{ji}$$

We must show that these variables satisfy the same relations as the generators $g_j\in\mathbb{Z}_2^N$. The self-adjointness being automatic, the relations to be checked are therefore:

$$G_i^2=1\quad,\quad G_iG_j=G_jG_i$$

In what regards the squares, we have the following formula:

$$G_i^2=\sum_{kl}g_kg_l\otimes u_{ik}u_{il}=1+\sum_{k<l}g_kg_l\otimes(u_{ik}u_{il}+u_{il}u_{ik})$$

Also, we have the following formula:

$$[G_i,G_j]=\sum_{k<l}g_kg_l\otimes(u_{ik}u_{jl}-u_{jk}u_{il}+u_{il}u_{jk}-u_{jl}u_{ik})$$

From the first relation we obtain $ab=0$ for $a\neq b$ on the same row of u, and by using the antipode, the same happens for the columns. From the second relation we obtain:

$$[u_{ik},u_{jl}]=[u_{jk},u_{il}]\quad,\forall k\neq l$$

Now by applying the antipode we obtain:

$$[u_{lj}, u_{ki}] = [u_{li}, u_{kj}]$$

By relabelling, this gives, for $j \neq i$:

$$[u_{ik}, u_{jl}] = [u_{il}, u_{jk}]$$

Thus for $i \neq j, k \neq l$ we must have:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] = 0$$

We are therefore led to $G \subset \bar{O}_N$, as claimed. □

In connection with the various extensions of our formalism, regarding colored graphs, or finite metric spaces, let us record as well the following result, also from [20]:

Theorem 9.17. *The quantum isometry group of the N-hypercube, regarded as a finite metric subspace of $\mathbb{R}^N$, is:*

$$G^+(\square_N) = \bar{O}_N$$

That is, we obtain the twisted orthogonal group $\bar{O}_N$.

Proof. We recall that the quantum symmetry group of a finite colored graph is produced by the formula $du = ud$. This construction applies in particular to the finite metric spaces, which can be regarded as complete graphs, with the edges colored by their lengths. The distance matrix of the cube has a color decomposition as follows:

$$d = d_1 + \sqrt{2}d_2 + \sqrt{3}d_3 + \ldots + \sqrt{N}d_N$$

Since the powers of d_1 can be computed by counting loops on the cube, we have formulae as follows, with $x_{ij} \in \mathbb{N}$ being certain positive integers:

$$\begin{aligned}
d_1^2 &= x_{21}1_N + x_{22}d_2 \\
d_1^3 &= x_{31}1_N + x_{32}d_2 + x_{33}d_3 \\
&\vdots \\
d_1^N &= x_{N1}1_N + x_{N2}d_2 + x_{N3}d_3 + \ldots + x_{NN}d_N
\end{aligned}$$

But this shows that we have the following equality of algebras:

$$< d > = < d_1 >$$

Now since d_1 is the adjacency matrix of $\square_N$, viewed as graph, this proves our claim, and we obtain the result from Theorem 9.16. □

Our purpose now is to understand which representation of O_N produces by twisting the magic representation of $\bar{O}_N$. In order to solve this question, we will need:

Proposition 9.18. *The Fourier transform over $\mathbb{Z}_2^N$ is the map*

$$\alpha : C(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N)$$

$$\delta_{g_1^{i_1}\ldots g_N^{i_N}} \to \frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>}g_1^{j_1}\ldots g_N^{j_N}$$

with the usual convention $<i,j>=\sum_k i_kj_k$, and its inverse is the map

$$\beta : C^*(\mathbb{Z}_2^N) \to C(\mathbb{Z}_2^N)$$

$$g_1^{i_1}\ldots g_N^{i_N} \to \sum_{j_1\ldots j_N}(-1)^{<i,j>}\delta_{g_1^{j_1}\ldots g_N^{j_N}}$$

with all the exponents being binary, $i_1,\ldots,i_N,j_1,\ldots,j_N\in\{0,1\}$.

Proof. Observe first that the group $\mathbb{Z}_2^N$ can be written as follows:

$$\mathbb{Z}_2^N = \left\{g_1^{i_1}\ldots g_N^{i_N}\Big|i_1,\ldots,i_N\in\{0,1\}\right\}$$

Thus both α,β are well-defined, and it is elementary to check that both are morphisms of algebras. We have as well $\alpha\beta=\beta\alpha=id$, coming from the standard formula:

$$\begin{aligned}\frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>} &= \prod_{k=1}^N\left(\frac{1}{2}\sum_{j_r}(-1)^{i_rj_r}\right)\\ &= \delta_{i0}\end{aligned}$$

Thus we have indeed a pair of inverse Fourier morphisms, as claimed. □

As an illustration here, at $N=1$, with $\mathbb{Z}_2=\{1,g\}$, the map α is given by:

$$\delta_1\to\frac{1}{2}(1+g)\quad,\quad\delta_g\to\frac{1}{2}(1-g)$$

As for its inverse β, this is given by the following formulae:

$$1\to\delta_1+\delta_g\quad,\quad g\to\delta_1-\delta_g$$

By using now these Fourier transforms, we obtain following formula:

Theorem 9.19. *The magic unitary for the embedding $\bar{O}_N\subset S_{2^N}^+$ is given by*

$$w_{i_1\ldots i_N,k_1\ldots k_N}=\frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}(-1)^{<i+k_b,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

where $k_b=(k_{b_1},\ldots,k_{b_N})$, with respect to multi-indices $i,k\in\{0,1\}^N$ as above.

Proof. By composing the coaction map Φ from Theorem 9.16 with the above Fourier transform isomorphisms α,β, we have a diagram as follows:

$$\begin{array}{ccc}
C^*(\mathbb{Z}_2^N) & \xrightarrow{\ \Phi\ } & C^*(\mathbb{Z}_2^N)\otimes C(\bar{O}_N)\\
\uparrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta\otimes id}\\
C(\mathbb{Z}_2^N) & \xrightarrow{\ \Psi\ } & C(\mathbb{Z}_2^N)\otimes C(\bar{O}_N)
\end{array}$$

In order to compute the composition on the bottom Ψ, we first recall from Theorem 9.16 above that the coaction map Φ is defined by the formula:

$$\Phi(g_b)=\sum_a g_a\otimes u_{ab}$$

Now by making products of such quantities, we obtain the following global formula for Φ, valid for any exponents $i_1,\ldots,i_N\in\{1,\ldots,N\}$:

$$\Phi(g_1^{i_1}\ldots g_N^{i_N})=\left(\frac{1}{N}\right)^{\#(0\in i)}\sum_{b_1\ldots b_N}g_{b_1}^{i_1}\ldots g_{b_N}^{i_N}\otimes u_{1b_1}^{i_1}\ldots u_{Nb_N}^{i_N}$$

The term on the right can be put in "standard form" as follows:

$$g_{b_1}^{i_1}\ldots g_{b_N}^{i_N}=g_1^{\sum_{b_x=1}i_x}\ldots g_N^{\sum_{b_x}i_x}$$

We therefore obtain the following formula for the coaction map Φ:

$$\Phi(g_1^{i_1}\ldots g_N^{i_N})=\left(\frac{1}{N}\right)^{\#(0\in i)}\sum_{b_1\ldots b_N}g_1^{\sum_{b_x=1}i_x}\ldots g_N^{\sum_{b_x=N}i_x}\otimes u_{1b_1}^{i_1}\ldots u_{Nb_N}^{i_N}$$

Now by applying the Fourier transforms, we obtain the following formula:

$$\begin{aligned}
&\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}})\\
=\ &(\beta\otimes id)\Phi\left(\frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>}g_1^{j_1}\ldots g_N^{j_N}\right)\\
=\ &\frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}(-1)^{<i,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}\beta\left(g_1^{\sum_{b_x=1}j_x}\ldots g_N^{\sum_{b_x=N}j_x}\right)\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}
\end{aligned}$$

By using now the formula of β from Proposition 9.18, we obtain:

$$\begin{aligned}\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}}) &= \frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\sum_{k_1\ldots k_N}\left(\frac{1}{N}\right)^{\#(0\in j)}\\ &\quad(-1)^{<i,j>}(-1)^{<(\sum_{b_x=1}j_x,\ldots,\sum_{b_x=N}j_x),(k_1,\ldots,k_N)>}\\ &\quad\delta_{g_1^{k_1}\ldots g_N^{k_N}}\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}\end{aligned}$$

Now observe that, with the notation $k_b=(k_{b_1},\ldots,k_{b_N})$, we have:

$$\left\langle\left(\sum_{b_x=1}j_x,\ldots,\sum_{b_x=N}j_x\right),(k_1,\ldots,k_N)\right\rangle=<j,k_b>$$

Thus, we obtain the following formula for our map Ψ:

$$\begin{aligned}&\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}})\\ =&\ \frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\sum_{k_1\ldots k_N}(-1)^{<i+k_b,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}\delta_{g_1^{k_1}\ldots g_N^{k_N}}\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}\end{aligned}$$

But this gives the formula in the statement for the corresponding magic unitary, with respect to the basis $\{\delta_{g_1^{i_1}\ldots g_N^{i_N}}\}$ of the algebra $C(\mathbb Z_2^N)$, and we are done. □

We can now solve our original question, namely understanding where the magic representation of $\bar{O}_N$ really comes from, with the following final answer to it, from [10]:

Theorem 9.20. *The magic representation of $\bar{O}_N$, coming from its action on the N-cube, corresponds to the antisymmetric representation of O_N, via twisting.*

Proof. This follows from the formula of w in Theorem 9.19, by computing the character, and then interpreting the result via twisting, as follows:

(1) By applying the trace to the formula of w, we obtain:

$$\chi=\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\left(\frac{1}{2^N}\sum_{i_1\ldots i_N}(-1)^{<i+i_b,j>}\right)\left(\frac{1}{N}\right)^{\#(0\in j)}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(2) By computing the Fourier sum in the middle, we are led to the following formula, with binary indices $j_1,\ldots,j_N\in\{0,1\}$, and plain indices $b_1,\ldots,b_N\in\{1,\ldots,N\}$:

$$\chi=\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\left(\frac{1}{N}\right)^{\#(0\in j)}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(3) With the notation $r=\#(1\in j)$ we obtain a decomposition of type:

$$\chi=\sum_{r=0}^N\chi_r$$

To be more precise, the variables χ_r are as follows:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{\#(1\in j)=r}\sum_{b_1\ldots b_N}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(4) Consider now the set $A\subset\{1,\ldots,N\}$ given by:

$$A=\{a|j_a=1\}$$

The binary multi-indices $j\in\{0,1\}^N$ satisfying $\#(1\in j)=r$ being in bijection with such subsets A, satisfying $|A|=r$, we can replace the sum over j with a sum over such subsets A. We obtain a formula as follows, where j is the index corresponding to A:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{|A|=r}\sum_{b_1\ldots b_N}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}\prod_{a\in A}u_{ab_a}$$

(5) Let us identify b with the corresponding function $b:\{1,\ldots,N\}\to\{1,\ldots,N\}$, via $b(a)=b_a$. Then for any $p\in\{1,\ldots,N\}$ we have:

$$\delta_{j_p,\sum_{b_x=p}j_x}=1\iff|b^{-1}(p)\cap A|=\chi_A(p)\ (\mathrm{mod}\ 2)$$

We conclude that the multi-indices $b\in\{1,\ldots,N\}^N$ which effectively contribute to the sum are those coming from the functions satisfying $b<A$. Thus, we have:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{|A|=r}\sum_{b<A}\prod_{a\in A}u_{ab_a}$$

(6) We can further split each χ_r over the sets $A\subset\{1,\ldots,N\}$ satisfying:

$$|A|=r$$

The point is that for each of these sets we have:

$$\frac{1}{N^{N-r}}\sum_{b<A}\prod_{a\in A}u_{ab_a}=\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

Thus, the magic character of $\bar{O}_N$ splits as:

$$\chi=\sum_{r=0}^N\chi_r$$

To be more precise, the components are:

$$\chi_r=\sum_{|A|=r}\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

(7) The twisting operation $O_N\to\bar{O}_N$ makes correspond the following products:

$$\varepsilon(\sigma)\prod_{a\in A}u_{a\sigma(a)}\to\prod_{a\in A}u_{a\sigma(a)}$$

Now by summing over sets A and permutations σ, we conclude that the twisting operation $O_N \to \bar{O}_N$ makes correspond the following quantities:

$$\sum_{|A|=r}\sum_{\sigma\in S_N^A}\varepsilon(\sigma)\prod_{a\in A}u_{a\sigma(a)}\to\sum_{|A|=r}\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

Thus, we are led to the conclusion in the statement. □

Let us go back now to the square problem. In order to present the correct, final solution to it, the idea will be that to look at the quantum group $G^+(|\ |)$ instead, which is equal to it, according to Theorem 9.2 (3). We will need the following result, from [49]:

Theorem 9.21. *Given closed subgroups $G\subset U_N^+$, $H\subset S_k^+$, with fundamental corepresentations u,v, the following construction produces a closed subgroup of U_{Nk}^+:*

$$C(G\wr_* H)=(C(G)^{*k}*C(H))/<[u_{ij}^{(a)},v_{ab}]=0>$$

In the case where G,H are classical, the classical version of $G\wr_* H$ is the usual wreath product $G\wr H$. Also, when G is a quantum permutation group, so is $G\wr_* H$.

Proof. Consider indeed the matrix $w_{ia,jb}=u_{ij}^{(a)}v_{ab}$, over the quotient algebra in the statement. Then w is unitary, and in the case $G\subset S_N^+$, this matrix is magic. With these observations in hand, it is routine to check that $G\wr_* H$ is indeed a quantum group, with fundamental corepresentation w, by constructing maps Δ,ε,S as in section 1 above, and in the case $G\subset S_N^+$, we obtain in this way a closed subgroup of S_{Nk}^+. See [49]. □

We refer to [13], [49], [132] for more details regarding the above construction. With this notion in hand, we can now formulate a non-trivial result, as follows:

Theorem 9.22. *Given a connected graph X, and $k\in\mathbb{N}$, we have the formulae*

$$G(kX)=G(X)\wr S_k$$

$$G^+(kX)=G^+(X)\wr_* S_k^+$$

where $kX=X\sqcup\ldots\sqcup X$ is the k-fold disjoint union of X with itself.

Proof. The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is elementary to check that we have an inclusion as follows, for any finite graph X:

$$G^+(X)\wr_* S_k^+\subset G^+(kX)$$

Regarding now the reverse inclusion, which requires X to be connected, this follows by doing some matrix analysis, by using the commutation with u. To be more precise, let us denote by w the fundamental corepresentation of $G^+(kX)$, and set:

$$u_{ij}^{(a)}=\sum_b w_{ia,jb}\quad,\quad v_{ab}=\sum_i v_{ab}$$

It is then routine to check, by using the fact that X is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus, we obtain:

$$G^+(kX) \subset G^+(X) \wr_* S_k^+$$

But this gives the result. See [13]. □

We are led in this way to the following result, from [20]:

Theorem 9.23. *Consider the graph consisting of N segments.*

1. *Its symmetry group is the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$.*
2. *Its quantum symmetry group is the quantum group $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$.*

Proof. Here the first assertion is clear from definitions, with the remark that the relation with the formula $H_N = G(\square_N)$ comes by viewing the N segments as being the $[-1, 1]$ segments on each of the N coordinate axes of $\mathbb{R}^N$. Indeed, a symmetry of the N-cube is the same as a symmetry of the N segments, and so, as desired:

$$G(\square_N) = \mathbb{Z}_2 \wr S_N$$

As for the second assertion, this follows from Theorem 9.22 above, applied to the segment graph. Observe also that (2) implies (1), by taking the classical version. □

Now back to the square, we have $G^+(\square) = H_2^+$, and our claim is that this is the "good" and final formula. In order to prove this, we must work out the easiness theory for H_N, H_N^+, and find a compatibility there. We first have the following result:

Proposition 9.24. *The algebra $C(H_N^+)$ can be presented in two ways, as follows:*

1. *As the universal algebra generated by the entries of a $2N \times 2N$ magic unitary having the "sudoku" pattern $w = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, with a, b being square matrices.*
2. *As the universal algebra generated by the entries of a $N \times N$ orthogonal matrix which is "cubic", in the sense that $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$, for any $j \neq k$.*

As for $C(H_N)$, this has similar presentations, among the commutative algebras.

Proof. We must prove that the algebras A_s, A_c coming from (1,2) coincide.

We can define a morphism $A_c \to A_s$ by the following formula:

$$\varphi(u_{ij}) = a_{ij} - b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij} = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \beta_{ij} = \frac{u_{ij}^2 - u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M = \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) \\ (\beta_{ij}) & (\alpha_{ij}) \end{pmatrix}$$

Thus we can define a morphism $A_s \to A_c$ by the following formula:

$$\psi(a_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \psi(b_{ij}) = \frac{u_{ij}^2 - u_{ij}}{2}$$

We check now the fact that ψ, φ are indeed inverse morphisms:

$$\begin{aligned}
\psi\varphi(u_{ij}) &= \psi(a_{ij} - b_{ij}) \\
&= \frac{u_{ij}^2 + u_{ij}}{2} - \frac{u_{ij}^2 - u_{ij}}{2} \\
&= u_{ij}
\end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned}
\varphi\psi(a_{ij}) &= \varphi\left(\frac{u_{ij}^2 + u_{ij}}{2}\right) \\
&= \frac{(a_{ij} - b_{ij})^2 + (a_{ij} - b_{ij})}{2} \\
&= a_{ij}
\end{aligned}$$

A similar computation gives $\varphi\psi(b_{ij}) = b_{ij}$, which completes the proof. □

We can now work out the easiness property of H_N, H_N^+, with respect to the cubic representations, and we are led to the following result, which is fully satisfactory:

Theorem 9.25. *The quantum groups H_N, H_N^+ are both easy, as follows:*

1. *H_N corresponds to the category P_{even}.*
2. *H_N^+ corresponds to the category NC_{even}.*

Proof. These assertions follow indeed from the fact that the cubic relations are implemented by the one-block partition in $P(2,2)$, which generates NC_{even}. □

As a final conclusion now, to the long story told here, the correct analogue of the hyperoctahedral group H_N is the quantum group H_N^+ constructed above, with $H_N \to H_N^+$ being a liberation, in the sense of easy quantum group theory.

10. Reflection groups

These quantum groups H_N and H_N^+ belong in fact to series, depending on a parameter $s \in \mathbb{N} \cup \{\infty\}$, as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N$$
$$H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

We discuss here, following [11], [44], the algebraic and analytic structure of these latter quantum groups. The main motivation comes from the cases $s = 1, 2, \infty$, where we recover respectively S_N, S_N^+ and H_N, H_N^+, and the full reflection groups K_N, K_N^+.

Let us start with a brief discussion concerning the classical case. The result that we will need, which is well-known and elementary, is as follows:

Proposition 10.1. *The group $H_N^s = \mathbb{Z}_s \wr S_N$ of $N \times N$ permutation-like matrices having as nonzero entries the s-th roots of unity is as follows:*

(1) *$H_N^1 = S_N$ is the symmetric group.*
(2) *$H_N^2 = H_N$ is the hyperoctahedral group.*
(3) *$H_N^\infty = K_N$ is the group of unitary permutation-like matrices.*

Proof. Everything here is clear from definitions. □

The free analogues of the reflection groups H_N^s can be constructed as follows:

Definition 10.2. *$C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements u_{ij}, subject to the following relations,*

(1) *$u = (u_{ij})$ is unitary,*
(2) *$u^t = (u_{ji})$ is unitary,*
(3) *$p_{ij} = u_{ij}u_{ij}^*$ is a projection,*
(4) *$u_{ij}^s = p_{ij}$,*

with Woronowicz algebra maps Δ, ε, S constructed by universality.

Here we allow the value $s = \infty$, with the convention that the last axiom simply disappears in this case. Observe that at $s < \infty$ the normality condition is actually redundant. This is because a partial isometry a subject to the relation $aa^* = a^s$ is normal. As a first result, making the connection with H_N^s, we have:

Theorem 10.3. *We have an inclusion as follows,*

$$H_N^s \subset H_N^{s+}$$

which is a liberation, in the sense that the classical version of H_N^{s+}, obtained by dividing by the commutator ideal, is H_N^s.

Proof. This follows as for $O_N \subset O_N^+$ or $S_N \subset S_N^+$, by using the Gelfand theorem. □

In analogy with the results in section 9, we have the following result:

Proposition 10.4. *The algebras $C(H_N^{s+})$ with $s=1,2,\infty$, and their presentation relations in terms of the entries of the matrix $u=(u_{ij})$, are as follows.*

1. *For $C(H_N^{1+})=C(S_N^+)$, the matrix u is magic: all its entries are projections, summing up to 1 on each row and column.*
2. *For $C(H_N^{2+})=C(H_N^+)$ the matrix u is cubic: it is orthogonal, and the products of pairs of distinct entries on the same row or the same column vanish.*
3. *For $C(H_N^{\infty+})=C(K_N^+)$ the matrix u is unitary, its transpose is unitary, and all its entries are normal partial isometries.*

Proof. The idea here is as follows:

(1) This follows from definitions and from standard operator algebra tricks.

(2) This follows as well from definitions and from standard operator algebra tricks.

(3) This is just a translation of the definition of $C(H_N^{s+})$, at $s=\infty$. □

Let us prove now that H_N^{s+} with $s<\infty$ is a quantum permutation group. For this purpose, we must change the fundamental representation. Let us start with:

Definition 10.5. *A (s,N)-sudoku matrix is a magic unitary of size sN, of the form*

$$m=\begin{pmatrix}a^0&a^1&\ldots&a^{s-1}\\ a^{s-1}&a^0&\ldots&a^{s-2}\\ \vdots&\vdots&&\vdots\\ a^1&a^2&\ldots&a^0\end{pmatrix}$$

where $a^0,\ldots,a^{s-1}$ are $N\times N$ matrices.

The basic examples of such sudoku matrices come from the group H_n^s. Indeed, with $w=e^{2\pi i/s}$, each of the N^2 matrix coordinates $u_{ij}:H_N^s\to\mathbb C$ takes values in the set:

$$S=\{0\}\cup\{1,w,\ldots,w^{s-1}\}$$

Thus this coordinate decomposes as follows:

$$u_{ij}=\sum_{r=0}^{s-1}w^ra_{ij}^r$$

Here each a_{ij}^r is a function taking values in $\{0,1\}$, and so a projection in the C^*-algebra sense, and it follows from definitions that these projections form a sudoku matrix. With this notion in hand, we have the following result:

Theorem 10.6. *The following happen:*

1. *The algebra $C(H_N^s)$ is isomorphic to the universal commutative C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*
2. *The algebra $C(H_N^{s+})$ is isomorphic to the universal C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*

Proof. The first assertion follows from the second one, via Theorem 10.3. In order to prove now the second assertion, consider the universal algebra in the statement, namely:

$$A=C^*\left(a_{ij}^p\Big|\left(a_{ij}^{q-p}\right)_{pi,qj}=(s,N)-\text{sudoku}\right)$$

Consider also the algebra $C(H_N^{s+})$. According to Definition 10.2, this is presented by certain relations R, that we will call here level s cubic conditions:

$$C(H_N^{s+})=C^*\left(u_{ij}\Big|u=N\times N\text{ level } s\text{ cubic}\right)$$

We will construct a pair of inverse morphisms between these algebras.

(1) Our first claim is that $U_{ij}=\sum_p w^{-p}a_{ij}^p$ is a level s cubic unitary. Indeed, by using the sudoku condition, the verification of (1-4) in Definition 10.2 is routine.

(2) Our second claim is that the elements $A_{ij}^p=\frac{1}{s}\sum_r w^{rp}u_{ij}^r$, with the convention $u_{ij}^0=p_{ij}$, form a level s sudoku unitary. Once again, the proof here is routine.

(3) According to the above, we can define a morphism $\Phi:C(H_N^{s+})\to A$ by the formula $\Phi(u_{ij})=U_{ij}$, and a morphism $\Psi:A\to C(H_N^{s+})$ by the formula $\Psi(a_{ij}^p)=A_{ij}^p$.

(4) We check now the fact that Φ,Ψ are indeed inverse morphisms:

$$\begin{aligned}\Psi\Phi(u_{ij})&=\sum_p w^{-p}A_{ij}^p\\&=\frac{1}{s}\sum_p w^{-p}\sum_r w^{rp}u_{ij}^r\\&=\frac{1}{s}\sum_{pr}w^{(r-1)p}u_{ij}^r\\&=u_{ij}\end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned}\Phi\Psi(a_{ij}^p)&=\frac{1}{s}\sum_r w^{rp}U_{ij}^r\\&=\frac{1}{s}\sum_r w^{rp}\sum_q w^{-rq}a_{ij}^q\\&=\frac{1}{s}\sum_q a_{ij}^q\sum_r w^{r(p-q)}\\&=a_{ij}^p\end{aligned}$$

Thus we have an isomorphism $C(H_N^{s+})=A$, as claimed. $\square$

Let us discuss now the interpretation of H_N^s,H_N^{s+} as classical and quantum symmetry groups of graphs. We will need the following simple fact:

Proposition 10.7. *A $sN\times sN$ magic unitary commutes with the matrix*

$$\Sigma=\begin{pmatrix}0&I_N&0&\ldots&0\\ 0&0&I_N&\ldots&0\\ \vdots&\vdots&&\ddots&\\ 0&0&0&\ldots&I_N\\ I_N&0&0&\ldots&0\end{pmatrix}$$

if and only if it is a sudoku matrix in the sense of Definition 10.5.

Proof. This follows from the fact that commutation with Σ means that the matrix is circulant. Thus, we obtain the sudoku relations from Definition 10.5 above. $\square$

Now let Z_s be the oriented cycle with s vertices, and consider the graph NZ_s consisting of N disjoint copies of it. Observe that, with a suitable labeling of the vertices, the adjacency matrix of this graph is the above matrix Σ. We obtain from this:

Theorem 10.8. *We have the following results:*

(1) *H_N^s is the symmetry group of NZ_s.*
(2) *H_N^{s+} is the quantum symmetry group of NZ_s.*

Proof. The idea here is as follows:

(1) This follows from definitions.

(2) This follows from Theorem 10.6 and Proposition 10.7, because $C(H_N^{s+})$ is the quotient of $C(S_{sN}^+)$ by the relations making the fundamental corepresentation commute with the adjacency matrix of NZ_s. $\square$

Next in line, we must talk about wreath products. We have here:

Theorem 10.9. *We have the following results:*

(1) *$H_N^s=\mathbb{Z}_s\wr S_N$.*
(2) *$H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$.*

Proof. This follows from the following formulae, valid for any connected graph X, and explained in the previous section, applied to Z_s:

$$\begin{aligned}G(NX)&=G(X)\wr S_N\\ G^+(NX)&=G^+(X)\wr_* S_N^+\end{aligned}$$

Alternatively, (1) follows from definitions, and (2) can be proved directly, by constructing a pair of inverse morphisms. For details here, we refer to [44]. $\square$

Regarding now the easiness property of the quantum groups H_N^s,H_N^{s+}, we already know that this happens at $s=1,2$. In general, we have the following result, from [11]:

Theorem 10.10. *The quantum groups H_N^s, H_N^{s+} are easy, the corresponding categories*

$$P^s \subset P$$

$$NC^s \subset NC$$

consisting of partitions having the property

$$\# \circ - \# \bullet = 0(s)$$

as a weighted sum, in each block.

Proof. Observe that the result holds at $s = 1$, trivially, and at $s = 2$ as well, where our condition is equivalent to $\# \circ + \# \bullet = 0(2)$, in each block. In general, this follows as in the proof for H_N, H_N^+, by using the one-block partition in $P(s,s)$. See [11]. □

The above proof was of course quite brief, but we will not be really interested here in the case $s \geq 3$, which is quite technical. In fact, the above result, dealing with the general case $s \in \mathbb{N}$, is here for providing an introduction to the case $s = \infty$, where we have:

Theorem 10.11. *The quantum groups K_N, K_N^+ are easy, the corresponding categories*

$$\mathcal{P}_{even} \subset P$$

$$\mathcal{NC}_{even} \subset NC$$

consisting of partitions having the property

$$\#\circ = \#\bullet$$

as a weighted equality, in each block.

Proof. This follows from Theorem 10.10, or rather by proving the result directly, a bit as in the $s = 1, 2$ cases, because the $s = \infty$ case is needed first, in order to discuss the general case, $s \in \mathbb{N} \cup \{\infty\}$. For details here, we refer once again to [11]. □

Let us discuss now, following [44], the classification of the irreducible representations of H_N^{s+}, and the computation of their fusion rules. For this purpose, let us go back to the elements u_{ij}, p_{ij} in Definition 10.2 above. We recall that, as a consequence of Proposition 10.4, the matrix $p = (p_{ij})$ is a magic unitary. We first have the following result:

Proposition 10.12. *The elements u_{ij} and p_{ij} satisfy:*

(1) $p_{ij} u_{ij} = u_{ij}$.
(2) $u_{ij}^* = u_{ij}^{s-1}$.
(3) $u_{ij} u_{ik} = 0$ *for* $j \neq k$.

Proof. We use the fact that in a C^*-algebra, $aa^* = 0$ implies $a = 0$.

(1) This follows from the following computation, with $a = (p_{ij} - 1)u_{ij}$:

$$\begin{aligned} aa^* &= (p_{ij} - 1)p_{ij}(p_{ij} - 1) \\ &= 0 \end{aligned}$$

(2) With $a=u_{ij}^*-u_{ij}^{s-1}$ we have $aa^*=0$, which gives the result.

(3) With $a=u_{ij}u_{ik}$ we have $aa^*=0$, which gives the result. □

In what follows, we make the convention $u_{ij}^0=p_{ij}$. We have then:

Theorem 10.13. *The algebra $C(H_N^{s+})$ has a family of N-dimensional corepresentations $\{u_k|k\in\mathbb Z\}$, satisfying the following conditions:*

(1) *$u_k=(u_{ij}^k)$ for any $k\geq 0$.*
(2) *$u_k=u_{k+s}$ for any $k\in\mathbb Z$.*
(3) *$\bar{u}_k=u_{-k}$ for any $k\in\mathbb Z$.*

Proof. The idea here is as follows:

(1) Let us set $u_k=(u_{ij}^k)$. By using Proposition 10.12 (3), we have:

$$\begin{aligned}\Delta(u_{ij}^k)&=\sum_{l_1\ldots l_k}u_{il_1}\ldots u_{il_k}\otimes u_{l_1j}\ldots u_{l_kj}\\&=\sum_l u_{il}^k\otimes u_{lj}^k\end{aligned}$$

We have as well, trivially, the following two formulae:

$$\varepsilon(u_{ij}^k)=\delta_{ij}$$

$$S(u_{ij}^k)=u_{ji}^{*k}$$

(2) This follows once again from Proposition 10.12 (3), as follows:

$$\begin{aligned}u_{ij}^{k+s}&=u_{ij}^ku_{ij}^s\\&=u_{ij}^kp_{ij}\\&=u_{ij}^k\end{aligned}$$

(3) This follows from Proposition 10.12 (2), and we are done. □

Let us compute now the intertwiners between the various tensor products between the above corepresentations u_i. For this purpose, we make the assumption $N\geq 4$, which brings linear independence. In order to simplify the notations, we will use:

Definition 10.14. *For $i_1,\ldots,i_k\in\mathbb Z$ we use the notation*

$$u_{i_1\ldots i_k}=u_{i_1}\otimes\ldots\otimes u_{i_k}$$

where $\{u_i|i\in\mathbb Z\}$ are the corepresentations in Theorem 10.13.

Observe that in the particular case $i_1,\ldots,i_k\in\{\pm1\}$, we obtain in this way all the possible tensor products between $u=u_1$ and $\bar{u}=u_{-1}$, known by [147] to contain any irreducible corepresentation of $C(H_N^{s+})$. Here is now our main result:

Theorem 10.15. *We have the following equality of linear spaces*

$$Hom(u_{i_1\ldots i_k}, u_{j_1\ldots j_l}) = span\left\{T_p\Big|p \in NC_s(i_1\ldots i_k, j_1\ldots j_l)\right\}$$

where the set on the right consists of elements of $NC(k,l)$ having the property that in each block, the sum of i indices equals the sum of j indices, modulo s.

Proof. This result is from [44], the idea of the proof being as follows:

(1) Our first claim is that, in order to prove $\supset$, we may restrict attention to the case $k = 0$. This follow indeed from the Frobenius duality isomorphism.

(2) Our second claim is that, in order to prove $\supset$ in the case $k = 0$, we may restrict attention to the one-block partitions. Indeed, this follows once again from a standard trick. Consider the following disjoint union:

$$NC_s = \bigcup_{k=0}^\infty \bigcup_{i_1\ldots i_k} NC_s(0, i_1\ldots i_k)$$

This is a set of labeled partitions, having property that each $p \in NC_s$ is noncrossing, and that for $p \in NC_s$, any block of p is in NC_s. But it is well-known that under these assumptions, the global algebraic properties of NC_s can be checked on blocks.

(3) Proof of $\supset$. According to the above considerations, we just have to prove that the vector associated to the one-block partition in $NC(l)$ is fixed by $u_{j_1\ldots j_l}$, when:

$$s|j_1 + \ldots + j_l$$

Consider the standard generators $e_{ab} \in M_N(\mathbb{C})$, acting on the basis vectors by:

$$e_{ab}(e_c) = \delta_{bc}e_a$$

The corepresentation $u_{j_1\ldots j_l}$ is given by the following formula:

$$u_{j_1\ldots j_l} = \sum_{a_1\ldots a_l}\sum_{b_1\ldots b_l} u_{a_1b_1}^{j_1}\ldots u_{a_lb_l}^{j_l} \otimes e_{a_1b_1} \otimes \ldots \otimes e_{a_lb_l}$$

As for the vector associated to the one-block partition, this is:

$$\xi_l = \sum_b e_b^{\otimes l}$$

By using now several times the relations in Proposition 10.12, we obtain, as claimed:

$$\begin{aligned}
u_{j_1\ldots j_l}(1 \otimes \xi_l) &= \sum_{a_1\ldots a_l}\sum_b u_{a_1b}^{j_1}\ldots u_{a_lb}^{j_l} \otimes e_{a_1} \otimes \ldots \otimes e_{a_l}\\
&= \sum_{ab} u_{ab}^{j_1+\ldots+j_l} \otimes e_a^{\otimes l}\\
&= 1 \otimes \xi_l
\end{aligned}$$

(4) Proof of $\subset$. The spaces on the right in the statement form a Tannakian category in the sense of [148], so they correspond to a certain Woronowicz algebra A.

This algebra is by definition the maximal model for the Tannakian category. In other words, it comes with a family of corepresentations $\{v_i\}$, such that:

$$Hom(v_{i_1\ldots i_k}, v_{j_1\ldots j_l}) = \text{span}\left\{T_p\Big|p \in NC_s(i_1\ldots i_k, j_1\ldots j_l)\right\}$$

On the other hand, the inclusion $\supset$ that we just proved shows that $C(H_N^{s+})$ is a model for the category. Thus we have a quotient map as follows:

$$A \to C(H_N^{s+})$$

$$v_i \to u_i$$

But this latter map can be shown to be an isomorphism, by suitably adapting the proof from the $s = 1$ case, for the quantum permutation group S_N^+. See [11], [44]. $\square$

As an illustration for the above result, we have the following statement:

Theorem 10.16. *The basic corepresentations $u_0, \ldots, u_{s-1}$ are as follows:*
(1) *$u_1, \ldots, u_{s-1}$ are irreducible.*
(2) *$u_0 = 1 + r_0$, with r_0 irreducible.*
(3) *$r_0, u_1, \ldots, u_{s-1}$ are distinct.*

Proof. We apply Theorem 10.15 with $k = l = 1$ and $i_1 = i, j_1 = j$. This gives:

$$\dim(Hom(u_i, u_j)) = \#NC_s(i,j)$$

We have two candidates for the elements of $NC_s(i,j)$, namely the two partitions in $NC(1,1)$. So, consider these two partitions, with the points labeled by i, j:

$$p = \begin{Bmatrix} i \\ | \\ | \\ j \end{Bmatrix} \qquad\qquad q = \begin{Bmatrix} i \\ | \\ \\ | \\ j \end{Bmatrix}$$

We have to check for each of these partitions if the sum of i indices equals or not the sum of j indices, modulo s, in each block. The answer is as follows:

$$p \in NC_s(i,j) \iff i = j$$
$$q \in NC_s(i,j) \iff i = j = 0$$

By collecting together these two answers, we obtain:

$$\#NC_s(i,j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \neq 0 \\ 2 & \text{if } i = j = 0 \end{cases}$$

Now (1) follows from the second equality, (2) follows from the third equality and from the fact that we have $1 \in u_s$, and (3) follows from the first equality. □

Let us record as well, as a second consequence, the following result:

Theorem 10.17. *We have the formula*

$$\#(1 \in u_{i_1} \otimes \ldots \otimes u_{i_k}) = \#NC_s(i_1 \ldots i_k)$$

where the set on the right consists of noncrossing partitions of $\{1, \ldots, k\}$ having the property that the sum of indices in each block is a multiple of s.

Proof. This is clear indeed from Theorem 10.15 above. □

We can now compute the fusion rules for H_N^{s+}. The result here, from [44], is as follows:

Theorem 10.18. *Let $F =< \mathbb{Z}_s >$ be the words over $\mathbb{Z}_s$, with involution*

$$(i_1 \ldots i_k)^- = (-i_k) \ldots (-i_1)$$

and with fusion product given by:

$$(i_1 \ldots i_k) \cdot (j_1 \ldots j_l) = i_1 \ldots i_{k-1}(i_k + j_1)j_2 \ldots j_l$$

The irreducible representations of H_N^{s+} can then be labeled r_x with $x \in F$, such that the involution and fusion rules are $\bar{r}_x = r_{\bar{x}}$ and

$$r_x \otimes r_y = \sum_{x=vz, y=\bar{z}w} r_{vw} + r_{v \cdot w}$$

and such that we have $r_i = u_i - \delta_{i0} 1$ for any $i \in \mathbb{Z}_s$.

Proof. This basically follows from Theorem 10.15, the idea being as follows:

(1) Consider the monoid $A = \{a_x | x \in F\}$, with multiplication $a_x a_y = a_{xy}$. We denote by $\mathbb{N}A$ the set of linear combinations of elements in A, with coefficients in $\mathbb{N}$, and we endow it with fusion rules as in the statement:

$$a_x \otimes a_y = \sum_{x=vz, y=\bar{z}w} a_{vw} + a_{v \cdot w}$$

With these notations, $(\mathbb{N}A, +, \otimes)$ is a semiring. We will use as well the set $\mathbb{Z}A$, formed by the linear combinations of elements of A, with coefficients in $\mathbb{Z}$. The above tensor product operation extends to $\mathbb{Z}A$, and $(\mathbb{Z}A, +, \otimes)$ is a ring.

(2) Our claim is that the fusion rules on $\mathbb{Z}A$ can be uniquely described by conversion formulae as follows, with C being positive integers, and D being integers:

$$a_{i_1} \otimes \ldots \otimes a_{i_k} = \sum_l \sum_{j_1 \ldots j_l} C_{i_1 \ldots i_k}^{j_1 \ldots j_l} a_{j_1 \ldots j_l}$$

$$a_{i_1 \ldots i_k} = \sum_l \sum_{j_1 \ldots j_l} D_{i_1 \ldots i_k}^{j_1 \ldots j_l} a_{j_1} \otimes \ldots \otimes a_{j_l}$$

Indeed, the existence and uniqueness of such decompositions follow from the definition of the tensor product operation, and by recurrence over k for the D coefficients.

(3) Our claim is that there is a unique morphism of rings $\Phi:\mathbb ZA\to R$, such that $\Phi(a_i)=r_i$ for any i. Indeed, consider the following elements of R:

$$r_{i_1\ldots i_k}=\sum_l\sum_{j_1\ldots j_l}D_{i_1\ldots i_k}^{j_1\ldots j_l}r_{j_1}\otimes\ldots\otimes r_{j_l}$$

In case we have a morphism as claimed, we must have $\Phi(a_x)=r_x$ for any $x\in F$. Thus our morphism is uniquely determined on A, so it is uniquely determined on $\mathbb ZA$.

In order to prove the existence, we can set $\Phi(a_x)=r_x$ for any $x\in F$, then extend Φ by linearity to the whole $\mathbb ZA$. Since Φ commutes with the above conversion formulae, which describe the fusion rules, it is indeed a morphism.

(4) Our claim is that Φ commutes with the linear forms $x\to\#(1\in x)$. Indeed, by linearity we just have to check the following equality:

$$\#(1\in a_{i_1}\otimes\ldots\otimes a_{i_k})=\#(1\in r_{i_1}\otimes\ldots\otimes r_{i_k})$$

Now remember that the elements r_i are defined as $r_i=u_i-\delta_{i0}1$. So, consider the elements $c_i=a_i+\delta_{i0}1$. Since the operations $r_i\to u_i$ and $a_i\to c_i$ are of the same nature, by linearity the above formula is equivalent to:

$$\#(1\in c_{i_1}\otimes\ldots\otimes c_{i_k})=\#(1\in u_{i_1}\otimes\ldots\otimes u_{i_k})$$

Now by using Theorem 10.15, what we have to prove is:

$$\#(1\in c_{i_1}\otimes\ldots\otimes c_{i_k})=\#NC_s(i_1\ldots i_k)$$

In order to prove this formula, consider the product on the left:

$$P=(a_{i_1}+\delta_{i_10}1)\otimes(a_{i_2}+\delta_{i_20}1)\otimes\ldots\otimes(a_{i_k}+\delta_{i_k0}1)$$

This quantity can be computed by using the fusion rules on A. A recurrence on k shows that the final components of type a_x will come from the different ways of grouping and summing the consecutive terms of the sequence $(i_1,\ldots,i_k)$, and removing some of the sums which vanish modulo s, as to obtain the sequence x. But this can be encoded by families of noncrossing partitions, and in particular the 1 components will come from the partitions in $NC_s(i_1\ldots i_k)$. Thus $\#(1\in P)=\#NC_s(i_1\ldots i_k)$, as claimed.

(5) Our claim now is that Φ is injective. Indeed, this follows from the result in the previous step, by using a standard positivity argument:

$$\begin{aligned}\Phi(\alpha)=0\quad&\implies\quad\Phi(\alpha\alpha^*)=0\\&\implies\quad\#(1\in\Phi(\alpha\alpha^*))=0\\&\implies\quad\#(1\in\alpha\alpha^*)=0\\&\implies\quad\alpha=0\end{aligned}$$

Here α is arbitrary in the domain of Φ, we use the notation $a_x^* = a_{\bar{x}}$, where $a \to \#(1, a)$ is the unique linear extension of the operation consisting of counting the number of 1's. Observe that this latter linear form is indeed positive definite, according to the identity $\#(1, a_x a_y^*) = \delta_{xy}$, which is clear from the definition of the product of $\mathbb{Z}A$.

(6) Our claim is that we have $\Phi(A) \subset R_{irr}$. This is the same as saying that $r_x \in R_{irr}$ for any $x \in F$, and we will prove it by recurrence on the length of x.

Assume that the assertion is true for all the words of length $< k$, and consider an arbitrary length k word, $x = i_1 \dots i_k$. We have:

$$a_{i_1} \otimes a_{i_2 \dots i_k} = a_x + a_{i_1+i_2, i_3 \dots i_k} + \delta_{i_1+i_2, 0} a_{i_3 \dots i_k}$$

By applying Φ to this decomposition, we obtain:

$$r_{i_1} \otimes r_{i_2 \dots i_k} = r_x + r_{i_1+i_2, i_3 \dots i_k} + \delta_{i_1+i_2, 0} r_{i_3 \dots i_k}$$

We have the following computation, which is valid for $y = i_1 + i_2, i_3 \dots i_k$, as well as for $y = i_3 \dots i_k$ in the case $i_1 + i_2 = 0$:

$$\begin{aligned}
\#(r_y \in r_{i_1} \otimes r_{i_2 \dots i_k}) &= \#(1, r_{\bar{y}} \otimes r_{i_1} \otimes r_{i_2 \dots i_k}) \\
&= \#(1, a_{\bar{y}} \otimes a_{i_1} \otimes a_{i_2 \dots i_k}) \\
&= \#(a_y \in a_{i_1} \otimes a_{i_2 \dots i_k}) \\
&= 1
\end{aligned}$$

Moreover, we know from the previous step that we have $r_{i_1+i_2, i_3 \dots i_k} \neq r_{i_3 \dots i_k}$, so we conclude that the following formula defines an element of R^+:

$$\alpha = r_{i_1} \otimes r_{i_2 \dots i_k} - r_{i_1+i_2, i_3 \dots i_k} - \delta_{i_1+i_2, 0} r_{i_3 \dots i_k}$$

On the other hand, we have $\alpha = r_x$, so we conclude that we have $r_x \in R^+$. Finally, the irreducibility of r_x follows from the following computation:

$$\begin{aligned}
\#(1 \in r_x \otimes \bar{r}_x) &= \#(1 \in r_x \otimes r_{\bar{x}}) \\
&= \#(1 \in a_x \otimes a_{\bar{x}}) \\
&= \#(1 \in a_x \otimes \bar{a}_x) \\
&= 1
\end{aligned}$$

(7) Summarizing, we have constructed an injective ring morphism:

$$\Phi : \mathbb{Z}A \to R$$

$$\Phi(A) \subset R_{irr}$$

The remaining fact to be proved, namely that we have $\Phi(A) = R_{irr}$, is clear from the general results in [147]. Indeed, since each element of $\mathbb{N}A$ is a sum of elements in A, by applying Φ we get that each element in $\Phi(\mathbb{N}A)$ is a sum of irreducible corepresentations in $\Phi(A)$. But since $\Phi(\mathbb{N}A)$ contains all the tensor powers between the fundamental corepresentation and its conjugate, we get $\Phi(A) = R_{irr}$, and we are done. $\square$

Let us discuss now the computation of the asymptotic laws of characters. We begin with a discussion for H_N, from [20], which has its own interest:

Theorem 10.19. *The asymptotic law of χ_t for the group H_N is given by*

$$b_t = e^{-t}\sum_{k=-\infty}^\infty \delta_k \sum_{p=0}^\infty \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

where δ_k is the Dirac mass at $k \in \mathbb{Z}$.

Proof. We regard the hyperoctahedral group H_N as being the symmetry group of the graph $I_N = \{I^1, \ldots, I^N\}$ formed by N segments. The diagonal coefficients are then:

$$u_{ii}(g) = \begin{cases} \ \ 0 \ \text{ if } g \text{ moves } I^i \\ +1 \text{ if } g \text{ fixes } I^i \\ -1 \text{ if } g \text{ returns } I^i \end{cases}$$

Let $s = [tN]$, and denote by $\uparrow g, \downarrow g$ the number of segments among $\{I^1, \ldots, I^s\}$ which are fixed, respectively returned by an element $g \in H_N$. With this notation, we have:

$$u_{11} + \ldots + u_{ss} = \uparrow g - \downarrow g$$

We denote by P_N probabilities computed over the group H_N. The density of the law of $u_{11} + \ldots + u_{ss}$ at a point $k \geq 0$ is given by the following formula:

$$\begin{aligned} D(k) &= P_N(\uparrow g - \downarrow g = k) \\ &= \sum_{p=0}^\infty P_N(\uparrow g = k + p, \downarrow g = p) \end{aligned}$$

Assume first $t = 1$. We use the fact that the probability of $\sigma \in S_N$ to have no fixed points is asymptotically $P_0 = \frac{1}{e}$. Thus the probability of $\sigma \in S_N$ to have m fixed points is asymptotically $P + m = \frac{1}{(em!)}$. In terms of probabilities over H_N, we obtain:

$$\begin{aligned} \lim_{N\to\infty} D(k) &= \lim_{N\to\infty}\sum_{p=0}^\infty (1/2)^{k+2p}\binom{k+2p}{k+p} P_N(\uparrow g + \downarrow g = k + 2p) \\ &= \sum_{p=0}^\infty (1/2)^{k+2p}\binom{k+2p}{k+p}\frac{1}{e(k+2p)!} \\ &= \frac{1}{e}\sum_{p=0}^\infty \frac{(1/2)^{k+2p}}{(k+p)!p!} \end{aligned}$$

The general case $t \in (0,1]$ follows by performing some modifications in the above computation. The asymptotic density is computed as follows:

$$\begin{aligned}
\lim_{N\to\infty} D(k) &= \lim_{N\to\infty}\sum_{p=0}^{\infty}(1/2)^{k+2p}\binom{k+2p}{k+p}P_N(\uparrow g + \downarrow g = k+2p)\\
&= \sum_{p=0}^{\infty}(1/2)^{k+2p}\binom{k+2p}{k+p}\frac{t^{k+2p}}{e^t(k+2p)!}\\
&= e^{-t}\sum_{p=0}^{\infty}\frac{(t/2)^{k+2p}}{(k+p)!p!}
\end{aligned}$$

On the other hand, we have $D(-k) = D(k)$, so we obtain the result. $\square$

Observe that the measure found above is of the form:

$$b_t = e^{-t}\sum_{k=-\infty}^{\infty}\delta_k\, f_k(t/2)$$

Here f_k is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty}\frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Next, we have the following result, once again from [20]:

Theorem 10.20. *The Bessel laws b_t have the additivity property*

$$b_s * b_t = b_{s+t}$$

so they form a truncated one-parameter semigroup with respect to convolution.

Proof. The Fourier transform of b_t is given by:

$$Fb_t(y) = e^{-t}\sum_{k=-\infty}^{\infty}e^{ky}\, f_k(t/2)$$

We compute now the derivative with respect to t:

$$Fb_t(y)' = -Fb_t(y) + \frac{e^{-t}}{2}\sum_{k=-\infty}^{\infty}e^{ky}\, f_k'(t/2)$$

On the other hand, the derivative of f_k with $k\geq 1$ is given by:

$$\begin{aligned}
f_k'(t) &= \sum_{p=0}^\infty\frac{(k+2p)t^{k+2p-1}}{(k+p)!p!}\\
&= \sum_{p=0}^\infty\frac{(k+p)t^{k+2p-1}}{(k+p)!p!}+\sum_{p=0}^\infty\frac{p\,t^{k+2p-1}}{(k+p)!p!}\\
&= \sum_{p=0}^\infty\frac{t^{k+2p-1}}{(k+p-1)!p!}+\sum_{p=1}^\infty\frac{t^{k+2p-1}}{(k+p)!(p-1)!}\\
&= \sum_{p=0}^\infty\frac{t^{(k-1)+2p}}{((k-1)+p)!p!}+\sum_{p=1}^\infty\frac{t^{(k+1)+2(p-1)}}{((k+1)+(p-1))!(p-1)!}\\
&= f_{k-1}(t)+f_{k+1}(t)
\end{aligned}$$

This computation works in fact for any k, so we get:

$$\begin{aligned}
Fb_t(y)' &= -Fb_t(y)+\frac{e^{-t}}{2}\sum_{k=-\infty}^\infty e^{ky}(f_{k-1}(t/2)+f_{k+1}(t/2))\\
&= -Fb_t(y)+\frac{e^{-t}}{2}\sum_{k=-\infty}^\infty e^{(k+1)y}f_k(t/2)+e^{(k-1)y}f_k(t/2)\\
&= -Fb_t(y)+\frac{e^y+e^{-y}}{2}Fb_t(y)\\
&= \left(\frac{e^y+e^{-y}}{2}-1\right)Fb_t(y)
\end{aligned}$$

Thus the log of the Fourier transform is linear in t, and we get the assertion. $\square$

In order to discuss now the free analogue β_t of the above measure b_t, as well as the s-analogues b_t^s,β_t^s of the measures b_t,β_t, we need some free probability. We have the following notion, extending the Poisson limit theory from section 5:

Definition 10.21. *Associated to any compactly supported positive measure ρ on $\mathbb R$ are the probability measures*

$$p_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}$$

$$\pi_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{\boxplus n}$$

where $c=mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho=\delta_t$ with $t>0$, which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

Proposition 10.22. *For $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$ we have*

$$F_{p_\rho}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right)$$

$$R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i}$$

where F, R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure in Definition 10.21, under the convolution signs:

$$\mu_n = \left(1 - \frac{c}{n}\right)\delta_0 + \frac{1}{n}$$

In the classical case, we have the following computation:

$$F_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^s c_i e^{iyz_i}$$

$$\implies \quad F_{\mu_n^{*n}}(y) = \left(\left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^s c_i e^{iyz_i}\right)^n$$

$$\implies \quad F_{p_\rho}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right)$$

In the free case now, we use a similar method. The Cauchy transform of μ_n is:

$$G_{\mu_n}(\xi) = \left(1 - \frac{c}{n}\right)\frac{1}{\xi} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{\xi - z_i}$$

Consider now the R-transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y) = nR_{\mu_n}(y)$$

The above formula of G_{μ_n} shows that the equation for $R = R_{\mu_n^{\boxplus n}}$ is as follows:

$$\left(1 - \frac{c}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{y^{-1} + R/n - z_i} = y$$

$$\implies \quad \left(1 - \frac{c}{n}\right)\frac{1}{1 + yR/n} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} = 1$$

Now by multiplying by n, rearranging the terms, and letting $n \to \infty$, we get:

$$\begin{aligned}
\frac{c+yR}{1+yR/n} &= \sum_{i=1}^s \frac{c_i}{1+yR/n-yz_i}\\
\implies \quad c+yR_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i}{1-yz_i}\\
\implies \quad R_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_iz_i}{1-yz_i}
\end{aligned}$$

This finishes the proof in the free case, and we are done. □

We have as well the following result, providing an alternative to Definition 10.21:

Theorem 10.23. *For $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$ we have*

$$p_\rho/\pi_\rho = \mathrm{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Let α be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/R-transform of α is given by the formulae in Proposition 10.22.

Indeed, by using some well-known Fourier transform formulae, we have:

$$\begin{aligned}
F_{\alpha_i}(y) = \exp(c_i(e^{iy}-1)) \quad &\implies \quad F_{z_i\alpha_i}(y) = \exp(c_i(e^{iyz_i}-1))\\
&\implies \quad F_\alpha(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)
\end{aligned}$$

Also, by using some well-known R-transform formulae, we have:

$$\begin{aligned}
R_{\alpha_i}(y) = \frac{c_i}{1-y} \quad &\implies \quad R_{z_i\alpha_i}(y) = \frac{c_iz_i}{1-yz_i}\\
&\implies \quad R_\alpha(y) = \sum_{i=1}^s \frac{c_iz_i}{1-yz_i}
\end{aligned}$$

Thus we have indeed the same formulae as those in Proposition 10.22. □

We can go back now to quantum reflection groups, and we have:

Theorem 10.24. *The asymptotic laws of truncated characters are as follows, where ε_s with $s \in \{1, 2, \ldots, \infty\}$ is the uniform measure on the s-th roots of unity:*

(1) *For H_N^s we obtain the compound Poisson law $b_t^s = p_{t\varepsilon_s}$.*
(2) *For H_N^{s+} we obtain the compound free Poisson law $\beta_t^s = \pi_{t\varepsilon_s}$.*

These measures are in Bercovici-Pata bijection.

Proof. This follows from easiness, and from the Weingarten formula. To be more precise, at $t=1$ this follows by counting the partitions, and at $t\in(0,1]$ general, this follows in the usual way, for instance by using cumulants. For details here, we refer to [11]. $\square$

The Bessel and free Bessel laws have particularly interesting properties at the parameter values $s=2,\infty$. So, let us record the precise statement here:

Theorem 10.25. *The asymptotic laws of truncated characters are as follows:*

(1) *For H_N we obtain the real Bessel law $b_t=p_{t\varepsilon_2}$.*
(2) *For K_N we obtain the complex Bessel law $B_t=p_{t\varepsilon_\infty}$.*
(3) *For H_N^+ we obtain the free real Bessel law $\beta_t=\pi_{t\varepsilon_2}$.*
(4) *For K_N^+ we obtain the free complex Bessel law $\mathfrak{B}_t=\pi_{t\varepsilon_\infty}$.*

Proof. This follows indeed from Theorem 10.24 above, at $s=2,\infty$. $\square$

Our next task will be that upgrading our results about π_t in this setting, using a parameter $s\in\mathbb{N}\cup\{\infty\}$. We have here the following result:

Theorem 10.26. *The moments of the various central limiting measures, namely*

$$\begin{tikzcd}
\beta_t^s \arrow[r, dash] \arrow[d, dash] & \gamma_t \arrow[r, dash] \arrow[d, dash] & \Gamma_t \arrow[d, dash] \\
b_t^s \arrow[r, dash] & g_t \arrow[r, dash] & G_t
\end{tikzcd}$$

are always given by the same formula, involving partitions, namely

$$M_k=\sum_{\pi\in D(k)}t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively

$$\begin{tikzcd}
NC^s \arrow[d] & NC_2 \arrow[l] \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\
P^s & P_2 \arrow[l] & \mathcal{P}_2 \arrow[l]
\end{tikzcd}$$

and where $|.|$ is the number of blocks.

Proof. This follows by putting together the various moment results that we have. $\square$

As already mentioned, in what regards the Bessel and free Bessel laws b_t^s,β_t^s, the important particular cases are $s=1,2,\infty$. It is therefore tempting to leave one of these 3 cases aside, and fold the corresponding diagram into a cube.

Quite surprisingly, in order to do so, in a correct way, the case which must be left aside is the most important one, namely $s = 1$, corresponding to the Poisson and free Poisson laws p_t, π_t. We will comment later on this, but let us just start by doing so:

Theorem 10.27. *The moments of the various central limiting measures, namely*

$$\begin{tikzcd}
& \mathfrak{B}_t \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \Gamma_t \arrow[dl, dash] \arrow[dd, dash] \\
\beta_t \arrow[rr, dash] \arrow[dd, dash] & & \gamma_t \arrow[dd, dash] & \\
& B_t \arrow[rr, dash] \arrow[dl, dash] & & G_t \arrow[dl, dash] \\
b_t \arrow[rr, dash] & & g_t &
\end{tikzcd}$$

are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively

$$\begin{tikzcd}
& \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
& \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

and where $|.|$ is the number of blocks.

Proof. This follows by putting together the various moment results that we have. $\square$

In addition to what has been said above, there are as well some interesting results about the Bessel and free Bessel laws involving the multiplicative convolution $\times$, and the multiplicative free convolution $\boxtimes$ from [136]. For details, we refer here to [11].

11. Liberation theory

We have seen in the previous section that the basic reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$ have free analogues $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$, and that the theory of these quantum groups, both classical and free, is very interesting, algebrically and analytically speaking.

The world of quantum reflection groups is in fact much wider than this. In the classical case already, the classification theorem for the complex reflection groups, a celebrated result by Shephard and Todd [128], from the 50s, is as follows:

Theorem 11.1. *The irreducible complex reflection groups are*

$$H_N^{sd} = \left\{ U \in H_N^s \Big| (\det U)^d = 1 \right\}$$

along with 34 *exceptional examples.*

Proof. This is something quite advanced, and we refer here to the paper of Shephard and Todd [128], and to the subsequent literature on the subject. □

In the general quantum case now, the axiomatization and classification of the quantum reflection groups is a key problem, which is not understood yet. We will be interested in what follows in the "twistable" case, where the theory is more advanced than in the general case. Let us start with the following definition:

Definition 11.2. *A closed subgroup $G \subset U_N^+$ is called:*
1. *Half-homogeneous, when it contains the alternating group, $A_N \subset G$.*
2. *Homogeneous, when it contains the symmetric group, $S_N \subset G$.*
3. *Twistable, when it contains the hyperoctahedral group, $H_N \subset G$.*

These notions are mostly motivated by the easy case. Here we have by definition $S_N \subset G \subset U_N^+$, and so our quantum group is automatically homogeneous. The point now is that the twistability assumption corresponds to the following condition, at the level of the associated category of partitions $D \subset P$:

$$D \subset P_{even}$$

We recognize here the condition which is needed for performing the Schur-Weyl twisting operation, explained in section 9 above, and based on the signature map:

$$\varepsilon : P_{even} \to \{\pm 1\}$$

As a conclusion, in the easy case our notion of twistability is the correct one. In general, there are of course more general twisting methods, usually requiring $\mathbb{Z}_2^N \subset G$ only. But in the half-homogeneous case, the condition $\mathbb{Z}_2^N \subset G$ is equivalent to $H_N \subset G$.

With this discussion done, let us formulate now the following definition:

Definition 11.3. *A twistable quantum reflection group is an intermediate subgroup*

$$H_N \subset K \subset K_N^+$$

between the group $H_N = \mathbb{Z}_2 \wr S_N$, and the quantum group $K_N^+ = \mathbb{T} \wr_* S_N^+$.

Here is now another definition, which is important for general compact quantum group purposes, and which provides motivations for our formalism from Definition 11.3:

Definition 11.4. *Given a closed subgroup $G \subset U_N^+$ which is twistable, in the sense that we have $H_N \subset G$, we define its associated reflection subgroup to be*

$$K = G \cap K_N^+$$

with the intersection taken inside U_N^+. We say that G appears as a soft liberation of its classical version $G_{class} = G \cap U_N$ when $G =< G_{class}, K >$.

These notions are important in the classification theory of compact quantum groups, and in connection with certain noncommutative geometry questions as well. As a first observation, with K being as above, we have an intersection diagram, as follows:

$$\begin{tikzcd}
K \arrow[r] & G \\
K_{class} \arrow[r] \arrow[u] & G_{class} \arrow[u]
\end{tikzcd}$$

The soft liberation condition states that this diagram must be a generation diagram. We will be back to this in a moment, with some further theoretical comments. Let us work out some examples. As a basic result, we have:

Theorem 11.5. *The reflection subgroups of the basic unitary quantum groups*

$$\begin{tikzcd}
U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\
O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

are as follows,

$$\begin{tikzcd}
K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\
H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^+ \arrow[u]
\end{tikzcd}$$

and these unitary quantum groups all appear via soft liberation.

Proof. The fact that the reflection subgroups of the quantum groups on the left are those on the right is clear in all cases, with the middle objects being by definition:

$$H_N^* = H_N \cap O_N^*$$
$$K_N^* = K_N \cap U_N^*$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case there is nothing to prove, because any classical group is by definition a soft liberation of itself.

(2) In the half-classical case the results are non-trivial, but can be proved by using the technology developed by Bichon and Dubois-Violette in [54].

(3) In the free case the results are highly non-trivial, and the only known proof so far uses the recurrence methods developed by Chirvasitu in [68]. □

Summarizing, we are here into recent and interesting quantum group theory. We will discuss a bit later the concrete applications of Theorem 11.5. There is a connection here as well with the notion of diagonal torus, introduced in section 1 above. We can indeed refine Definition 11.4, in the following way:

Definition 11.6. *Given $H_N \subset G \subset U_N^+$, the diagonal tori $T = G \cap \mathbb{T}_N^+$ and reflection subgroups $K = G \cap K_N^+$ for G and for $G_{class} = G \cap U_N$ form a diagram as follows:*

$$\begin{tikzcd}
T \arrow[r] & K \arrow[r] & G \\
T_{class} \arrow[r] \arrow[u] & K_{class} \arrow[r] \arrow[u] & G_{class} \arrow[u]
\end{tikzcd}$$

We say that G appears as a soft/hard liberation when it is generated by G_{class} and by K/T, which means that the right square/whole rectangle should be generation diagrams.

It is in fact possible to further complicate the picture, by adding free versions as well, with these free versions being by definition given by the following formula:

$$G_{free} = < G, S_N^+ >$$

Importantly, we can equally add the parameter $N \in \mathbb{N}$ to the picture, the idea being that we have a kind of "ladder", whose steps are the diagrams in Definition 11.6, perhaps extended with the free versions too, at fixed values of $N \in \mathbb{N}$.

The various generation and intersection properties of this ladder are important properties of $G = (G_N)$ itself, with subtle relations between them. In fact, as already mentioned in the proof of Theorem 11.5 above, the proof of the soft generation property for O_N^+, U_N^+ uses in fact this ladder, via the recurrence methods developed in [68].

All this is quite technical, so as a concrete result in connection with the above hard liberation notion, we have the following statement, improving Theorem 11.5:

Theorem 11.7. *The diagonal tori of the basic unitary quantum groups*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

are as follows,

$$\begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[r] \arrow[u] & T_N^* \arrow[r] \arrow[u] & T_N^+ \arrow[u] \end{tikzcd}$$

and these unitary quantum groups all appear via hard liberation.

Proof. The first assertion is something that we already know, from section 1 above. As for the second assertion, this can be proved by carefully examining the proof of Theorem 11.5, and performing some suitable modifications, where needed. □

As an interesting remark, some subtleties appear in the following way:

Proposition 11.8. *The diagonal tori of the basic quantum reflection groups*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^+ \arrow[u] \end{tikzcd}$$

are as follows,

$$\begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[r] \arrow[u] & T_N^* \arrow[r] \arrow[u] & T_N^+ \arrow[u] \end{tikzcd}$$

and these quantum reflection groups do not all appear via hard liberation.

Proof. The first assertion is clear, for instance as a consequence of Theorem 11.7, because the diagonal torus is the same for a quantum group, and for its reflection subgroup:

$$G\cap\mathbb T_N^+=(G\cap K_N^+)\cap\mathbb T_N^+$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case the hard liberation property definitely holds, because any classical group is by definition a hard liberation of itself.

(2) In the half-classical case the answer is again positive, and this can be proved by using the technology developed by Bichon and Dubois-Violette in [54].

(3) In the free case the hard liberation property fails, due to some intermediate quantum groups $H_N^{[\infty]}$, $K_N^{[\infty]}$, where "hard liberation stops". We will be back to this. □

As a conjectural solution to these latter difficulties, coming from Proposition 11.8, we have the notion of Fourier liberation, that we will discuss now. Let us first discuss the group dual subgroups of the arbitrary compact quantum groups $G\subset U_N^+$. To start with, we have the following basic statement:

Proposition 11.9. *Let $G\subset U_N^+$ be a compact quantum group, and consider the group dual subgroups $\widehat{\Lambda}\subset G$, also called toral subgroups, or simply "tori".*

1. *In the classical case, where $G\subset U_N$ is a compact Lie group, these are the usual tori, where by torus we mean here closed abelian subgroup.*
2. *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group, these are the duals of the various quotients $\Gamma\to\Lambda$.*

Proof. Both these assertions are elementary, as follows:

(1) This follows indeed from the fact that a closed subgroup $H\subset U_N^+$ is at the same time classical, and a group dual, precisely when it is classical and abelian.

(2) This follows from the general propreties of the Pontrjagin duality, and more precisely from the fact that the subgroups $\widehat{\Lambda}\subset\widehat{\Gamma}$ correspond to the quotients $\Gamma\to\Lambda$. □

At a more concrete level now, most of the tori that we met appear as diagonal tori. However, for certain quantum groups like the bistochastic ones, or the quantum permutation group ones, this torus collapses to $\{1\}$, and so it cannot be of use in the study of G. In order to deal with this issue, the idea, from [41], will be that of using:

Proposition 11.10. *Given a closed subgroup $G\subset U_N^+$ and a matrix $Q\in U_N$, we let $T_Q\subset G$ be the diagonal torus of G, with fundamental representation spinned by Q:*

$$C(T_Q)=C(G)\Big/\left\langle(QuQ^*)_{ij}=0\Big|\forall i\neq j\right\rangle$$

This torus is then a group dual, $T_Q=\widehat{\Lambda}_Q$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=(QuQ^*)_{ii}$, which are unitaries inside $C(T_Q)$.

Proof. This follows indeed from our results, because, as said in the statement, T_Q is by definition a diagonal torus. Equivalently, since $v = QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i = v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q) = C^*(\Lambda_Q)$, where $\Lambda_Q =< g_1, \ldots, g_N >$ is the group in the statement, and this gives the result. □

Summarizing, associated to any closed subgroup $G \subset U_N^+$ is a whole family of tori, indexed by the unitaries $U \in U_N$. As a first result regarding these tori, we have:

Theorem 11.11. *Any torus $T \subset G$ appears as follows, for a certain $Q \in U_N$:*

$$T \subset T_Q \subset G$$

In other words, any torus appears inside a standard torus.

Proof. Given a torus $T \subset G$, we have an inclusion $T \subset G \subset U_N^+$. On the other hand, we know that each torus $T = \widehat{\Lambda} \subset U_N^+$, coming from a discrete group $\Lambda =< g_1, \ldots, g_N >$, has a fundamental corepresentation as follows, with $Q \in U_N$:

$$u = Qdiag(g_1, \ldots, g_N)Q^*$$

But this shows that we have $T \subset T_Q$, and this gives the result. □

Let us do now some computations. In the classical case, the result is as follows:

Proposition 11.12. *For a closed subgroup $G \subset U_N$ we have*

$$T_Q = G \cap (Q^* \mathbb{T}^N Q)$$

where $\mathbb{T}^N \subset U_N$ is the group of diagonal unitary matrices.

Proof. This is indeed clear at $Q = 1$, where Γ_1 appears by definition as the dual of the compact abelian group $G \cap \mathbb{T}^N$. In general, this follows by conjugating by Q. □

In the group dual case now, we have the following result, from [47]:

Proposition 11.13. *Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, consider its dual compact quantum group $G = \widehat{\Gamma}$, diagonally embedded into U_N^+. We have then*

$$\Lambda_Q = \Gamma \Big/ \left\langle g_i = g_j \Big| \exists k, Q_{ki} \neq 0, Q_{kj} \neq 0 \right\rangle$$

with the embedding $T_Q \subset G = \widehat{\Gamma}$ coming from the quotient map $\Gamma \to \Lambda_Q$.

Proof. Assume indeed that $\Gamma =< g_1, \ldots, g_N >$ is a discrete group, with $\widehat{\Gamma} \subset U_N^+$ coming via $u = diag(g_1, \ldots, g_N)$. With $v = QuQ^*$, we have:

$$\begin{aligned}\sum_s \bar{Q}_{si} v_{sk} &= \sum_{st} \bar{Q}_{si} Q_{st} \bar{Q}_{kt} g_t \\ &= \sum_t \delta_{it} \bar{Q}_{kt} g_t \\ &= \bar{Q}_{ki} g_i\end{aligned}$$

Thus $v_{ij} = 0$ for $i \neq j$ gives $\bar{Q}_{ki} v_{kk} = \bar{Q}_{ki} g_i$, which is the same as saying that $Q_{ki} \neq 0$ implies $g_i = v_{kk}$. But this latter equality reads:

$$g_i = \sum_j |Q_{kj}|^2 g_j$$

We conclude from this that $Q_{ki} \neq 0, Q_{kj} \neq 0$ implies $g_i = g_j$, as desired. As for the converse, this is elementary to establish as well. □

In view of the above, we can expect the collection $\{T_Q | Q \in U_N\}$ to encode various algebraic and analytic properties of G. We have the following result, from [41]:

Theorem 11.14. *The following results hold, both for the compact Lie groups, and for the duals of the finitely generated discrete groups:*

(1) *Generation: any closed quantum subgroup $G \subset U_N^+$ has the generation property $G =< T_Q | Q \in U_N >$. In other words, G is generated by its tori.*
(2) *Characters: if G is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that P becomes nonzero, when mapped into $C(T_Q)$.*
(3) *Amenability: a closed subgroup $G \subset U_N^+$ is coamenable if and only if each of the tori T_Q is coamenable, in the usual discrete group sense.*
(4) *Growth: assuming $G \subset U_N^+$, the discrete quantum group $\widehat{G}$ has polynomial growth if and only if each the discrete groups $\widehat{T_Q}$ has polynomial growth.*

Proof. In the classical case, where $G \subset U_N$, the proof is elementary, based on standard facts from linear algebra, and goes as follows:

(1) Generation. We use the following formula, established above:

$$T_Q = G \cap Q^* \mathbb{T}^N Q$$

Since any group element $U \in G$ is diagonalizable, $U = Q^* D Q$ with $Q \in U_N, D \in \mathbb{T}^N$, we have $U \in T_Q$ for this value of $Q \in U_N$, and this gives the result.

(2) Characters. We can take here $Q \in U_N$ to be such that $QTQ^* \subset \mathbb{T}^N$, where $T \subset U_N$ is a maximal torus for G, and this gives the result.

(3) Amenability. This conjecture holds trivially in the classical case, $G \subset U_N$, due to the fact that these latter quantum groups are all coamenable.

(4) Growth. This is something nontrivial, well-known from the theory of compact Lie groups, and we refer here for instance to the literature.

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix Q. □

The various statements above are conjectured to hold for any compact quantum group. We refer to [41] and to subsequent papers for a number of verifications, notably covering many basic examples of easy quantum groups, as well as half-liberations.

Let us focus now on the generation property. We will need:

Proposition 11.15. *Given a closed subgroup $G \subset U_N^+$ and a matrix $Q \in U_N$, the corresponding standard torus and its Tannakian category are given by*

$$T_Q = G \cap \mathbb{T}_Q$$

$$C_{T_Q} = < C_G, C_{\mathbb{T}_Q} >$$

where $\mathbb{T}_Q \subset U_N^+$ is the dual of the free group $F_N = < g_1, \ldots, g_N >$, with the fundamental corepresentation of $C(\mathbb{T}_Q)$ being the matrix $u = Qdiag(g_1, \ldots, g_N)Q^*$.

Proof. The first assertion comes from the well-known fact that given two closed subgroups $G, H \subset U_N^+$, the corresponding quotient algebra $C(U_N^+) \to C(G \cap H)$ appears by dividing by the kernels of both the quotient maps

$$C(U_N^+) \to C(G) \quad , \quad C(U_N^+) \to C(H)$$

Indeed, the construction of T_Q amounts precisely in performing this operation, with $H = \mathbb{T}_Q$, and so we obtain $T_Q = G \cap \mathbb{T}_Q$, as claimed. As for the Tannakian category formula, this follows from this, and from the general duality formula $C_{G \cap H} = < C_G, C_H >$. □

We have the following Tannakian reformulation of the generation property:

Theorem 11.16. *Given a closed subgroup $G \subset U_N^+$, the subgroup*

$$G' = < T_Q | Q \in U_N >$$

generated by its standard tori has the following Tannakian category:

$$C_{G'} = \bigcap_{Q \in U_N} < C_G, C_{\mathbb{T}_Q} >$$

In particular we have $G = G'$ when this intersection reduces to C_G.

Proof. Consider indeed the subgroup $G' \subset G$ constructed in the statement. We have:

$$C_{G'} = \bigcap_{Q \in U_N} C_{T_Q}$$

Together with the formula in Proposition 11.15, this gives the result. □

The above result can be used for investigating the toral generation conjecture, but the combinatorics is quite difficult, and there are no results yet, along these lines. Let us further discuss now the toral generation property, with some modest results, regarding its behaviour with respect to product operations. We first have:

Proposition 11.17. *Given two closed subgroups $G,H\subset U_N^+$, and $Q\in U_N$, we have:*

$$<T_Q(G),T_Q(H)>\subset T_Q(<G,H>)$$

Also, the toral generation property is stable under the operation $<\,,>$.

Proof. The first assertion can be proved either by using Theorem 11.16, or directly. For the direct proof, which is perhaps the simplest, we have:

$$T_Q(G)=G\cap\mathbb{T}_Q\subset<G,H>\cap\mathbb{T}_Q=T_Q(<G,H>)$$

We have as well the following computation:

$$T_Q(H)=H\cap\mathbb{T}_Q\subset<G,H>\cap\mathbb{T}_Q=T_Q(<G,H>)$$

Now since $A,B\subset C$ implies $<A,B>\subset C$, this gives the result. Regarding now the second assertion, we have the following computation:

$$\begin{aligned}<G,H> &= <<T_Q(G)|Q\in U_N>,<T_Q(H)|Q\in U_N>>\\ &= <T_Q(G),T_Q(H)|Q\in U_N>\\ &= <<T_Q(G),T_Q(H)>|Q\in U_N>\\ &\subset <T_Q(<G,H>)|Q\in U_N>\end{aligned}$$

Thus the quantum group $<G,H>$ is generated by its tori, as claimed. □

We have as well the following result:

Proposition 11.18. *We have the following formula, for any G,H and R,S:*

$$T_{R\otimes S}(G\times H)=T_R(G)\times T_S(H)$$

Also, the toral generation property is stable under usual products $\times$.

Proof. The product formula is clear. Regarding now the second assertion, we have:

$$\begin{aligned}<T_Q(G\times H)|Q\in U_{MN}> &\supset <T_{R\otimes S}(G\times H)|R\in U_M,S\in U_N>\\ &= <T_R(G)\times T_S(H)|R\in U_M,S\in U_N>\\ &= <T_R(G)\times\{1\},\{1\}\times T_S(H)|R\in U_M,S\in U_N>\\ &= <T_R(G)|R\in U_M>\times<T_G(H)|H\in U_N>\\ &= G\times H\end{aligned}$$

Thus the quantum group $G\times H$ is generated by its tori, as claimed. □

Let us go back now to the quantum permutation groups. In relation with the tori, let us start with the following basic fact, which generalizes the embedding $\widehat{D}_\infty \subset S_4^+$ that we met in section 2 above, when proving that we have $S_4^+ \neq S_4$:

Proposition 11.19. *Consider a discrete group generated by elements of finite order, written as a quotient group, as follows:*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

We have then an embedding of quantum groups $\widehat{\Gamma} \subset S_N^+$, where $N = N_1 + \ldots + N_k$.

Proof. We have a sequence of embeddings and isomorphisms as follows:

$$\begin{aligned}
\widehat{\Gamma} &\subset \mathbb{Z}_{N_1} \widehat{* \ldots *} \mathbb{Z}_{N_k} \\
&= \widehat{\mathbb{Z}_{N_1}} \,\hat{*}\, \ldots \,\hat{*}\, \widehat{\mathbb{Z}_{N_k}} \\
&\simeq \mathbb{Z}_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{N_k} \\
&\subset S_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k} \\
&\subset S_{N_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k}^+ \\
&\subset S_N^+
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is quite abstract, and it is worth working out the details, with an explicit formula for the associated magic matrix. Let us start with a study of the simplest situation, where $k = 1$, and where $\Gamma = \mathbb{Z}_{N_1}$. The result here is as follows:

Proposition 11.20. *The magic matrix for the quantum permutation group*

$$\widehat{\mathbb{Z}}_N \simeq \mathbb{Z}_N \subset S_N \subset S_N^+$$

with standard Fourier isomorphism on the left, is given by the formula

$$u = FIF^*$$

where $F = \frac{1}{\sqrt{N}}(w^{ij})$ with $w = e^{2\pi i/N}$ is the Fourier matrix, and where

$$I = \begin{pmatrix} 1 &&& \\ & g && \\ && \ddots & \\ &&& g^{N-1} \end{pmatrix}$$

is the diagonal matrix formed by the elements of $\mathbb{Z}_N$, regarded as elements of $C^*(\mathbb{Z}_N)$.

Proof. The magic matrix for the quantum group $\mathbb{Z}_N \subset S_N \subset S_N^+$ is given by:

$$\begin{aligned}
v_{ij} &= \chi\left(\sigma \in \mathbb{Z}_N \middle| \sigma(j) = i\right) \\
&= \delta_{i-j}
\end{aligned}$$

Let us apply now the Fourier transform. According to our Pontrjagin duality conventions from section 1 above, we have a pair of inverse isomorphisms, as follows:

$$\Phi : C(\mathbb{Z}_N) \to C^*(\mathbb{Z}_N) \quad , \quad \delta_i \to \frac{1}{N}\sum_k w^{ik} g^k$$

$$\Psi : C^*(\mathbb{Z}_N) \to C(\mathbb{Z}_N) \quad , \quad g^i \to \sum_k w^{-ik}\delta_k$$

Here $w = e^{2\pi i/N}$, and we use the standard Fourier analysis convention that the indices are $0, 1, \ldots, N-1$. With $F = \frac{1}{\sqrt{N}}(w^{ij})$ and $I = diag(g^i)$ as above, we have:

$$\begin{aligned} u_{ij} &= \Phi(v_{ij}) \\ &= \frac{1}{N}\sum_k w^{(i-j)k} g^k \\ &= \frac{1}{N}\sum_k w^{ik} g^k w^{-jk} \\ &= \sum_k F_{ik} I_{kk} (F^*)_{kj} \\ &= (FIF^*)_{ij} \end{aligned}$$

Thus, the magic matrix that we are looking for is $u = FIF^*$, as claimed. □

With the above result in hand, we can refine Proposition 11.19, as follows:

Theorem 11.21. *Given a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, we have an embedding $\widehat{\Gamma} \subset S_N^+$, with $N = N_1 + \ldots + N_k$, with magic matrix given by the formula*

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* & & \\ & \ddots & \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix}$$

where $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$ with $w_N = e^{2\pi i/N}$ are Fourier matrices, and where

$$I_r = \begin{pmatrix} 1 & & & \\ & g_r & & \\ & & \ddots & \\ & & & g_r^{N_r - 1} \end{pmatrix}$$

with $g_1, \ldots, g_k$ being the standard generators of Γ.

Proof. This follows indeed from Proposition 11.19 and Proposition 11.20. □

As explained in [51], as a consequence of the orbit theory developed there, any group dual subgroup $\widehat{\Gamma}\subset S_N^+$ appears in the above way, from a partition $N=N_1+\ldots+N_k$, and then a quotient group $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}\to\Gamma$. We will be back to this in section 13 below, when discussing the orbit theory developed in [51], and its applications.

In the meantime, we can recover this result, that we will need in what follows, by using our maximal torus method. Following [12], we have indeed the following result:

Theorem 11.22. *For the quantum permutation group S_N^+, we have:*

(1) *Given $Q\in U_N$, the quotient $F_N\to\Lambda_Q$ comes from the following relations:*

$$\begin{cases}g_i=1&\text{if }\sum_lQ_{il}\neq0\\ g_ig_j=1&\text{if }\sum_lQ_{il}Q_{jl}\neq0\\ g_ig_jg_k=1&\text{if }\sum_lQ_{il}Q_{jl}Q_{kl}\neq0\end{cases}$$

(2) *Given a decomposition $N=N_1+\ldots+N_k$, for the matrix $Q=diag(F_{N_1},\ldots,F_{N_k})$, where $F_N=\frac{1}{\sqrt N}(\xi^{ij})_{ij}$ with $\xi=e^{2\pi i/N}$ is the Fourier matrix, we obtain:*

$$\Lambda_Q=\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}$$

(3) *Given an arbitrary matrix $Q\in U_N$, there exists a decomposition $N=N_1+\ldots+N_k$, such that Λ_Q appears as quotient of $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}$.*

Proof. This is more or less equivalent to the classification of the group dual subgroups $\widehat{\Gamma}\subset S_N^+$ from [51], and can be proved by a direct computation, as follows:

(1) Fix a unitary matrix $Q\in U_N$, and consider the following quantities:

$$\begin{cases}c_i=\sum_lQ_{il}\\ c_{ij}=\sum_lQ_{il}Q_{jl}\\ d_{ijk}=\sum_l\bar Q_{il}\bar Q_{jl}Q_{kl}\end{cases}$$

We write $w=QvQ^*$, where v is the fundamental corepresentation of $C(S_N^+)$. Assume $X\simeq\{1,\ldots,N\}$, and let α be the coaction of $C(S_N^+)$ on $C(X)$. Let us set:

$$\varphi_i=\sum_l\bar Q_{il}\delta_l\in C(X)$$

Also, let $g_i=(QvQ^*)_{ii}\in C^*(\Lambda_Q)$. If β is the restriction of α to $C^*(\Lambda_Q)$, then:

$$\beta(\varphi_i)=\varphi_i\otimes g_i$$

Now recall that $C(X)$ is the universal C^*-algebra generated by elements $\delta_1,\ldots,\delta_N$ which are pairwise orthogonal projections. Writing these conditions in terms of the linearly independent elements φ_i by means of the formulae $\delta_i = \sum_l Q_{il}\varphi_l$, we find that the universal relations for $C(X)$ in terms of the elements φ_i are as follows:

$$\begin{cases}\sum_i c_i\varphi_i = 1\\ \varphi_i^* = \sum_j c_{ij}\varphi_j\\ \varphi_i\varphi_j = \sum_k d_{ijk}\varphi_k\end{cases}$$

Let $\tilde{\Lambda}_Q$ be the group in the statement. Since β preserves these relations, we get:

$$\begin{cases}c_i(g_i - 1) = 0\\ c_{ij}(g_ig_j - 1) = 0\\ d_{ijk}(g_ig_j - g_k) = 0\end{cases}$$

We conclude from this that Λ_Q is a quotient of $\tilde{\Lambda}_Q$. On the other hand, it is immediate that we have a coaction map as follows:

$$C(X) \to C(X) \otimes C^*(\tilde{\Lambda}_Q)$$

Thus $C(\tilde{\Lambda}_Q)$ is a quotient of $C(S_N^+)$. Since w is the fundamental corepresentation of S_N^+ with respect to the basis $\{\varphi_i\}$, it follows that the generator w_{ii} is sent to $\tilde{g}_i \in \tilde{\Lambda}_Q$, while w_{ij} is sent to zero. We conclude that $\tilde{\Lambda}_Q$ is a quotient of Λ_Q. Since the above quotient maps send generators on generators, we conclude that $\Lambda_Q = \tilde{\Lambda}_Q$, as desired.

(2) We apply the result found in (1), with the N-element set X used in the proof there chosen to be the following set:

$$X = \mathbb{Z}_{N_1} \sqcup \ldots \sqcup \mathbb{Z}_{N_k}$$

With this choice, we have $c_i = \delta_{i0}$ for any i. Also, we have $c_{ij} = 0$, unless i, j, k belong to the same block to Q, in which case $c_{ij} = \delta_{i+j,0}$, and also $d_{ijk} = 0$, unless i, j, k belong to the same block of Q, in which case $d_{ijk} = \delta_{i+j,k}$. We conclude from this that Λ_Q is the free product of k groups which have generating relations as follows:

$$g_ig_j = g_{i+j} \quad , \quad g_i^{-1} = g_{-i}$$

But this shows that our group is $\Lambda_Q = \mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$, as stated.

(3) This follows indeed from (2). See [47]. □

As already mentioned, there are many other possible ways of recovering the above results, the standard way being via orbit theory. We will be back to this in section 13 below, when discussing the orbit theory developed in [51], and its applications.

In connection with our liberation questions now, in the quantum permutation group case, the standard tori parametrized by Fourier matrices play a special role.

This suggests the following definition:

Definition 11.23. *Consider a closed subgroup $G\subset U_N^+$.*

1. *Its standard tori T_F, with $F=F_{N_1}\otimes\ldots\otimes F_{N_k}$, and $N=N_1+\ldots+N_k$ being regarded as a partition, are called Fourier tori.*
2. *In the case where we have $G_N=<G_N^c,(T_F)_F>$, we say that G_N appears as a Fourier liberation of its classical version G_N^c.*

The conjecture is that the easy quantum groups should appear as Fourier liberations. With respect to the basic examples, the situation in the free case is as follows:

(1) O_N^+,U_N^+ are diagonal liberations, so they are Fourier liberations as well.

(2) B_N^+,C_N^+ are Fourier liberations too, with this being standard.

(3) S_N^+ is a Fourier liberation too, being generated by its tori [62], [69].

(4) H_N^+,K_N^+ remain to be investigated, by using the general theory in [125].

As a word of warning here, observe that an arbitrary classical group $G_N\subset U_N$ is not necessarily generated by its Fourier tori, and nor is an arbitrary discrete group dual, with spinned embedding. Thus, the Fourier tori, and the related notion of Fourier liberation, remain something quite technical, in connection with the easy case.

As an application of all this, let us go back to quantum permutation groups, and more specifically to the quantum symmetry groups of finite graphs, from section 9 above. One interesting question is whether $G^+(X)$ appears as a Fourier liberation of $G(X)$. Generally speaking, this is something quite difficult, because for the empty graph itself we are in need of the above-mentioned technical results from [62], [69].

In order to discuss this, let us begin with the following elementary statement:

Theorem 11.24. *In order for a closed subgroup $G\subset U_K^+$ to appear as $G=G^+(X)$, for a certain graph X having N vertices, the following must happen:*

1. *We must have a representation $G\subset U_N^+$.*
2. *This representation must be magic, $G\subset S_N^+$.*
3. *We must have a graph X having N vertices, such that $d\in End(u)$.*
4. *X must be in fact such that the Tannakian category of G is precisely $<d>$.*

Proof. This is more of an empty statement, coming from the definition of the quantum automorphism group $G^+(X)$, as formulated in section 9 above. □

In the group dual case, forgetting about Fourier transforms, and imagining that we are at step (1) in the general strategy outlined in Theorem 11.24, we must compute the Tannakian category of $\widehat{\Gamma}\subset U_N^+$, diagonally embedded, for the needs of (3,4). We have:

Proposition 11.25. *Given a discrete group $\Gamma =< g_1,\ldots,g_N >$, embed diagonally $\widehat{\Gamma}\subset U_N^+$, via the unitary matrix $u=diag(g_1,\ldots,g_N)$. We have then the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=\left\{T=(T_{j_1\ldots j_l,i_1\ldots i_k})\Big|g_{i_1}\ldots g_{i_k}\neq g_{j_1}\ldots g_{j_l}\implies T_{j_1\ldots j_l,i_1\ldots i_k}=0\right\}$$

and in particular, with $k=l=1$, we have the formula

$$End(u)=\left\{T=(T_{ji})\Big|g_i\neq g_j\implies T_{ji}=0\right\}$$

with the linear maps being identified with the corresponding scalar matrices.

Proof. This is well-known, and elementary, with the first assertion coming from:

$$\begin{aligned}T\in Hom(u^{\otimes k},u^{\otimes l})&\iff Tu^{\otimes k}=u^{\otimes l}T\\&\iff (Tu^{\otimes k})_{j_1\ldots j_l,i_1\ldots i_k}=(u^{\otimes l}T)_{j_1\ldots j_l,i_1\ldots i_k}\\&\iff T_{j_1\ldots j_l,i_1\ldots i_k}g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}T_{j_1\ldots j_l,i_1\ldots i_k}\\&\iff T_{j_1\ldots j_l,i_1\ldots i_k}(g_{i_1}\ldots g_{i_k}-g_{j_1}\ldots g_{j_l})=0\end{aligned}$$

As for the second assertion, this follows from the first one. □

Let us go ahead now, with respect to the general strategy outlined in Theorem 11.24, and apply [51] in order to solve (2), and then reformulate (3,4), by using Proposition 11.25, and by choosing to put the multi-Fourier transform on the graph part. We are led in this way into the following refinement of Theorem 11.24, in the group dual setting:

Theorem 11.26. *In order for a group dual $\widehat{\Gamma}$ to appear as $G=G^+(X)$, for a certain graph X having N vertices, the following must happen:*

1. *First, we need a quotient map $\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}\to\Gamma$.*
2. *Let $u=diag(I_1,\ldots,I_k)$, with $I_r=diag(\mathbb{Z}_{N_r})$, for any r.*
3. *Consider also the matrix $F=diag(F_{N_1},\ldots,F_{N_k})$.*
4. *We must then have a graph X having N vertices.*
5. *This graph must be such that $F^*dF\neq 0\implies I_i=I_j$.*
6. *In fact, $< F^*dF >$ must be the category in Proposition 11.25.*

Proof. This is something rather informal, the idea being as follows:

(1) This is what comes out from the classification result in [51], explained above, modulo a unitary base change, as explained before.

(2) This is just a notation, with $I_r=diag(\mathbb{Z}_{N_r})$ meaning that I_r is the diagonal matrix formed by $1,g,g^2,\ldots,g^{N_r-1}$, with $g\in\mathbb{Z}_{N_r}$ being the standard generator.

(3) This is another notation, with each Fourier matrix F_{N_r} being the standard one, namely $F_{N_r}=\frac{1}{\sqrt{N_r}}(w^{ij})$, with $w=e^{2\pi i/N_r}$, and with indices $0,1,\ldots,N_r-1$.

(4) This is a just a statement, with the precise graph formalism to be clarified later on, in view of the fact that X will get Fourier-transformed anyway.

(5) This is an actual result, our claim being that the condition $d \in End(u)$ from Theorem 11.24 (3) is equivalent to the condition $F^*dF \neq 0 \implies I_i = I_j$ in the statement. Indeed, we know that with F, I being as in the statement, we have $u = FIF^*$. Now with this formula in hand, we have the following equivalences:

$$\begin{aligned}\widehat{\Gamma} \curvearrowright X &\iff du = ud \\ &\iff dFIF^* = FIF^*d \\ &\iff [F^*dF, I] = 0\end{aligned}$$

Also, since the matrix I is diagonal, with $M = F^*dF$ have:

$$\begin{aligned}MI = IM &\iff (MI)_{ij} = (IM)_{ij} \\ &\iff M_{ij}I_j = I_iM_{ij} \\ &\iff [M_{ij} \neq 0 \implies I_i = I_j]\end{aligned}$$

We therefore conclude that we have, as desired:

$$\widehat{\Gamma} \curvearrowright X \iff [F^*dF \neq 0 \implies I_i = I_j]$$

(6) This is the Tannakian condition in Theorem 11.24 (4), with reference to the explicit formula for the Tannakian category of $G = \widehat{\Gamma}$ given in Proposition 11.25. □

Going ahead now, in connection with the Fourier tori, we have:

Proposition 11.27. *The Fourier tori of $G^+(X)$ are the biggest quotients*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

whose duals act on the graph, $\widehat{\Gamma} \curvearrowright X$.

Proof. We have indeed the following computation, at $F = 1$:

$$\begin{aligned}C(T_1(G^+(X))) &= C(G^+(X))/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < [d,u] = 0 >]/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < u_{ij} = 0, \forall i \neq j >]/ < [d,u] = 0 > \\ &= C(T_1(S_N^+))/ < [d,u] = 0 >\end{aligned}$$

Thus, we obtain the result, with the remark that the quotient that we are interested in appears via relations of type $d_{ij} = 1 \implies g_i = g_j$. The proof in general is similar. □

In connection now with the above-mentioned questions, we have:

Theorem 11.28. *Consider the following conditions:*

(1) *We have $G(X) = G^+(X)$.*
(2) *$G(X) \subset G^+(X)$ is a Fourier liberation.*
(3) *$\widehat{\Gamma} \curvearrowright X$ implies that Γ is abelian.*

We have then $(1) \iff (2) + (3)$.

Proof. This is something elementary, the proof being as follows:

(1) $\implies$ (2, 3) Here both the implications are trivial.

(2, 3) $\implies$ (1) Assuming $G(X) \neq G^+(X)$, from (2) we know that $G^+(X)$ has at least one non-classical Fourier torus, and this contradicts (3). □

With this in hand, our question is whether (3) $\implies$ (1) holds. We believe that this is a good question, which in practice would make connections between the various conjectures that can be made about a given graph X, and its quantum symmetry group $G^+(X)$.

As an illustration for the potential interest of such considerations, it is known from [115] that the random graphs have no quantum symmetries, with this being something highly non-trivial. Our point now is that, assuming that one day the general compact quantum Lie group theory will solve its Weyl-type questions in relation with the tori, and in particular know, as a theorem, that any $G^+(X)$ appears as a Fourier liberation of $G(X)$, this deep graph result from [115] would become accessible as well via its particular case for the group dual subgroups, which is something elementary, as follows:

Proposition 11.29. *For a graph X having N vertices, the probability for having an action*

$$\widehat{\Gamma} \curvearrowright X$$

with Γ being a non-abelian group goes to 0 with $N \to \infty$.

Proof. This is something quite elementary, the idea being as follows:

(1) First of all, the graphs X having a fixed number $N \in \mathbb{N}$ of vertices correspond to the matrices $d \in M_N(0,1)$ which are symmetric, and have 0 on the diagonal. The probability mentioned in the statement is the uniform one on such 0-1 matrices.

(2) Regarding now the proof, our claim is that this should come in a quite elementary way, from the $du = ud$ condition, as reformulated before. Indeed, observe first that in the cyclic case, where $F = F_N$ is a usual Fourier matrix, associated to a cyclic group $\mathbb{Z}_N$, we have the following formula, with $w = e^{2\pi i/N}$:

$$\begin{aligned}
(F^*dF)_{ij} &= \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} \\
&= \sum_{kl} w^{lj-ik} d_{kl} \\
&= \sum_{k \sim l} w^{lj-ik}
\end{aligned}$$

(3) In the general setting now, where we have a quotient map $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N_1 + \ldots + N_k = N$, the computation is similar, as follows, with $w_i = e^{2\pi i/N_i}$:

$$\begin{aligned}(F^*dF)_{ij} &= \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} \\ &= \sum_{k\sim l} (F^*)_{ik} F_{lj} \\ &= \sum_{k:i,l:j,k\sim l} (w_{N_i})^{-ik} (w_{N_j})^{lj}\end{aligned}$$

Here the conditions $k : i$ and $l : j$ refer to the fact that k, l must belong respectively to the same matrix blocks as i, j, with respect to the partition $N_1 + \ldots + N_k = N$, and $k \sim l$ means as usual that there is an edge between k, l, in the graph X.

(4) The point now is that with the partition $N_1 + \ldots + N_k = N$ fixed, and with $d \in M_N(0,1)$ being random, we have $(F^*dF)_{ij} \neq 0$ almost everywhere in the $N \to \infty$ limit, and so we obtain $I_i = I_j$ almost everywhere, and so abelianity of Γ, with $N \to \infty$. $\square$

12. Twisted reflections

We discuss here the twisted analogues of the complex reflection groups, obtained by using generalized quantum permutation groups, S_F^+ with F being an arbitrary finite quantum space. Let us first recall from section 4 above that we have:

Definition 12.1. *A finite quantum space F is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(F) = B$$

The number of elements of such a space is by definition the number $|F| = \dim B$. By decomposing the algebra B, we have a formula of the following type:

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \ldots = n_k = 1$ we obtain in this way the space $F = \{1, \ldots, k\}$. Also, when $k = 1$ the equation is $C(F) = M_n(\mathbb{C})$, and the solution will be denoted $F = M_n$.

As explained in section 4, each such finite quantum space F has a counting measure, corresponding as the algebraic level to the following integration functional, obtained by applying the regular representation, and then the normalized matrix trace:

$$tr : C(F) \to B(l^2(F)) \to \mathbb{C}$$

As basic examples, for both $F = \{1, \ldots, N\}$ and $F = M_N$ we obtain the usual trace. In general, with $C(F) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, the weights of tr are:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Let us also mention that the canonical trace is precisely the one making $\mathbb{C} \subset B$ a Markov inclusion. Equivalently, the counting measure is the one making $F \to \{.\}$ a Markov fibration. For a discussion of these facts, see [2], and also [5], [22].

We will also need the definition and main properties of the quantum symmetry groups S_F^+ of such spaces F. The result here, from section 4 as well, is as follows:

Theorem 12.2. *Given a finite quantum space F, there is a universal compact quantum group S_F^+ acting on F, leaving the counting measure invariant. We have*

$$C(S_F^+) = C(U_N^+) \Big/ \Big\langle \mu \in Hom(u^{\otimes 2}, u), \eta \in Fix(u) \Big\rangle$$

where $N = |F|$ and where μ, η are the multiplication and unit maps of $C(F)$. For $F = \{1, \ldots, N\}$ we have $S_F^+ = S_N^+$. Also, for the space $F = M_2$ we have $S_F^+ = SO_3$.

Proof. This is something that we know as well from section 4, with the proof being based on the fact that the coaction axioms for a map $\Phi : C(F) \to C(F) \otimes C(G)$, written as $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$, correspond to the fact that $u = (u_{ij})$ must be a corepresentation, satisfying the conditions $\mu \in Hom(u^{\otimes 2}, u)$ and $\eta \in Fix(u)$ in the statement. $\square$

For our purposes here it will be very useful to have bases and indices. We will use a single index approach, based on the following formalism:

Definition 12.3. *Given a finite quantum space F, we let $\{e_i\}$ be the standard multimatrix basis of $B = C(F)$, so that the multiplication, involution and unit of B are given by*

$$e_ie_j = e_{ij} \quad , \quad e_i^* = e_{\bar{i}} \quad , \quad 1 = \sum_{i=\bar{i}} e_i$$

where $(i,j) \to ij$ is the standard partially defined multiplication on the indices, with the convention $e_\emptyset = 0$, and where $i \to \bar{i}$ is the standard involution on the indices.

To be more precise, let $\{e_{ab}^r\} \subset B$ be the multimatrix basis. We set then $i = (abr)$, and with this convention, the multiplication, coming from $e_{ab}^r e_{cd}^p = \delta_{rp}\delta_{bc}e_{ad}^r$, is given by:

$$(abr)(cdp) = \begin{cases} (adr) & \text{if } r = p,\ b = c \\ \emptyset & \text{otherwise} \end{cases}$$

As for the involution, coming from $(e_{ab}^r)^* = e_{ba}^r$, this is given by:

$$\overline{(a,b,r)} = (b,a,r)$$

Finally, the unit formula comes from the following formula for the unit $1 \in B$:

$$1 = \sum_{ar} e_{aa}^r$$

Regarding now the generalized quantum permutation groups S_F^+, the construction in Theorem 12.2 reformulates as follows, by using the single index formalism:

Theorem 12.4. *Given a finite quantum space F, with basis $\{e_i\} \subset C(F)$ as above, the algebra $C(S_F^+)$ is generated by variables u_{ij} with the following relations,*

$$\sum_{ij=p} u_{ik}u_{jl} = u_{p,kl} \quad , \quad \sum_{kl=p} u_{ik}u_{jl} = u_{ij,p}$$

$$\sum_{i=\bar{i}} u_{ij} = \delta_{j\bar{j}} \quad , \quad \sum_{j=\bar{j}} u_{ij} = \delta_{i\bar{i}}$$

$$u_{ij}^* = u_{\bar{i}\bar{j}}$$

with the fundamental corepresentation being the matrix $u = (u_{ij})$. We call a matrix $u = (u_{ij})$ satisfying the above relations "generalized magic".

Proof. Once again, this is something that we know from section 4 above, the idea being that the relations $\mu \in Hom(u^{\otimes 2}, u)$ and $\eta \in Fix(u)$ in Theorem 12.2 produce the 1st and 4th relations, then the biunitarity of u gives the 5th relation, and finally the 2nd and 3rd relations follow from the 1st and 4th relations, by using the antipode. $\square$

As an illustration, consider the case $F=\{1,\ldots,N\}$. Here the index multiplication is $ii=i$ and $ij=\emptyset$ for $i\neq j$, and the involution is $\bar{i}=i$. Thus, our relations read:

$$u_{ik}u_{il}=\delta_{kl}u_{ik}\quad,\quad u_{ik}u_{jk}=\delta_{ij}u_{ik}$$

$$\sum_i u_{ij}=1\quad,\quad \sum_j u_{ij}=1$$

$$u_{ij}^*=u_{ij}$$

We recognize here the standard magic conditions on a matrix $u=(u_{ij})$.

Getting now to the point where we wanted to get, namely the quantum symmetries of the finite "quantum" graphs, and the generalized quantum reflection groups, let us start with the following straightforward extension of the usual notion of finite graph, from [91], obtained by using a finite quantum space as set of vertices:

Definition 12.5. *We call "finite quantum graph" a pair of type*

$$X=(F,d)$$

with F being a finite quantum space, and with $d\in M_N(\mathbb{C})$.

Such a quantum graph can be represented as a colored oriented graph on $\{1,\ldots,N\}$, with the vertices being decorated by single indices i as above, and with the colors being complex numbers, namely the entries of d. This is quite similar to the formalism from section 9 above, but there is a discussion here in what regards the exact choice of the colors, which are normally irrelevant in connection with our $G^+(X)$ problematics, and so can be true colors instead of complex numbers. More on this later.

With the above notion in hand, we have the following definition, also from [91]:

Definition 12.6. *The quantum automorphism group of $X=(F,d)$ is the subgroup*

$$G^+(X)\subset S_F^+$$

obtained via the relations $du=ud$.

We refer to [91] and to [132] for more on this notion, and for a number of advanced computations, in relation with the free wreath products.

At an elementary level, a first problem is that of working out the basics of the correspondence $X\to G^+(X)$, following [6]. There are 5 things to be done here, namely simplices, complementation, color independence, multi-simplices, and reflections.

Let us start with the simplices. The result here is as follows:

Theorem 12.7. *Given a finite quantum space F, we have*

$$G^+(F_{empty}) = G^+(F_{full}) = S_F^+$$

where F_{empty} is the empty graph, coming from the matrix $d = 0$, and where F_{full} is the simplex, coming from the matrix $d = NP_1 - 1_N$.

Proof. This is something quite tricky, the idea being as follows:

(1) First of all, the formula $G^+(F_{empty}) = S_F^+$ is clear from definitions, because the commutation of u with the matrix $d = 0$ is automatic.

(2) Regarding $G^+(F_{full}) = S_F^+$, let us first discuss the classical case, $F = \{1, \ldots, N\}$. Here the simplex F_{full} is the graph having having edges between any two vertices, whose adjacency matrix is $d = \mathbb{I}_N - 1_N$, where $\mathbb{I}_N$ is the all-1 matrix. The commutation of u with 1_N being automatic, and the commutation with $\mathbb{I}_N$ being automatic too, u being bistochastic, we have $[u, d] = 0$, and so $G^+(F_{full}) = S_F^+$ in this case, as stated.

(3) In the general case now, we know from Theorem 12.2 that we have $\eta \in Fix(u)$, with $\eta : \mathbb{C} \to C(F)$ being the unit map. Thus we have $P_1 \in End(u)$, and so $[u, P_1] = 0$ is automatic. Together with the fact that in the classical case we have the formula $\mathbb{I}_N = NP_1$, this suggests to define the adjacency matrix of the simplex as being $d = NP_1 - 1_N$, and with this definition, we have $G^+(F_{full}) = S_F^+$, as claimed.

(4) Thus, we have the result, and the only piece of discussion still needed concerns the understanding of what the simplex F_{full} really is, say pictorially speaking. According to our conventions, the adjacency matrix of the simplex is:

$$\begin{aligned} d_{ij} &= (NP_1 - 1_N)_{ij} \\ &= \bar{1}_i 1_j - \delta_{ij} \\ &= \delta_{i\bar{i}}\delta_{j\bar{j}} - \delta_{ij} \end{aligned}$$

(5) For $F = \{1, \ldots, N\}$, where the involution on the index set is $\bar{i} = i$, we obtain $d_{ij} = 1 - \delta_{ij}$, as we should. In the case $F = M_n$ now, by using double indices we have:

$$\begin{aligned} d_{ab,cd} &= \delta_{ab,ba}\delta_{cd,dc} - \delta_{ab,cd} \\ &= \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} \end{aligned}$$

Thus, we obtain a matrix $d \in M_N(-1, 0, 1)$, which generically has 0 entries. This matrix is symmetric. It has not 0 on the diagonal, the self-edges, worth 1, appearing at the off-diagonal points of F. The case of edges worth -1 is possible too. □

With the above result in hand, we can now talk about complementation, as follows:

Theorem 12.8. *For any finite quantum graph X we have the formula*

$$G^+(X) = G^+(X^c)$$

where $X \to X^c$ is the complementation operation, given by $d_X + d_{X^c} = d_{F_{full}}$.

Proof. This follows from Theorem 12.7, and more specifically from the fact that the condition $[u, d_{F_{full}}] = 0$ is automatic, as explained there. There is of course still some discussion here to be done, in what concerns the pictorial representation of X^c, as a continuation of the discussion in (4) from the proof of Theorem 12.7. □

As basic examples, with $F = \{1, \ldots, N\}$ we recover the usual $G^+(X)$ quantum groups. For $F = M_2$ we have $S_F^+ = S_F = SO_3$, and for the few graphs here, having $|M_2| = 4$ vertices, we recover certain subgroups of SO_3, to be determined.

Before getting into color independence, let us discuss some interesting examples, coming from the Cayley graphs. Let us start with the following well-known fact:

Proposition 12.9. *For any finite quantum group G, the counting measure is the Haar measure.*

Proof. Given an arbitrary finite quantum group G, we must prove that the counting measure is left and right invariant, in the sense that we have:

$$(tr \otimes id)\Delta = (id \otimes tr)\Delta = tr$$

But this is something well-known, which follows from the definition of the canonical trace, as being the following composition:

$$tr : C(G) \to B(l^2(G)) \to \mathbb{C}$$

Indeed, this composition is left and right invariant, as desired. □

As a consequence, we have a Cayley theorem in the present setting, as follows:

Theorem 12.10. *For any finite quantum group G, we have:*

$$G \subset S_G^+$$

That is, the Cayley theorem holds, in the present setting.

Proof. We have an action $G \curvearrowright G$, well-known to leave invariant the Haar measure. Now since by Proposition 12.9 the Haar measure is the counting measure, we conclude that $G \curvearrowright G$ leaves invariant the counting measure, and so we have $G \subset S_G^+$, as claimed. □

By adding now edges, we are led to the following result:

Theorem 12.11. *Given a finite quantum group G, the following happen:*

(1) *We have $G \subset G^+(X_G)$ with $X_G = (G, d)$ when the matrix $d \in M_G(\mathbb{C})$ belongs to the image of the right regular representation.*
(2) *In this context, we can always arrange as for the inclusion $G \subset G^+(X_G)$ to be "optimal", in the sense that $\{d\}' = End(u)$.*
(3) *In fact, we can always arrange as for having a formula of type $G = G^+(X_G)$, for a certain quantum graph X_G.*

Proof. This follows from Theorem 12.10, and from the basic properties of the left and right regular representation, for the finite quantum groups. □

Following now [6], let us discuss an important point, namely the "independence on the colors" question. The idea indeed is that given a classical graph X with edges colored by complex numbers, or by other types of colors, $G(X)$ does not change when changing the colors. This is obvious, and a quantum analogue of this fact, involving $G^+(X)$, can be shown to hold as well, as explained in [6], and in section 9 above.

In the quantum graph setting things are more complicated, with the independence on the colors not necessarily being true. Let us start with the following definition:

Definition 12.12. *We say that a quantum graph $X=(F,d)$ is washable if, whenever we have another quantum graph $X'=(F,d')$ with same color scheme, in the sense that*

$$d_{ij}=d_{kl}\iff d'_{ij}=d'_{kl}$$

we have $G^+(X)=G^+(X')$.

As already mentioned, it was proved in [6] that in the classical case, $F=\{1,\ldots,N\}$, all graphs are washable. This is a key result, and this for several reasons: (1) first of all, it gives some intuition on what is going on with respect to colors, in analogy with what happens for $G(X)$, which is very intuitive, and trivial, (2) also, it allows the use of true colors (black, blue, red...) when drawing colored graphs, instead of complex numbers, and (3) this can be combined with the fact that $G^+(X)$ is invariant as well via similar changes in the spectral decomposition of d, at the level of eigenvalues, with all this leading to some powerful combinatorial methods for the computation of $G^+(X)$.

All these things do not necessarily hold in general, and to start with, we have:

Proposition 12.13. *There are quantum graphs, such as the simplex in the homogeneous quantum space case, $F=M_K\times\{1,\ldots,L\}$ with $K,L\neq 2$, which are not washable.*

Proof. We know that the simplex, in the case $F=M_K\times\{1,\ldots,L\}$, has as adjacency matrix a certain matrix $d\in M_N(-1,0,1)$, with $N=K^2L$. Moreover, assuming $K,L\geq 2$ as in the statement, entries of all types, $-1,0,1$, are possible. Now assuming that this simplex is washable, it would follow that we have $\dim(End(u))\geq 3$, a contradiction. □

In order to come up with some positive results as well, the idea will be that of using the method in [6]. Let us start with the following statement, coming from there:

Proposition 12.14. *The following matrix belongs to $End(u)$, for any $n\in\mathbb N$:*

$$d_{ij}^{\times n}=\sum_{i=k_1\ldots k_n}\sum_{j=l_1\ldots l_n}d_{k_1l_1}\ldots d_{k_nl_n}$$

In particular, in the classical case, $F=\{1,\ldots,N\}$, all graphs are washable.

Proof. We have two assertions here, the idea being as follows:

(1) Consider the multiplication and comultiplication maps of the algebra $C(F)$, which in single index notation are given by:

$$\mu(e_i \otimes e_j) = e_{ij}$$

$$\gamma(e_i) = \sum_{i=jk} e_j \otimes e_k$$

Observe that we have $\mu^* = \gamma$, with the adjoint taken with respect to the scalar product coming from the canonical trace. We conclude that we have:

$$\mu \in Hom(u^{\otimes 2}, u)$$

$$\gamma \in Hom(u, u^{\otimes 2})$$

The point now is that we can consider the iterations $\mu^{(n)}, \gamma^{(n)}$ of μ, γ, constructed in the obvious way, and we have then:

$$\mu^{(n)} \in Hom(u^{\otimes n}, u)$$

$$\gamma^{(n)} \in Hom(u, u^{\otimes n})$$

Now if we assume that we have $d \in End(u)$, we have $d^{\otimes n} \in End(u^{\otimes n})$ for any n, and we conclude that we have:

$$\mu^{(n)} d^{\otimes n} \gamma^{(n)} \in End(u)$$

In single index notation, we have the following formula:

$$(\mu^{(n)} d^{\otimes n} \gamma^{(n)})_{ij} = \sum_{i=k_1\ldots k_n} \sum_{j=l_1\ldots l_n} d_{k_1 l_1} \ldots d_{k_n l_n}$$

Thus, we are led to the conclusion in the statement.

(2) Assuming that we are in the case $F = \{1, \ldots, N\}$, the matrix $d^{\times n}$ in the statement is simply the componentwise n-th power of d, given by:

$$d_{ij}^{\times n} = d_{ij}^n$$

As explained in [6], a simple analytic argument, using $n \to \infty$ and then a recurrence on the number of colors, shows from this that we have washability indeed. □

In order to exploit now the findings in Proposition 12.14, we will assume that we are in the case $F = M_K \times \{1, \ldots, L\}$, and we will use an idea which is familiar in random matrices and quantum information, namely assuming that d is "split".

To be more precise, we have the following result:

Theorem 12.15. *Assuming that we are in the case $F = M_K \times \{1, \ldots, L\}$, and that the adjacency matrix is split, in the sense that one of the following happens,*

$$d_{ab,cd} = e_{ab} f_{cd}$$
$$d_{ab,cd} = e_{ac} f_{bd}$$
$$d_{ab,cd} = e_{ad} f_{bc}$$

the quantum graph is washable.

Proof. The idea here is that of computing first the matrix $d^{\times n}$ from Proposition 12.14, and then adapting the proof from the $K = 1$ case, from [6], explained above. We know from Proposition 12.14 that we have the following formula, in single index notation:

$$d^{\times n}_{ij} = \sum_{i=k_1 \ldots k_n} \sum_{j=l_1 \ldots l_n} d_{k_1 l_1} \ldots d_{k_n l_n}$$

In double index notation, which is more convenient for our purposes here, we have:

$$\begin{aligned} d^{\times n}_{ab,cd} &= \sum_{x_1 \ldots x_{n-1}} \sum_{y_1 \ldots y_{n-1}} d_{ax_1, cy_1} d_{x_1 x_2, y_1 y_2} d_{x_2 x_3, y_2 y_3} \ldots \ldots \\ &\qquad \ldots \ldots d_{x_{n-2} x_{n-1}, y_{n-2} y_{n-1}} d_{x_{n-1} b, y_{n-1} d} \end{aligned}$$

We have 3 cases to be investigated, and here are the computations of this matrix:

(1) In the case $d_{ab,cd} = e_{ab} f_{cd}$ we have the following computation:

$$\begin{aligned} d^{\times n}_{ab,cd} &= \sum_{x_1 \ldots x_{n-1}} \sum_{y_1 \ldots y_{n-1}} e_{ax_1} f_{cy_1} e_{x_1 x_2} f_{y_1 y_2} e_{x_2 x_3} f_{y_2 y_3} \ldots \ldots \\ &\qquad \ldots \ldots e_{x_{n-2} x_{n-1}} f_{y_{n-2} y_{n-1}} e_{x_{n-1} b} f_{y_{n-1} d} \\ &= \sum_{x_1 \ldots x_{n-1}} e_{ax_1} e_{x_1 x_2} e_{x_2 x_3} \ldots \ldots e_{x_{n-2} x_{n-1}} e_{x_{n-1} b} \\ &\quad \sum_{y_1 \ldots y_{n-1}} f_{cy_1} f_{y_1 y_2} f_{y_2 y_3} \ldots \ldots f_{y_{n-2} y_{n-1}} f_{y_{n-1} d} \\ &= (e^n)_{ab} (f^n)_{cd} \end{aligned}$$

(2) In the case $d_{ab,cd} = e_{ac} f_{bd}$ we have the following computation, where the $\times$ operation at the end is the usual componentwise product of the square matrices, and where E is the total sum of the entries of a given square matrix:

$$\begin{aligned} d^{\times n}_{ab,cd} &= \sum_{x_1 \ldots x_{n-1}} \sum_{y_1 \ldots y_{n-1}} e_{ac} f_{x_1 y_1} e_{x_1 y_1} f_{x_2 y_2} e_{x_2 y_2} f_{x_3 y_3} \ldots \ldots \\ &\qquad \ldots \ldots e_{x_{n-2} y_{n-2}} f_{x_{n-1} y_{n-1}} e_{x_{n-1} y_{n-1}} f_{bd} \\ &= e_{ac} f_{bd} \sum_{x_1 \ldots x_{n-1}} \sum_{y_1 \ldots y_{n-1}} (e \times f)_{x_1 y_1} (e \times f)_{x_2 y_2} \ldots \ldots (e \times f)_{x_{n-1} y_{n-1}} \\ &= e_{ac} e_{bd} E\left[(e \times f)^{n-1}\right] \end{aligned}$$

(3) In the case $d_{ab,cd}=e_{ad}f_{bc}$ we have the following computation, in a rough form, with the general case depending on the parity of n:

$$\begin{aligned}
d_{ab,cd}^{\times n} &= \sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}} e_{ay_1}f_{x_1c}e_{x_1y_2}f_{x_2y_1}e_{x_2y_3}f_{x_3y_2}\ldots\ldots\\
&\qquad\qquad\qquad\ldots\ldots e_{x_{n-2}y_{n-1}}f_{x_{n-1}y_{n-2}}e_{x_{n-1}d}f_{by_{n-1}}\\
&= \sum_{x_1\ldots x_{n-1}}\sum_{y_1\ldots y_{n-1}} e_{ay_1}(f^t)_{y_1x_2}e_{x_2y_3}\ldots\ldots(f^t)_{cx_1}e_{x_1y_2}(f^t)_{x_2y_3}\ldots\ldots\\
&= \left[(ef^t)^{n/2}\right]_{ad}\left[(f^te)^{n/2}\right]_{cb}
\end{aligned}$$

With these formulae in hand, we are led to the conclusion in the statement. □

Let us discuss now some basic examples of quantum symmetry groups of quantum graphs. We first have the following result:

Theorem 12.16. *We have the following results:*

(1) $G_i\curvearrowright X_i$ *implies* $\hat{*}\,G_i\curvearrowright\sqcup X_i$.
(2) $G\curvearrowright X$ *implies* $G\wr_* S_N^+\curvearrowright NX$.

Proof. These results are standard, following the proofs from the usual graph case, where $F=\{1,\ldots,N\}$, and we refer here to [91], and to [132]. □

Let us also mention that under suitable connectedness assumptions on X, similar to those in the classical case, taken in a functional analytic sense, the action in (2) above can be shown to be universal, when taking $G=G^+(X)$, and so we have:

$$G^+(NX)=G^+(X)\wr_* S_N^+$$

For more on this material, we refer to [91], [132] and related papers.

With the above technology in hand, we can talk about multi-simplices, and quantum reflections. The idea is that the quantum automorphism groups $S_{F\to E}^+$ of the Markov fibrations $F\to E$, which correspond by definition to the Markov inclusions of finite dimensional C^*-algebras $C(E)\subset C(F)$, were studied in [4].

As explained in [6] and in subsequent papers, in the classical case, $F=\{1,\ldots,N\}$, the quantum groups $S_{F\to E}^+$ are of the form $G^+(X)$, with X being a multi-simplex, obtained as a union of simplices, and correspond to the notion of quantum reflection group.

In the general case, the quantum groups of type $S_{F\to E}^+$, with $F\to E$ being a Markov fibration, can be thought of as being "generalized quantum reflection groups".

In order to discuss this, let us start with the following definition:

Definition 12.17. *Let $B \subset D$ be an inclusion of finite dimensional $\mathbb{C}^*$-algebras and let φ be a state on D. We define the universal $\mathbb{C}^*$-algebra $A_{aut}(B \subset D)$ generated by the coefficients v_{ij} of a unitary matrix v subject to the conditions*

$$m \in Hom(v^{\otimes 2}, v) \quad , \quad u \in Hom(1, v) \quad , \quad e \in End(v)$$

where $m : D \otimes D \to D$ is the multiplication, $u : \mathbb{C} \to D$ is the unit and $e : D \to D$ is the projection onto B, with respect to the scalar product $< x, y >= \varphi(xy^*)$.

By universality we can construct maps Δ, ε, S, and so we have a Woronowicz algebra in the sense of section 1 above. The matrix v is a corepresentation of $A_{aut}(B \subset D)$ on the Hilbert space D. The three "Hom" conditions translate into the fact that v corresponds to a coaction of $A_{aut}(B \subset D)$ on the $\mathbb{C}^*$-algebra D, which leaves φ and B invariant.

Let us discuss now the corepresentation theory of $A_{aut}(B \subset D)$. Following [4], we will prove that under Markov assumptions on $B \subset D$, the corresponding Tannakian category is the Fuss-Catalan category, introduced by Bisch and Jones in [57].

The Fuss-Catalan category, as well as other categories to be used in what follows, is a tensor $\mathbb{C}^*$-category having $(\mathbb{N}, +)$ as monoid of objects. Such a tensor category will be called a $\mathbb{N}$-algebra. If C is a $\mathbb{N}$-algebra we use the following notations:

$$C(m, n) = Hom_C(m, n) \quad , \quad C(m) = End_C(m)$$

As a first class of examples, which is very wide, associated to any object O in a tensor $\mathbb{C}^*$-category is the $\mathbb{N}$-algebra $\mathbb{N}O$ given by the following formula:

$$\mathbb{N}O(m, n) = Hom(O^{\otimes m}, O^{\otimes n})$$

Let us first discuss in detail the Temperley-Lieb algebra, as a continuation of the material presented in sections 1-4 above. We have the following definition:

Definition 12.18. *The $\mathbb{N}$-algebra TL^2 of index $\delta > 0$ is defined as follows:*

1. *The space $TL^2(m, n)$ consists of linear combinations of noncrossing pairings between $2m$ points and $2n$ points:*

$$TL^2(m,n) = \left\{ \sum \alpha \begin{matrix} \cdots\cdots & \leftarrow & 2m \text{ points} \\ \mathfrak{W} & \leftarrow & m+n \text{ strings} \\ \cdot\cdot & \leftarrow & 2n \text{ points} \end{matrix} \right\}$$

2. *The operations $\circ, \otimes, *$ are induced by the vertical and horizontal concatenation and the upside-down turning of diagrams:*

$$A \circ B = \begin{pmatrix} B \\ A \end{pmatrix} \quad , \quad A \otimes B = AB \quad , \quad A^* = \forall$$

3. *With the rule $\bigcirc = \delta$, erasing a circle is the same as multiplying by δ.*

Our first task will be that of finding a suitable presentation for this algebra. Consider the following two elements $u\in TL^2(0,1)$ and $m\in TL^2(2,1)$:

$$u=\delta^{-\frac{1}{2}}\cap\quad,\quad m=\delta^{\frac{1}{2}}\,|\cup|$$

With this convention, we have the following result:

Theorem 12.19. *The following relations are a presentation of TL^2 by the above rescaled diagrams $u\in TL^2(0,1)$ and $m\in TL^2(2,1)$:*

(1) $mm^*=\delta^2$.
(2) $u^*u=1$.
(3) $m(m\otimes 1)=m(1\otimes m)$.
(4) $m(1\otimes u)=m(u\otimes 1)=1$.
(5) $(m\otimes 1)(1\otimes m^*)=(1\otimes m)(m^*\otimes 1)=m^*m$.

Proof. This is something very standard, well-known, and elementary, which follows by drawing diagrams. □

In more concrete terms, the above result says that u,m satisfy the above relations, which is something clear, and that if C is a $\mathbb{N}$-algebra and $v\in C(0,1)$ and $n\in C(2,1)$ satisfy the same relations then there exists a $\mathbb{N}$-algebra morphism as follows:

$$TL^2\to C\quad,\quad u\to v\quad,\quad m\to n$$

Now let D be a finite dimensional $\mathbb{C}^*$-algebra with a state φ on it. We have a scalar product $<x,y>=\varphi(xy^*)$ on D, so D is an object in the category of finite dimensional Hilbert spaces. Consider the unit u and the multiplication m of D:

$$u\in\mathbb{N}D(0,1)\quad,\quad m\in\mathbb{N}D(2,1)$$

The relations in Theorem 12.19 are then satisfied if and only if the first one, namely $mm^*=\delta^2$, is satisfied, and this is automatic when φ is the standard trace. One can deduce from Theorem 12.19 that in this case, the category of corepresentations of the Hopf algebra $A_{aut}(D)$ is the completion of TL^2, in the sense of [148].

Getting now to Fuss-Catalan algebras, we have here:

Definition 12.20. *A Fuss-Catalan diagram is a planar diagram formed by an upper row of $4m$ points, a lower row of $4n$ points, both colored*

$$\circ\bullet\bullet\circ\circ\bullet\bullet\ldots$$

and by $2m+2n$ noncrossing strings joining these $4m+4n$ points, with the rule that the points which are joined must have the same color.

Fix $\beta>0$ and $\omega>0$. The $\mathbb N$-algebra FC is defined as follows. The spaces $FC(m,n)$ consist of linear combinations of Fuss-Catalan diagrams:

$$FC(m,n)=\left\{\sum\alpha\begin{array}{ccc}\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots&\leftarrow&4m\ \text{ colored points}\\&&m+n\text{ black strings}\\\mathfrak W&\leftarrow&\text{and}\\&&m+n\text{ white strings}\\\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots&\leftarrow&4n\text{ colored points}\end{array}\right\}$$

As before with the Temperley-Lieb algebra, the operations $\circ$, $\otimes$, $*$ are induced by vertical and horizontal concatenation and upside-down turning of diagrams, but this time with the rule that erasing a black/white circle is the same as multiplying by β/ω:

$$A\circ B=\begin{pmatrix}B\\A\end{pmatrix}\quad,\quad A\otimes B=AB\quad,\quad A^*=\forall$$

$$\text{black}\to\bigcirc=\beta\qquad,\qquad\text{white}\to\bigcirc=\omega$$

Let $\delta=\beta\omega$. We have the following bicolored analogues of the elements u,m:

$$u=\delta^{-\frac{1}{2}}\ \textstyle\bigcap\!\!\!\!\!\!\cap\quad,\quad m=\delta^{\frac{1}{2}}\ ||\bigcup\!\!\!\!\!\!\cup||$$

These elements generate in FC a $\mathbb N$-subalgebra which is isomorphic to TL^2.

Consider also the black and white Jones projections, namely:

$$e=\omega^{-1}\ |\,{}^{\cup}_{\cap}\,|\quad,\quad f=\beta^{-1}\ |||\,{}^{\cup}_{\cap}\,|||$$

We have $f=\beta^{-2}(1\otimes me)m^*$, so we won't need f for presenting FC. For simplifying writing we identify x and $x\otimes 1$, for any x. We have the following result:

Theorem 12.21. *The following relations, with $f=\beta^{-2}(1\otimes me)m^*$, are a presentation of FC by $m\in FC(2,1)$, $u\in FC(0,1)$ and $e\in FC(1)$:*

1. *The relations in Theorem 12.19, with $\delta=\beta\omega$.*
2. *$e=e^2=e^*$, $f=f^*$ and $(1\otimes f)f=f(1\otimes f)$.*
3. *$eu=u$.*
4. *$mem^*=m(1\otimes e)m^*=\beta^2$.*
5. *$mm(e\otimes e\otimes e)=emm(e\otimes 1\otimes e)$.*

Proof. As for any presentation result, we have to prove two assertions:

(I) The elements m,u,e satisfy the relations (1-5) and generate the $\mathbb N$-algebra FC.

(II) If M, U and E in a $\mathbb N$-algebra C satisfy the relations (1-5), then there exists a morphism of $\mathbb N$-algebras $FC\to C$ sending $m\to M$, $u\to U$, $e\to E$.

The proof will be based on the results from the paper of Bisch and Jones [57], plus some diagrammatic computations for (I), and some purely algebraic computations for (II).

(I) First, the relations (1-5) are easily verified by drawing pictures.

Let us show now that the $\mathbb{N}$-subalgebra $C =< m, u, e >$ of FC is equal to FC. First, C contains the infinite sequence of black and white Jones projections:

$$p_1 = e = \omega^{-1} \; | \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} |$$

$$p_2 = f = \beta^{-1} \; ||| \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} |||$$

$$p_3 = 1 \otimes e = \omega^{-1} \; ||||| \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} |$$

$$p_4 = 1 \otimes f = \beta^{-1} \; ||||||| \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} |||$$

$$\vdots$$

The algebra C contains as well the infinite sequence of bicolored Jones projections:

$$e_1 = uu^* = \delta^{-1} \begin{matrix}\bigcup\\ \bigcap\end{matrix}$$

$$e_2 = \delta^{-2} m^* m = \delta^{-1} \; || \begin{matrix}\bigcup\\ \bigcap\end{matrix} ||$$

$$e_3 = 1 \otimes uu^* = \delta^{-1} \; |||| \begin{matrix}\bigcup\\ \bigcap\end{matrix}$$

$$e_4 = \delta^{-2}(1 \otimes m^* m) = \delta^{-1} \; |||||| \begin{matrix}\bigcup\\ \bigcap\end{matrix} ||$$

$$\vdots$$

By the results of Bisch and Jones in [57], these latter projections generate the diagonal $\mathbb{N}$-algebra ΔFC. Thus we have inclusions as follows:

$$\Delta FC \subset C \subset FC$$

By definition of C, we have as well the following equality:

$$\Delta FC = \Delta C$$

Also, the existence of semicircles shows that the objects of C and FC are self-dual, and by Frobenius reciprocity we obtain that for $m + n$ even, we have:

$$\begin{aligned}
\dim(C(m,n)) &= \dim\left(C\left(\frac{m+n}{2}\right)\right) \\
&= \dim\left(FC\left(\frac{m+n}{2}\right)\right) \\
&= \dim(FC(m,n))
\end{aligned}$$

By tensoring with u and u^* we get embeddings as follows:

$$C(m,n)\subset C(m,n+1)$$

$$FC(m,n)\subset FC(m,n+1)$$

But this shows that the above dimension equalities hold for any m and n. Together with $\Delta FC\subset C\subset FC$, this shows that $C=FC$.

(II) Assume that M,U,E in a $\mathbb{N}$-algebra C satisfy the relations (1-5). We have to construct a morphism $FC\to C$ sending:

$$m\to M\quad,\quad u\to U\quad,\quad e\to E$$

As a first task, we would like to construct a morphism $\Delta FC\to\Delta C$ sending:

$$m^*m\to M^*M\quad,\quad uu^*\to UU^*\quad,\quad e\to E$$

By constructing the corresponding Jones projections E_i and P_i, we must send:

$$e_i\to E_i\quad,\quad p_i\to P_i\quad (i=1,2,3,\ldots)$$

In order to construct these maps, we use now the fact, from [57], that the following relations are a presentation of ΔFC:

(a) $e_i^2=e_i$, $e_ie_j=e_je_i$ if $|i-j|\geq 2$ and $e_ie_{i\pm 1}e_i=\delta^{-2}e_i$.

(b) $p_i^2=p_i$ and $p_ip_j=p_jp_i$.

(c) $e_ip_i=p_ie_i=e_i$ and $p_ie_j=e_jp_i$ if $|i-j|\geq 2$.

(d) $e_{2i\pm 1}p_{2i}e_{2i\pm 1}=\beta^{-2}e_{2i\pm 1}$ and $e_{2i}p_{2i\pm 1}e_{2i}=\omega^{-2}e_{2i}$.

(e) $p_{2i}e_{2i\pm 1}p_{2i}=\beta^{-2}p_{2i\pm 1}p_{2i}$ and $p_{2i\pm 1}e_{2i}p_{2i\pm 1}=\omega^{-2}p_{2i}p_{2i\pm 1}$.

Thus, it remains to verify that we have the following implication, where m,u,e are now abstract objects, and we are no longer allowed to draw pictures:

$$(1-5)\implies (a-e)$$

First, by using $e_{n+2}=1\otimes e_n$ and $p_{n+2}=1\otimes p_n$, these relations to be checked reduce to the following new collection of relations:

(α) $e_i^2=e_i$ for $i=1,2$, $e_1e_2e_1=\delta^{-2}e_1$ and $e_2e_1e_2=\delta^{-2}e_2$.

(β) $p_i^2=p_i$ for $i=1,2$ and $[p_1,p_2]=[1\otimes p_1,p_2]=[1\otimes p_2,p_2]=0$.

(γ) $[e_2,1\otimes p_2]=[p_2,1\otimes e_2]=0$ and $e_ip_i=p_ie_i=e_i$ for $i=1,2$.

($\delta 1$) $e_1p_2e_1=\beta^{-2}e_1$ and $(1\otimes e_1)p_2(1\otimes e_1)=\beta^{-2}(1\otimes e_1)$.

($\delta 2$) $e_2p_1e_2=e_2(1\otimes p_1)e_2=\omega^{-2}e_2$.

($\varepsilon 1$) $\beta^2p_2e_1p_2=\omega^2p_1e_2p_1=p_1p_2$.

($\varepsilon 2$) $\beta^2p_2(1\otimes e_1)p_2=\omega^2(1\otimes p_1)e_2(1\otimes p_1)=(1\otimes p_1)p_2$.

With $e_1 = uu^*$, $e_2 = \delta^{-2} m^* m$, $p_1 = e$ and $p_2 = f$ we can see that most of these relations are trivial. What is left can be reformulated in the following way:

(x) $em^* me = \beta^2 f^* e$.

(y) $(1 \otimes e)m^* m(1 \otimes e) = \beta^2 f^*(1 \otimes e)$.

(z) $f^* = f^* f$.

(t) $[e, f] = [1 \otimes e, f] = [m^* m, 1 \otimes f] = [f, 1 \otimes m^* m] = 0$.

By multiplying the relation (5) by u and by $1 \otimes 1 \otimes u$ to the right we obtain the following useful formula, to be used many times in what follows:

$$m(e \otimes e) = em(1 \otimes e) = eme$$

Let us verify now the above conditions (x-t). First, we have:

$$\beta^2 f^* e = m(e \otimes e)(1 \otimes m^*)$$

By replacing $m(e \otimes e)$ with eme we get $em^* me$, so (x) is true. Also, we have:

$$(1 \otimes e)m^* m(1 \otimes e) = m(1 \otimes (em(1 \otimes e))^*)$$

By replacing $em(1 \otimes e)$ with eme we get $\beta^2 f^*(1 \otimes e)$, so (y) is true. We have:

$$f^* f = \beta^{-4} m(1 \otimes em^* me)m^*$$

By replacing $em^* me$ with $eme(1 \otimes m^*)$, then eme with $m(e \otimes e)$ we get f^*, so (z) is true. The first two commutators are zero, because fe and $f(1 \otimes e)$ are self-adjoint. The same happens for the others, because of the following formulae:

$$mm^*(1 \otimes ff^*) = \beta^{-4}(1 \otimes 1 \otimes me)m^* m^* mm(1 \otimes 1 \otimes em^*)$$

$$(1 \otimes m^* m)ff^* = \beta^{-4}(1 \otimes m^* me)m^* m(1 \otimes em^* m)$$

The conclusion is that we constructed a certain $\mathbb{N}$-algebra morphism, as follows:

$$\Delta J : \Delta FC \to \Delta C$$

We have to extend now this morphism into a morphism $J : FC \to C$ sending $u \to U$ and $m \to M$. We will use a standard argument. For $w \geq k, l$ we define:

$$\phi : FC(l, k) \to FC(w)$$

$$x \to (u^{\otimes (w-k)} \otimes 1_k)\, x\, ((u^*)^{\otimes (w-l)} \otimes 1_l)$$

We can define as well a morphism as follows:

$$\theta : FC(w) \to FC(l, k)$$

$$x \to ((u^*)^{\otimes (w-k)} \otimes 1_k)\, x\, (u^{\otimes (w-l)} \otimes 1_l)$$

Here $1_k=1^{\otimes k}$, and the convention $x=x\otimes 1$ is no longer used. We define Φ and Θ in C by similar formulae. We have $\theta\phi=\Theta\Phi=Id$. We define a map J by:

$$\begin{tikzcd}
FC(l,k) \arrow[r, "J"] \arrow[d, "\phi"] & C(l,k) \\
FC(w) \arrow[r, "\Delta J"] & C(w) \arrow[u, "\Theta"]
\end{tikzcd}$$

Since the element $J(a)$ does now depend on the choice of w, these J maps are the components of a global map, as follows:

$$J:FC\to C$$

This map J extends ΔJ and sends $u\to U$ and $m\to M$. It remains to prove that J is a morphism. We have:

$$Im(\phi)=\left\{x\in FC(w)\Big|x=((uu^*)^{\otimes(w-k)}\otimes 1_k)\,x\,((uu^*)^{\otimes(w-l)}\otimes 1_l)\right\}$$

We have as well as a similar description of $Im(\Phi)$, and so J sends:

$$Im(\phi)\to Im(\Phi)$$

We have also $\Theta\Phi=Id$, so $\Phi\Theta=Id$ on $Im(\Phi)$. Thus the following diagram commutes:

$$\begin{tikzcd}
FC(l,k) \arrow[r, "J"] \arrow[d, "\phi"] & C(l,k) \arrow[d, "\Phi"] \\
FC(w) \arrow[r, "\Delta J"] & C(w)
\end{tikzcd}$$

It follows that J is multiplicative, because we have:

$$\begin{aligned}
J(ab) &= \Theta(\Delta J\phi(a)\Delta J\phi(b))\\
&= \Theta(\Phi J(a)\Phi J(b))\\
&= \Theta\Phi(J(a)J(b))\\
&= J(a)J(b)
\end{aligned}$$

In order to finish, it remains to prove that we have:

$$J(a\otimes b)=J(a)\otimes J(b)$$

Since we have $a\otimes b=(a\otimes 1_s)(1_t\otimes b)$ for certain s and t, it is enough to prove the above formula for pairs (a,b) of the form $(1_t,b)$ or $(a,1_s)$. For $(a,1_s)$ this is clear, so it remains to prove that the following set equals FC:

$$B=\left\{b\in FC\Big|J(1_t\otimes b)=1_t\otimes J(b),\,\forall\,t\in\mathbb N\right\}$$

For this purpose, observe first that ΔJ being a $\mathbb{N}$-algebra morphism, we have:

$$\Delta FC \subset B$$

On the other hand, a direct computation gives:

$$J(1_t \otimes u \otimes 1_s) = 1_t \otimes U \otimes 1_s$$

Also, J being involutive and multiplicative, B is stable by involution and multiplication. Thus B contains the compositions of elements of ΔFC with $1_t \otimes u \otimes 1_s$ and $1_t \otimes u^* \otimes 1_s$ maps. But any $b \in FC$ is equal to $\theta\phi(b)$, so it is of this form, and we are done. $\square$

Getting back now to the inclusions $B \subset D$ of finite dimensional $\mathbb{C}^*$-algebras, as in Definition 12.17 above, we have the following result:

Theorem 12.22. *If φ is a (β, ω)-form on $B \subset D$ then:*

$$< m, u, e >= FC$$

Proof. It is routine to check that the linear maps m, u, e associated to an inclusion $B \subset D$ as in the statement satisfy the relations (1-5) in Theorem 12.21. Thus, we obtain a certain $\mathbb{N}$-algebra surjective morphism, as follows:

$$J : FC \to < m, u, e >$$

It remains to prove that this morphism J is faithful. For this purpose, consider the following map, where $v = m^*u \in FC(0, 2)$:

$$\phi_n : FC(n) \to FC(n-1)$$

$$x \to (1^{\otimes(n-1)} \otimes v^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes v)$$

Consider as well the following map, where $v = m^*u \in FC(0, 2)$ is as above:

$$\psi_n : C(n) \to C(n-1)$$

$$x \to (1^{\otimes(n-1)} \otimes J(v)^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes J(v))$$

These maps make then the following diagram commutative:

$$\begin{tikzcd} FC(n) \arrow[r, "J"] \arrow[d, "\phi_n"] & C(n) \arrow[d, "\psi_n"] \\ FC(n-1) \arrow[r, "J"] & C(n-1) \end{tikzcd}$$

By gluing such diagrams we get a factorization by J of the composition on the left of conditional expectations, which is the Markov trace. By positivity J is faithful on ΔFC, then by Frobenius reciprocity faithfulness has to hold on the whole FC. $\square$

Getting back now to quantum groups, we have:

Theorem 12.23. *If φ is a (β,ω)-form on $B\subset D$ then the tensor $\mathbb C^*$-category of finite dimensional corepresentations of $A_{aut}(B\subset D)$ is the completion of FC.*

Proof. The algebra $A_{aut}(B\subset D)$ being by definition presented by the relations corresponding to m,u,e, its tensor category of corepresentations has to be completion of the tensor category $< m,u,e >$. On the other hand, the linear form φ being a (β,ω)-form, Theorem 12.22 applies and gives an isomorphism $< m,u,e >\simeq FC$. □

We refer to [4] and related papers for more on these topics.

13. Orbits, orbitals

The notions of orbits, and of transitivity, for the subgroups $G\subset S_N^+$ go back to Bichon's paper [51]. Bichon constructed there the orbits, and used them for classifying the group dual subgroups $\widehat{\Gamma}\subset S_N^+$. We will explain here this material. Let us start with:

Theorem 13.1. *Given a closed subgroup $G\subset S_N^+$, with standard coordinates denoted $u_{ij}\in C(G)$, the following defines an equivalence relation on $\{1,\ldots,N\}$,*

$$i\sim j\iff u_{ij}\neq 0$$

that we call orbit decomposition associated to the corresponding action:

$$G\curvearrowright\{1,\ldots,N\}$$

In the classical case, $G\subset S_N$, this is the usual orbit equivalence.

Proof. We first check the fact that we have indeed an equivalence relation:

(1) The reflexivity axiom $i\sim i$ follows by using the counit, as follows:

$$\begin{aligned}\varepsilon(u_{ij})=\delta_{ij}\quad&\implies\quad\varepsilon(u_{ii})=1\\&\implies\quad u_{ii}\neq 0\end{aligned}$$

(2) The symmetry axiom $i\sim j\implies j\sim i$ follows by using the antipode:

$$S(u_{ij})=u_{ji}\implies[u_{ij}\neq 0\implies u_{ji}\neq 0]$$

(3) As for the transitivity axiom $i\sim j,j\sim k\implies i\sim k$, this follows by using the comultiplication. Consider indeed the following formula:

$$\Delta(u_{ik})=\sum_j u_{ij}\otimes u_{jk}$$

On the right we have a sum of projections, and we obtain from this:

$$\begin{aligned}u_{ij}\neq 0,u_{jk}\neq 0\quad&\implies\quad u_{ij}\otimes u_{jk}>0\\&\implies\quad\Delta(u_{ik})>0\\&\implies\quad u_{ik}\neq 0\end{aligned}$$

Finally, in the classical case, where $G\subset S_N$, the standard coordinates are:

$$u_{ij}=\chi\left(\sigma\in G\Big|\sigma(j)=i\right)$$

Thus $u_{ij}\neq 0$ is equivalent to the existence of an element $\sigma\in G$ such that $\sigma(j)=i$. But this means that i,j must be in the same orbit under the action of G, as claimed. □

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, also from [51]:

Theorem 13.2. *Given a closed subgroup $G\subset S_N^+$, with magic matrix denoted $u=(u_{ij})$, consider the associated coaction map, on the space $X=\{1,\ldots,N\}$:*

$$\Phi:C(X)\to C(X)\otimes C(G)\quad,\quad e_i\to\sum_je_j\otimes u_{ji}$$

The following three subalgebras of $C(X)$ are then equal

$$Fix(u)=\left\{\xi\in C(X)\Big|u\xi=\xi\right\}$$

$$Fix(\Phi)=\left\{\xi\in C(X)\Big|\Phi(\xi)=\xi\otimes1\right\}$$

$$F=\left\{\xi\in C(X)\Big|i\sim j\implies\xi(i)=\xi(j)\right\}$$

where $\sim$ is the orbit equivalence relation constructed in Theorem 13.1.

Proof. The fact that we have $Fix(u)=Fix(\Phi)$ is standard, with this being valid for any corepresentation $u=(u_{ij})$. Regarding now the equality with F, we know from Theorem 13.1 that the magic unitary $u=(u_{ij})$ is block-diagonal, with respect to the orbit decomposition there. But this shows that the algebra $Fix(u)=Fix(\Phi)$ decomposes as well with respect to the orbit decomposition, and so in order to prove the result, we are left with a study in the transitive case, where the result is clear. See [51]. □

We have as well a useful analytic result, as follows:

Theorem 13.3. *Given a closed subgroup $G\subset S_N^+$, consider the following matrix:*

$$P_{ij}=\int_Gu_{ij}$$

Then P is the orthogonal projection onto the linear space

$$F=\left\{\xi\in\mathbb C^N\Big|i\sim j\implies\xi_i=\xi_j\right\}$$

and so the orbits and their sizes can be deduced from the knowledge of P.

Proof. This follows from the above results, and from the standard fact, coming from the Peter-Weyl theory, that P is the orthogonal projection onto $Fix(u)$. □

There are of course many explicit formulae that can be deduced from Theorem 13.3, and we will work out some of them in the next section, in connection with the transitive case, the idea being that $G\subset S_N^+$ is transitive precisely when the following happens:

$$\int_Gu_{ij}=\frac{1}{N}$$

As another comment, the result in Theorem 13.3 makes it clear that the various notions in relation with the orbit decomposition, coming from Theorem 13.1, in the quantum permutation group case, $G\subset S_N^+$, can be normally extended, for instance by using an analytic approach, to the general quantum symmetry group case:

$$G\subset S_X^+$$

There is quite some work to be done here, but instead of getting into this subject, which is quite technical, let us stay with the usual quantum permutation groups, $G\subset S_N^+$, and try to understand how the orbit theory can be further developed.

As a main application of the above orbit theory, we can recover the group duals results from section 11 above. Let us first recall from there that we have:

Proposition 13.4. *Given a quotient group $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}\to\Gamma$, we have an embedding $\widehat{\Gamma}\subset S_N^+$, with $N=N_1+\ldots+N_k$, with magic matrix given by the formula*

$$u=\begin{pmatrix}F_{N_1}I_1F_{N_1}^*\\&\ddots\\&&F_{N_k}I_kF_{N_k}^*\end{pmatrix}$$

where $F_N=\frac{1}{\sqrt N}(w_N^{ij})$ with $w_N=e^{2\pi i/N}$ are Fourier matrices, and where

$$I_r=\begin{pmatrix}1\\&g_r\\&&\ddots\\&&&g_r^{N_r-1}\end{pmatrix}$$

with $g_1,\ldots,g_k$ being the standard generators of Γ.

Proof. This is something that we already know. To be more precise, given a quotient group $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}\to\Gamma$ as in the statement, we have an embedding as follows:

$$\begin{aligned}\widehat{\Gamma}&\subset\widehat{\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}}\\&=\widehat{\mathbb Z_{N_1}}\,\hat*\,\ldots\,\hat*\,\widehat{\mathbb Z_{N_k}}\\&\simeq\mathbb Z_{N_1}\,\hat*\,\ldots\,\hat*\,\mathbb Z_{N_k}\\&\subset S_{N_1}\,\hat*\,\ldots\,\hat*\,S_{N_k}\\&\subset S_{N_1}^+\,\hat*\,\ldots\,\hat*\,S_{N_k}^+\\&\subset S_N^+\end{aligned}$$

Here all the embedings and identifications are standard, with the $\simeq$ sign standing for a multi-Fourier transform, and when working out what happens at the level of the correseponding magic unitaries, we are led to the formula in the statement. $\square$

We have also seen in section 11 that any group dual subgroup $\widehat{\Gamma}\subset S_N^+$ appears as above, as a consequence of the maximal torus theory developed there, and of a direct computation. We can now recover this result in a more conceptual way, as follows:

Theorem 13.5. *Consider a quotient group as follows, with $N = N_1 + \ldots + N_k$:*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

We have then $\widehat{\Gamma} \subset S_N^+$, and any group dual subgroup of S_N^+ appears in this way.

Proof. In one sense, this is something that we already know, from Proposition 13.4. Conversely now, assume that we have a group dual subgroup $\widehat{\Gamma} \subset S_N^+$. By Peter-Weyl, the corresponding magic unitary must be of the following form, with $U \in U_N$:

$$u = U \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix} U^*$$

Now if we denote by $N = N_1 + \ldots + N_k$ the orbit decomposition for $\widehat{\Gamma} \subset S_N^+$, coming from Theorem 13.1, we conclude that u has a $N = N_1 + \ldots + N_k$ block-diagonal pattern, and so that U has as well this $N = N_1 + \ldots + N_k$ block-diagonal pattern.

But this discussion reduces our problem to its $k = 1$ particular case, with the statement here being that the cyclic group $\mathbb{Z}_N$ is the only transitive group dual $\widehat{\Gamma} \subset S_N^+$. The proof of this latter fact being elementary, we obtain the result. See [51]. □

Summarizing, we have now a second proof for the classification of the group dual subgroups $\widehat{\Gamma} \subset S_N^+$. We should mention that the story is not over here, and we will be back to this key result, in the finite quantum group case, with a third proof as well.

Following Lupini, Mančinska, Roberson [115], let us discuss now the higher orbitals. To start with, we have the following standard result, from the classical case:

Proposition 13.6. *Given a subgroup $G \subset S_N$, consider its magic unitary:*

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

We have then the following equivalence,

$$u_{i_1 j_1} \ldots u_{i_k j_k} \neq 0 \iff \exists \sigma \in G,\ \sigma(i_1) = j_1, \ldots, \sigma(i_k) = j_k$$

and these conditions produce an equivalence relation

$$(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k)$$

whose equivalence classes are the k-orbitals of G.

Proof. The fact that we have indeed an equivalence as in the statement, which produces an equivalence relation, is indeed clear from definitions. □

In the quantum case, the situation is more complicated. We follow the approach to the orbits and orbitals developed in [115]. We first have:

Theorem 13.7. *Let $G\subset S_N^+$ be a closed subgroup, with magic unitary $u=(u_{ij})$, and let $k\in\mathbb N$. The relation*

$$(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)\iff u_{i_1j_1}\ldots u_{i_kj_k}\neq0$$

is then reflexive, symmetric, and transitive at $k=1,2$.

Proof. This is known from [115], the proof being as follows:

(1) The reflexivity simply follows by using the counit:

$$\begin{aligned}\varepsilon(u_{i_ri_r})=1,\forall r&\implies\varepsilon(u_{i_1i_1}\ldots u_{i_ki_k})=1\\&\implies u_{i_1i_1}\ldots u_{i_ki_k}\neq0\\&\implies(i_1,\ldots,i_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(2) The symmetry follows by applying the antipode, and then the involution:

$$\begin{aligned}(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)&\implies u_{i_1j_1}\ldots u_{i_kj_k}\neq0\\&\implies u_{j_ki_k}\ldots u_{j_1i_1}\neq0\\&\implies u_{j_1i_1}\ldots u_{j_ki_k}\neq0\\&\implies(j_1,\ldots,j_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(3) The transitivity is something more tricky. We need to prove that we have:

$$u_{i_1j_1}\ldots u_{i_kj_k}\neq0\ ,\ u_{j_1l_1}\ldots u_{j_kl_k}\neq0\implies u_{i_1l_1}\ldots u_{i_kl_k}\neq0$$

In order to do so, we use the following formula:

$$\Delta(u_{i_1l_1}\ldots u_{i_kl_k})=\sum_{s_1\ldots s_k}u_{i_1s_1}\ldots u_{i_ks_k}\otimes u_{s_1l_1}\ldots u_{s_kl_k}$$

At $k=1$ the result is clear, because on the right we have a sum of projections, which is therefore strictly positive when one of these projections is nonzero.

At $k=2$ now, the result follows from the following trick, from [115]:

$$\begin{aligned}&(u_{i_1j_1}\otimes u_{j_1l_1})\Delta(u_{i_1l_1}u_{i_2l_2})(u_{i_2j_2}\otimes u_{j_2l_2})\\=&\sum_{s_1s_2}u_{i_1j_1}u_{i_1s_1}u_{i_2s_2}u_{i_2j_2}\otimes u_{j_1l_1}u_{s_1l_1}u_{s_2l_2}u_{j_2l_2}\\=&\ u_{i_1j_1}u_{i_2j_2}\otimes u_{j_1l_1}u_{j_2l_2}\end{aligned}$$

Indeed, we obtain from this that we have $u_{i_1l_1}u_{i_2l_2}\neq0$, as desired. $\square$

In general, the above equivalence relation is not transitive, the basic counterexample at $k=3$ being the Kac-Paljutkin quantum group. For a proof of this latter fact, and for further orbital theory, with examples and counterexamples, we refer to [119].

In view of the results that we have so far, we can formulate:

Definition 13.8. *Given a closed subgroup $G\subset S_N^+$, consider the relation defined by:*

$$(i_1,\ldots,i_k)\sim_k(j_1,\ldots,j_k)\iff u_{i_1j_1}\ldots u_{i_kj_k}\neq 0$$

1. *The equivalence classes with respect to $\sim_1$ are called orbits of G.*
2. *The equivalence classes with respect to $\sim_2$ are called orbitals of G.*

In the case where $\sim_k$ with $k\geq 3$ happens to be transitive, and so is an equivalence relation, we call its equivalence classes the algebraic k-orbitals of G.

Summarizing, things are quite complicated in the quantum group case.

We have as well an analytic approach to this higher orbital problematics, which is particularly useful when $\sim_k$ is not transitive, that we will explain now.

Let us begin with the following standard result:

Proposition 13.9. *For a subgroup $G\subset S_N$, which fundamental corepresentation denoted $u=(u_{ij})$, the following numbers are equal:*

1. *The number of k-orbitals.*
2. *The dimension of space $Fix(u^{\otimes k})$.*
3. *The number $\int_G\chi^k$, where $\chi=\sum_i u_{ii}$.*

Proof. This is well-known, the proof being as follows:

(1) = (2) Given $\sigma\in G$ and vector $\xi=\sum_{i_1\ldots i_k}\alpha_{i_1\ldots i_k}e_{i_1}\otimes\ldots\otimes e_{i_k}$, we have:

$$\begin{aligned}\sigma^{\otimes k}\xi&=\sum_{i_1\ldots i_k}\alpha_{i_1\ldots i_k}e_{\sigma(i_1)}\otimes\ldots\otimes e_{\sigma(i_k)}\\ \xi&=\sum_{i_1\ldots i_k}\alpha_{\sigma(i_1)\ldots\sigma(i_k)}e_{\sigma(i_1)}\otimes\ldots\otimes e_{\sigma(i_k)}\end{aligned}$$

Thus $\sigma^{\otimes k}\xi=\xi$ holds for any $\sigma\in G$ precisely when α is constant on the k-orbitals of G, and this gives the equality between the numbers in (1) and (2).

(2) = (3) This follows from the Peter-Weyl theory, because $\chi=\sum_i u_{ii}$ is the character of the fundamental corepresentation u. □

In the quantum case now, $G\subset S_N^+$, by the general Peter-Weyl type results established by Woronowicz in [147], we still have the following formula:

$$\dim Fix(u^{\otimes k})=\int_G\chi^k$$

The problem is that of understanding the k-orbital interpretation of this number. We first have the following result, basically coming from [51], [115]:

Proposition 13.10. *Given a closed subgroup $G \subset S_N^+$, and a number $k \in \mathbb{N}$, consider the following linear space:*

$$F_k = \left\{ \xi \in (\mathbb{C}^N)^{\otimes k} \Big| \xi_{i_1 \ldots i_k} = \xi_{j_1 \ldots j_k}, \forall (i_1, \ldots, i_k) \sim (j_1, \ldots, j_k) \right\}$$

1. *We have $F_k \subset Fix(u^{\otimes k})$.*
2. *At $k = 1, 2$ we have $F_k = Fix(u^{\otimes k})$.*
3. *In the classical case, we have $F_k = Fix(u^{\otimes k})$.*
4. *For $G = S_N^+$ with $N \geq 4$ we have $F_3 \neq Fix(u^{\otimes 3})$.*

Proof. The tensor power $u^{\otimes k}$ being the corepresentation $(u_{i_1, \ldots i_k, j_1 \ldots j_k})_{i_1 \ldots i_k, j_1 \ldots j_k}$, the corresponding fixed point space $Fix(u^{\otimes k})$ consists of the vectors ξ satisfying:

$$\sum_{j_1 \ldots j_k} u_{i_1 j_1} \ldots u_{i_k j_k} \xi_{j_1 \ldots j_k} = \xi_{i_1 \ldots i_k} \quad , \quad \forall i_1, \ldots, i_k$$

With this formula in hand, the proof goes as follows:

(1) Assuming $\xi \in F_k$, the above fixed point formula holds indeed, because:

$$\sum_{j_1 \ldots j_k} u_{i_1 j_1} \ldots u_{i_k j_k} \xi_{j_1 \ldots j_k} = \sum_{j_1 \ldots j_k} u_{i_1 j_1} \ldots u_{i_k j_k} \xi_{i_1 \ldots i_k} = \xi_{i_1 \ldots i_k}$$

(2) This is something more tricky, coming from the following formulae:

$$u_{ik} \left(\sum_j u_{ij} \xi_j - \xi_i \right) = u_{ik} (\xi_k - \xi_i)$$

$$u_{i_1 k_1} \left(\sum_{j_1 j_2} u_{i_1 j_1} u_{i_2 j_2} \xi_{j_1 j_2} - \xi_{i_1 i_2} \right) u_{i_2 k_2} = u_{i_1 k_1} u_{i_2 k_2} (\xi_{k_1 k_2} - \xi_{i_1 i_2})$$

(3) This follows indeed from Proposition 13.9 above.

(4) This follows from the representation theory of S_N^+ with $N \geq 4$, and from some elementary computations, the dimensions of the two spaces involved being $4 < 5$. To be more precise, let us start with the symmetric group S_N. It follows from definitions that the k-orbitals are indexed by the partitions $\pi \in P(k)$, as follows:

$$C_\pi = \left\{ (i_1, \ldots, i_k) \Big| \ker i = \pi \right\}$$

In particular at $k = 3$ we have 5 such orbitals, corresponding to:

$$\sqcap\!\sqcap \quad , \quad \sqcap | \quad , \quad | \sqcap \quad , \quad \sqcap\!\!\!\!\!\mid\,\, \quad , \quad |||$$

Regarding now S_N^+, the 3-orbitals are exactly as for S_N, except for the fact that the $\sqcap\!\!\!\!\!\mid\,\,$ and $|||$ 3-orbitals get merged. Thus, we have 4 such orbitals, corresponding to:

$$\sqcap\!\sqcap \quad , \quad \sqcap | \quad , \quad | \sqcap \quad , \quad \dot{|}\dot{|}\dot{|}$$

On the other hand, the number of analytic orbitals is the same as for S_N, namely 5. $\square$

The above considerations suggest formulating the following definition:

Definition 13.11. *Given a closed subgroup $G\subset U_N^+$, the integer*

$$\dim Fix(u^{\otimes k})=\int_G\chi^k$$

is called number of analytic k-orbitals.

To be more precise, in the classical case the situation is of course well understood, and this is the number of k-orbitals. The same goes for the general case, with $k=1,2$, where this is the number of k-orbitals. At $k=3$ and higher, however, even in the case where the algebraic 3-orbitals are well-defined, their number is not necessarily the above one.

In the particular case $k=3$, we have as well the following result, which brings some more support for the above definition:

Proposition 13.12. *For a closed subgroup $G\subset S_N^+$, and an integer $k\leq 3$, the following conditions are equivalent:*

1. *G is k-transitive, in the sense that $Fix(u^{\otimes k})$ has dimension $1,2,5$.*
2. *The k-th moment of the main character is $\int_G\chi^k=1,2,5$.*
3. *$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\frac{(N-k)!}{N!}$ for distinct indices i_r and distinct indices j_r.*
4. *$\int_G u_{i_1j_1}\ldots u_{i_kj_k}$ equals $\frac{(N-|\ker i|)!}{N!}$ when $\ker i=\ker j$, and equals 0, otherwise.*

Proof. Most of these implications are known, the idea being as follows:

(1) $\iff$ (2) This follows from the Peter-Weyl type theory from [147], because the k-th moment of the character counts the number of fixed points of $u^{\otimes k}$.

(2) $\iff$ (3) This follows from the Schur-Weyl duality results for S_N,S_N^+ and from $P(k)=NC(k)$ at $k\leq 3$.

(3) $\iff$ (4) Once again this follows from $P(k)=NC(k)$ at $k\leq 3$, and from a standard integration result for S_N. □

As a conclusion to all these considerations, we have:

Theorem 13.13. *For a closed subgroup $G\subset S_N^+$, and an integer $k\in\mathbb N$, the number $\dim(Fix(u^{\otimes k}))=\int_G\chi^k$ of "analytic k-orbitals" has the following properties:*

1. *In the classical case, this is the number of k-orbitals.*
2. *In general, at $k=1,2$, this is the number of k-orbitals.*
3. *At $k=3$, when this number is minimal, G is 3-transitive in the above sense.*

Proof. This follows indeed from the above considerations. □

Let us discuss now an alternative take on these questions, in the finite quantum group case. We start with the following standard definition:

Definition 13.14. *Associated to any finite quantum group F is its dual finite quantum group $G=\widehat{F}$, given by $C(G)=C(F)^*$, with Hopf C^*-algebra structure as follows:*

1. *Multiplication $(\varphi\psi)a=(\varphi\otimes\psi)\Delta(a)$.*
2. *Unit $1=\varepsilon$.*
3. *Involution $\varphi^*(a)=\overline{\varphi(S(a)^*)}$.*
4. *Comultiplication $(\Delta\varphi)(a\otimes b)=\varphi(ab)$.*
5. *Counit $\varepsilon(\varphi)=\varphi(1)$.*
6. *Antipode $(S\varphi)a=\varphi(S(a))$.*

Our aim in what follows will be that of reformulating in terms of $G=\widehat{F}$ the condition $F\subset S_N^+$. We will see later how this can potentially helps, by dropping the assumption that F,G are finite, in connection with various quantum permutation group questions.

In order to get started, we have the following well-known fact:

Proposition 13.15. *Given F and $G=\widehat{F}$ as in Definition 13.14, the formula*

$$\pi:C(G)\to M_N(\mathbb{C})$$

$$\varphi\to[\varphi(u_{ij})]_{ij}$$

defines a $*$-algebra representation precisely when u is a corepresentation.

Proof. In one sense, the fact that π is multiplicative follows from the fact that u is comultiplicative, in the sense that $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$, as follows:

$$\begin{aligned}
\pi(\varphi\psi) &= [(\varphi\psi)u_{ij}]_{ij}\\
&= [(\varphi\otimes\psi)\Delta(u_{ij})]_{ij}\\
&= \Big[\sum_k\varphi(u_{ik})\psi(u_{kj})\Big]_{ij}\\
&= [\varphi(u_{ij})]_{ij}[\psi(u_{ij})]_{ij}\\
&= \pi(\varphi)\pi(\psi)
\end{aligned}$$

The fact that π is unital is clear, coming from the fact that u is counital, in the sense that $\varepsilon(u_{ij})=\delta_{ij}$, as follows:

$$\pi(\varepsilon)=[\varepsilon(u_{ij})]_{ij}=1$$

Regarding now the fact that π is involutive, observe first that we have:

$$\varphi^*(u_{ij})=\overline{\varphi(S(u_{ij})^*)}=\overline{\varphi(u_{ji})}$$

Thus, we can prove that π is indeed involutive, as follows, using the fact that u is coinvolutive, in the sense that $S(u_{ij}) = u_{ji}^*$, as follows:

$$\begin{aligned}\pi(\varphi^*) &= [\varphi^*(u_{ij})]_{ij} \\ &= [\overline{\varphi(u_{ji})}]_{ij} \\ &= \Big[[\varphi(u_{ij})]_{ij}\Big]^* \\ &= \pi(\varphi)^*\end{aligned}$$

Finally, the proof in the other sense follows from exactly the same computations. □

In order to reach now to the condition $F \subset S_N^+$, we must impose several conditions on the matrix $u = (u_{ij})$.

Let us start with the bistochasticity condition. We have here:

Proposition 13.16. *Given F and $G = \widehat{F}$ as in Proposition 13.15, the matrix $u = (u_{ij})$ is bistochastic, in the sense that all the row and column sums are 1, precisely when the associated $*$-algebra representation $\pi : C(G) \to M_N(\mathbb{C})$ satisfies the conditions*

$$\pi(\varphi)\xi = \varphi(1)\xi$$

$$\pi(\varphi)^t\xi = \varphi(1)\xi$$

where $\xi \in \mathbb{C}^N$ is the all-one vector.

Proof. We want the following two conditions to be satisfied:

$$\sum_j u_{ij} = 1$$

$$\sum_i u_{ij} = 1$$

In what regards the condition $\sum_j u_{ij} = 1$, observe that in terms of π, we have:

$$\begin{aligned}\sum_j \pi(\varphi)_{ij} &= \sum_j \varphi(u_{ij}) \\ &= \varphi\left(\sum_j u_{ij}\right)\end{aligned}$$

Thus, we want this quantity to be $\varphi(1)$, for any i, and this leads to the condition $\pi(\varphi)\xi = \varphi(1)\xi$ in the statement. As for the second condition, namely $\sum_i u_{ij} = 1$, this leads to the second condition in the statement, namely $\pi(\varphi)^t\xi = \varphi(1)\xi$. □

Independently of the above result, we must impose the condition that the coordinates u_{ij} are self-adjoint. The result here is as follows:

Proposition 13.17. *Given F and $G = \widehat{F}$ as in Proposition 13.15, we have $u_{ij} = u_{ij}^*$ precisely when the associated $*$-algebra representation $\pi : C(G) \to M_N(\mathbb{C})$ satisfies:*

$$\pi S(\varphi) = \pi(\varphi)^t$$

Proof. According to the antipode formula $(S\varphi)a = \varphi(S(a))$ from Definition 13.14, we have the following computation:

$$\begin{aligned} \pi S(\varphi) &= [S\varphi(u_{ij})]_{ij} \\ &= [\varphi(u_{ji}^*)]_{ij} \end{aligned}$$

Now $u_{ij} = u_{ij}^*$ means that this latter matrix should be $[\varphi(u_{ji})]_{ij} = \pi(\varphi)^t$, as claimed. □

Let us put now what we have together. We are led to the following statement:

Proposition 13.18. *Given F and $G = \widehat{F}$ as in Proposition 13.15, the matrix $u = (u_{ij})$ is bistochastic, with self-adjoint entries, precisely when associated $*$-algebra representation*

$$\pi : C(G) \to M_N(\mathbb{C})$$

$$\varphi \to [\varphi(u_{ij})]_{ij}$$

satisfying the following conditions,

$$\pi(\varphi)\xi = \varphi(1)\xi$$

$$\pi(\varphi)^t\xi = \varphi(1)\xi$$

$$\pi S(\varphi) = \pi(\varphi)^t$$

with $\xi \in \mathbb{C}^N$ being the all-one vector.

Proof. This follows indeed from Proposition 13.16 and Proposition 13.17. □

In order to reach now to $F \subset S_N^+$, we must impose one final condition, stating that the entries of $u = (u_{ij})$ are idempotents, $u_{ij}^2 = u_{ij}$. This is something more technical:

Proposition 13.19. *Given F and $G = \widehat{F}$ as in Proposition 13.18, we have $u_{ij}^2 = u_{ij}$ precisely when the associated $*$-algebra representation $\pi : C(G) \to M_N(\mathbb{C})$ satisfies*

$$m(\pi \otimes \pi)\Delta(\varphi) = \pi(\varphi)m$$

as an equality of maps $\mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N$, where m is the multiplication of $\mathbb{C}^N$.

Proof. This is something which is quite routine. We have indeed the following computation, valid for any indices i,j, by using the Sweedler notation:

$$\begin{aligned}
m(\pi\otimes\pi)\Delta(\varphi)(e_i\otimes e_j) &= m(\pi\otimes\pi)\left(\sum\varphi_1\otimes\varphi_2\right)(e_i\otimes e_j)\\
&= m\left(\sum\pi(\varphi_1)\otimes\pi(\varphi_2)\right)(e_i\otimes e_j)\\
&= m\left(\sum\pi(\varphi_1)e_i\otimes\pi(\varphi_2)e_j\right)\\
&= m\left(\sum\sum_{kl}\varphi_1(u_{ki})e_k\otimes\varphi_2(u_{lj})e_l\right)\\
&= \sum\sum_{kl}\varphi_1(u_{ki})e_k\varphi_2(u_{lj})e_l\\
&= \sum\sum_k\varphi_1(u_{ki})\varphi_2(u_{kj})e_k\\
&= \sum\sum_k(\varphi_1\otimes\varphi_2)(u_{ki}\otimes u_{kj})e_k\\
&= \sum_k\Delta(\varphi)(u_{ki}\otimes u_{kj})e_k\\
&= \sum_k\varphi(u_{ki}u_{kj})e_k
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}
\pi(\varphi)m(e_i\otimes e_j) &= \pi(\varphi)\delta_{ij}e_i\\
&= \big[\varphi(u_{ij})\big]_{ij}\delta_{ij}e_i\\
&= \delta_{ij}\sum_k\varphi(u_{ki})e_k
\end{aligned}$$

Thus, the condition in the statement simply reads, for any i,j,k:

$$u_{ki}u_{kj}=\delta_{ij}u_{ki}$$

In particular with $i=j$ we obtain, as desired, the idempotent condition:

$$u_{ki}^2=u_{ki}$$

Conversely now, if this idempotent condition is satisfied, then $u=(u_{ij})$ follows to be a matrix of projections, which is bistochastic. Thus this matrix is magic, and so we have $u_{ki}u_{kj}=\delta_{ij}u_{ki}$ for any i,j,k, and this leads to the formula in the statement. □

Let us put now what we have together. We are led to the following statement:

Theorem 13.20. *Given F and $G = \widehat{F}$ as in Definition 13.14, we have $F \subset S_N^+$, with associated magic matrix $u = (u_{ij})$, precisely when we have a $*$-algebra representation*

$$\pi : C(G) \to M_N(\mathbb{C})$$
$$\varphi \to [\varphi(u_{ij})]_{ij}$$

satisfying the following conditions,

$$\pi(\varphi)\xi = \varphi(1)\xi$$
$$\pi(\varphi)^t\xi = \varphi(1)\xi$$
$$\pi S(\varphi) = \pi(\varphi)^t$$
$$m(\pi \otimes \pi)\Delta(\varphi) = \pi(\varphi)m$$

where $\xi \in \mathbb{C}^N$ is the all-one vector, and m is the multiplication of $\mathbb{C}^N$.

Proof. This follows indeed from Proposition 13.18 and Proposition 13.19, and from the well-known fact, already mentioned in the proof of Proposition 13.19, that a magic matrix $u = (u_{ij})$ is the same as a matrix of projections which is bistochastic. □

Let us discuss now some basic illustrations of Theorem 13.20.

In the classical case, the result, or rather illustration of our result, is something very simple, as follows:

Proposition 13.21. *Given a closed subgroup $F \subset U_N$, the associated $*$-algebra representation constructed in Theorem 13.20 is given by*

$$\pi : C^*(F) \to M_N(\mathbb{C})$$
$$\sum_g \lambda_g g \to \sum_g \lambda_g g$$

and we have $F \subset S_N$ precisely when the conditions in Theorem 13.20 are satisfied.

Proof. Here the first assertion is clear from definitions. As for the second assertion, this is something that we know from Theorem 13.20, but here is a direct check as well:

(1) For $\varphi \in C^*(F)$ given by $\varphi = \sum_g \lambda_g g$ we have $\pi(\varphi) = \sum_g \lambda_g g$, and also $\varphi(1) = \sum_g \lambda_g$ via $C^*(F) \simeq C(F)^*$ so the bistochasticity condition $F \subset C_N$ corresponds indeed to the conditions $\pi(\varphi)\xi = \varphi(1)\xi$ and $\pi(\varphi)^t\xi = \varphi(1)\xi$ from Theorem 13.20.

(2) Once again with $\varphi = \sum_g \lambda_g g$, we have the following formulae:

$$\pi S\varphi = \pi\left(\sum_g \lambda_g g^{-1}\right) = \sum_g \lambda_g g^{-1}$$

$$\pi(\varphi)^t = \left(\sum_g \lambda_g g\right)^t = \sum_g \lambda_g g^t$$

Thus $F\subset O_N$, which is the same as saying that $g^{-1}=g^t$, for any $g\in F$, is indeed equivalent to the condition $\pi S(\varphi)=\pi(\varphi)^t$ from Theorem 13.20.

(3) As before with $\varphi=\sum_g\lambda_g g$, assuming $F\subset S_N$, we have the following formula:

$$\begin{aligned}
m(\pi\otimes\pi)\Delta(\varphi)(e_i\otimes e_j) &= m\left(\sum_g\lambda_g g\otimes g\right)(e_i\otimes e_j)\\
&= m\left(\sum_g\lambda_g e_{g(i)}\otimes e_{g(j)}\right)\\
&= \delta_{ij}\sum_{g\in G}\lambda_g e_{g(i)}
\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
\pi(\varphi)m(e_i\otimes e_j) &= \left(\sum_g\lambda_g g\right)m(e_i\otimes e_j)\\
&= \left(\sum_g\lambda_g g\right)(\delta_{ij}e_i)\\
&= \delta_{ij}\sum_g\delta_g e_{g(i)}
\end{aligned}$$

Thus the condition $m(\pi\otimes\pi)\Delta(\varphi)=\pi(\varphi)m$ in Theorem 13.20 must be indeed satisfied, and the proof of the converse is similar, using the same computations. □

In the group dual case now, the result is a priori something more subtle, related to Bichon's classification in [51] of the group dual subgroups $\widehat{\Gamma}\subset S_N^+$.

However, and here comes our point, in the present dual setting everything drastically simplifies, and the complete result, with complete proof, is as follows:

Theorem 13.22. *Given a finite group G, and setting $F=\widehat{G}$, the associated $*$-algebra representation constructed in Theorem 13.20 appears as follows, for a certain family of generators $g_1,\ldots,g_N\in H$, and for a certain unitary $U\in U_N$,*

$$\pi:C(G)\to M_N(\mathbb{C})$$

$$\varphi\to Udiag(g_1,\ldots,g_N)U^*$$

and we have $F\subset S_N^+$ precisely when the conditions in Theorem 13.20 are satisfied, which in turn mean that the representation π appears as in [51].

Proof. Here the first assertion is standard, coming from Woronowicz's Peter-Weyl type theory from [146]. As for the second assertion, this is a priori something which is less obvious, related to Bichon's classification of the group dual subgroups $\widehat{\Gamma}\subset S_N^+$ in [51]. However, in our dual formulation this is something clear, because the algebra $C(G)$ is commutative, so its matrix representation π must appear diagonally, spinned by a unitary. Thus, we obtain the result, without a single computation needed. $\square$

There are many things that can be done with finite quantum permutation groups, that can be sometimes simpler by using the present dual formalism.

In order to discuss this, let us start with:

Proposition 13.23. *Given $G=\widehat{F}$ as in Theorem 13.20, the $*$-algebra representation*

$$\pi:C(G)\to M_N(\mathbb{C})$$

gives rise to a family of $*$-algebra representations as follows, for any $k\in\mathbb{N}$,

$$\pi^k:C(G)\to M_N(\mathbb{C})^{\otimes k}$$
$$\pi^k=\pi^{\otimes k}\Delta^{(k)}$$

that we will still denote by π, when there is no confusion, which are given by

$$\pi_{i_1\ldots i_k,j_1\ldots j_k}(\varphi)=\varphi(u_{i_1j_1}\ldots u_{i_kj_k})$$

in standard multi-index notation for the elements of $M_N(\mathbb{C})^{\otimes k}$.

Proof. Let us begin with the following computation, in Sweedler notation:

$$\begin{aligned}
& <\pi^{\otimes k}\Delta^{(k)}(\varphi)(e_{j_1}\otimes\ldots\otimes e_{j_k}),e_{i_1}\otimes\ldots\otimes e_{i_k}>\\
=\ & \left\langle\pi^{\otimes k}\left(\sum\varphi_1\otimes\ldots\otimes\varphi_k\right)(e_{j_1}\otimes\ldots\otimes e_{j_k}),e_{i_1}\otimes\ldots\otimes e_{i_k}\right\rangle\\
=\ & \sum<(\pi(\varphi_1)\otimes\ldots\otimes\pi(\varphi_k))(e_{j_1}\otimes\ldots\otimes e_{j_k}),e_{i_1}\otimes\ldots\otimes e_{i_k}>\\
=\ & \sum<\pi(\varphi_1)e_{j_1}\otimes\ldots\otimes\pi(\varphi_k)e_{j_k},e_{i_1}\otimes\ldots\otimes e_{i_k}>\\
=\ & \sum<\pi(\varphi_1)e_{j_1},e_{i_1}>\ldots<\pi(\varphi_k)e_{j_k},e_{i_k}>\\
=\ & \sum\varphi_1(u_{i_1j_1})\ldots\varphi_k(u_{i_kj_k})\\
=\ & \sum(\varphi_1\otimes\ldots\otimes\varphi_k)(u_{i_1j_1}\otimes\ldots\otimes u_{i_kj_k})\\
=\ & \Delta^{(k)}(\varphi)(u_{i_1j_1}\ldots u_{i_kj_k})\\
=\ & \varphi(u_{i_1j_1}\ldots u_{i_kj_k})
\end{aligned}$$

Thus, we have the following formula, valid at any $k\in\mathbb{N}$:

$$\pi^{\otimes k}\Delta^{(k)}(\varphi)=\left[\varphi(u_{i_1j_1}\ldots u_{i_kj_k})\right]_{i_1\ldots i_k,j_1\ldots j_k}$$

Equivalently, the representation $\pi = \pi^{\otimes k}\Delta^{(k)}$ is given by the following formula:

$$\pi_{i_1\ldots i_k,j_1\ldots j_k}(\varphi) = \varphi(u_{i_1j_1}\ldots u_{i_kj_k})$$

Thus, we are led to the conclusion in the statement. $\square$

Following [26], let us discuss now integration results. We have:

Theorem 13.24. *The polynomial integrals over G are given by*

$$\left[\int u_{i_1j_1}\ldots u_{i_kj_k}\right]_{i_1\ldots i_k,j_1\ldots j_k} = \pi^{\otimes k}\Delta^{(k)}(\textstyle\int)$$

and the moments of the main character $\chi = \sum_i u_{ii}$ are given by

$$\int \chi^k = Tr(\pi^{\otimes k}\Delta^{(k)}(\textstyle\int))$$

where $\int \in C(G)$ is the Haar integration functional.

Proof. The first formula is clear from Proposition 13.23. Regarding now the moments of the main character, observe first that we have the following general formula:

$$\begin{aligned}
Tr(\pi^{\otimes k}\Delta^{(k)}(\varphi)) &= \sum_{i_1\ldots i_k}\varphi(u_{i_1i_1}\ldots u_{i_ki_k})\\
&= \varphi\left(\sum_{i_1\ldots i_k}u_{i_1i_1}\ldots u_{i_ki_k}\right)\\
&= \varphi(\chi^k)
\end{aligned}$$

In particular, with $\varphi = \int$, the Haar integration, we obtain:

$$Tr(\pi^{\otimes k}\Delta^{(k)}(\textstyle\int)) = \displaystyle\int \chi^k$$

Thus, we are led to the conclusions in the statement. $\square$

As a second topic, which is of key interest, let us discuss the orbit and orbital theory, following [51], [115]. Regarding the orbits, following [51] we have:

Theorem 13.25. *The orbits of $F \subset S_N^+$ can be defined dually by $i \sim j$ when*

$$\pi_{ij}(\varphi) > 0$$

for a certain positive linear form $\varphi > 0$.

Proof. We know from [51] that $i \sim j$ when $u_{ij} \neq 0$ is an equivalence relation on $\{1,\ldots,N\}$. Here is a proof of this fact, using our present, dual formalism:

(1) The reflexivity of $\sim$ as defined in the statement is clear, coming from:

$$\pi(1) = 1 \implies \pi_{ii} \neq 0$$

(2) The symmetry is clear too, coming from $\pi S(\varphi)=\pi(\varphi)^t$. Alternatively:

$$\pi(\varphi^*)=\pi(\varphi)^* \implies \pi_{ij}(\varphi^*)=\overline{\pi_{ij}(\varphi)}$$

(3) Regarding now the transitivity, things are a bit more tricky. We have:

$$\pi_{ij}(\varphi\psi)=\sum_k\pi_{ik}(\varphi)\pi_{kj}(\psi)$$

Now since $\varphi\geq 0$ implies $\varphi(u_{ij})\geq 0$ for any i,j, we obtain the result. □

Regarding the orbitals, following [115], we first have:

Proposition 13.26. *The relation on $\{1,\ldots,N\}^k$ given by $i\sim j$ when*

$$\pi_{i_1\ldots i_k,j_1\ldots j_k}(\varphi)>0$$

for a certain positive linear form $\varphi>0$, is reflexive and symmetric.

Proof. The reflexivity is clear exactly as at $k=1$, coming from:

$$\pi(1)=1 \implies \pi_{i_1\ldots i_k,i_1\ldots i_k}\neq 0$$

The symmetry is clear too, coming from $\pi S(\varphi)=\pi(\varphi)^t$. Alternatively:

$$\pi(\varphi^*)=\pi(\varphi)^* \implies \pi_{i_1\ldots i_k,j_1\ldots j_k}(\varphi^*)=\overline{\pi_{i_1\ldots i_k,j_1\ldots j_k}(\varphi)}$$

Thus, we are led to the conclusion in the statement. □

Regarding the transitivity of the relation constructed in Proposition 13.26, things here are fairly tricky, and we were unable to find something simple, using our dual formalism.

Nevertheless, following [115], [119], let us formulate an informal statement, as follows:

Theorem 13.27. *The relation constructed in Proposition 13.26 is transitive at $k=2$, not necessarily transitive at $k\geq 3$, and these results can be both recovered in dual form.*

Proof. As mentioned, all this is quite complicated, the situation being as follows:

(1) It is known from [115] that we have transitivity at $k=2$, the proof being something tricky, but fairly short, as follows:

$$\begin{aligned}
&(u_{i_1j_1}\otimes u_{j_1l_1})\Delta(u_{i_1l_1}u_{i_2l_2})(u_{i_2j_2}\otimes u_{j_2l_2})\\
=&\sum_{s_1s_2}u_{i_1j_1}u_{i_1s_1}u_{i_2s_2}u_{i_2j_2}\otimes u_{j_1l_1}u_{s_1l_1}u_{s_2l_2}u_{j_2l_2}\\
=&\ u_{i_1j_1}u_{i_2j_2}\otimes u_{j_1l_1}u_{j_2l_2}
\end{aligned}$$

Indeed, we obtain from this that we have $u_{i_1l_1}u_{i_2l_2}\neq 0$, as desired.

(2) The challenge now is to reformulate the above proof from [115] in the dual setting, somehow by applying linear forms on both sides. This is something non-trivial, and a quite technical proof, using a conditioning method, and totalling about 1 page or so, can be found in [119]. We do not know how to simplify that proof.

(3) As good news however, the fact that we don't necessarily have transitivity at $k=3$, which was something conjectured in [115] and in follow-up papers, and not accessible with the methods there, was recently done in [119], using dual methods. $\square$

14. Transitive groups

We have seen in the previous section that a theory of orbits and orbitals can be developed for the closed subgroups $G \subset S_N^+$, and that all this is particularly interesting in connection with tori. In this section we restrict the attention to the transitive case.

Let us first review the basic theory, that we will need in what follows. The notion of transitivity, which goes back to Bichon's paper [51], can be introduced as follows:

Definition 14.1. *Let $G \subset S_N^+$ be a closed subgroup, with magic unitary $u = (u_{ij})$, and consider the equivalence relation on $\{1, \ldots, N\}$ given by $i \sim j \iff u_{ij} \neq 0$.*

1. *The equivalence classes under $\sim$ are called orbits of G.*
2. *G is called transitive when the action has a single orbit.*

In other words, we call a subgroup $G \subset S_N^+$ transitive when $u_{ij} \neq 0$, for any i, j.

This transitivity notion is standard, coming in a straightforward way from the orbit theory. In the classical case, we obtain of course the usual notion of transitivity.

We will need as well the following result, once again coming from [51]:

Theorem 14.2. *For a closed subgroup $G \subset S_N^+$, the following are equivalent:*

1. *G is transitive.*
2. *$Fix(u) = \mathbb{C}\xi$, where ξ is the all-one vector.*
3. *$\int_G u_{ij} = \frac{1}{N}$, for any i, j.*

Proof. This is well-known in the classical case. In general, the proof is as follows:

(1) $\iff$ (2) We use the standard fact that the fixed point space of a corepresentation coincides with the fixed point space of the associated coaction:

$$Fix(u) = Fix(\Phi)$$

As explained in section 13 above, the fixed point space of the magic corepresentation $u = (u_{ij})$ has the following interpretation, in terms of orbits:

$$Fix(u) = \left\{\xi \in C(X)\Big| i \sim j \implies \xi(i) = \xi(j)\right\}$$

In particular, the transitivity condition corresponds to $Fix(u) = \mathbb{C}\xi$, as stated.

(2) $\iff$ (3) This is clear from the general properties of the Haar integration, and more precisely from the fact that $(\int_G u_{ij})_{ij}$ is the projection onto $Fix(u)$. □

Let us recall now that in the classical case, in the situation where we have a transitive subgroup $G \subset S_N$, by setting $H = \{\sigma \in G | \sigma(1) = 1\}$ we have:

$$G/H = \{1, \ldots, N\}$$

Conversely, any subgroup $H \subset G$ produces an action $G \curvearrowright G/H$, given by $g(hH) = (gh)H$, and so a morphism $G \to S_N$, where $N = [G : H]$.

This latter morphism is injective when the following condition is satisfied:

$$hgh^{-1}\in H,\forall h\in G\implies g=1$$

In the quantum case now, it is quite unclear how to generalize this structure result. To be more precise, the various examples from [15] show that we cannot expect to have an elementary generalization of the above $G/H=\{1,\ldots,N\}$ isomorphism.

However, we can at least try to extend the obvious fact that $G=N|H|$ must be a multiple of N. And here, we have the following result, from [25]:

Theorem 14.3. *If $G\subset S_N^+$ is finite and transitive, then N divides $|G|$. Moreover:*

1. *The case $|G|=N$ comes from the classical finite groups, of order N, acting on themselves.*
2. *The case $|G|=2N$ is possible, in the non-classical setting, an example here being the Kac-Paljutkin quantum group, at $N=4$.*

Proof. In order to prove the first assertion, we use the coaction of $C(G)$ on the algebra $\mathbb C^N=C(1,\ldots,N)$. In terms of the standard coordinates u_{ij}, the formula is:

$$\Phi:\mathbb C^N\to\mathbb C^N\otimes C(G)\quad,\quad e_i\to\sum_je_j\otimes u_{ji}$$

For $a\in\{1,\ldots,N\}$ consider the evaluation map $ev_a:\mathbb C^N\to\mathbb C$ at a. By composing Φ with $ev_a\otimes id$ we obtain a $C(G)$-comodule map, as follows:

$$I_a:\mathbb C^N\to C(G)\quad,\quad e_i\to u_{ia}$$

Our transitivity assumption on G ensures that this map I_a is injective. In other words, we have realized $\mathbb C^N$ as a coideal subalgebra of $C(G)$.

We recall now that a finite dimensional Hopf algebra is free as a module over a coideal subalgebra A provided that the latter is Frobenius, in the sense that there exists a non-degenerate bilinear form $b:A\otimes A\to\mathbb C$ satisfying $b(xy,z)=b(x,yz)$.

We can apply this result to the coideal subalgebra $I_a(\mathbb C^N)\subset C(G)$, with the remark that $\mathbb C^N$ is indeed Frobenius, with bilinear form as follows:

$$b(fg)=\frac{1}{N}\sum_{i=1}^Nf(i)g(i)$$

Thus $C(G)$ is a free module over the N-dimensional algebra $\mathbb C^N$, and this gives the result. Regarding now the remaining assertions, the proof here goes as follows:

(1) Since $C(G)=<u_{ij}>$ is of dimension N, and its commutative subalgebra $<u_{i1}>$ is of dimension N already, $C(G)$ must be commutative. Thus G must be classical, and by transitivity, the inclusion $G\subset S_N$ must come from the action of G on itself.

(2) The closed subgroups $G\subset S_4^+$ are fully classified, and among them we have indeed the Kac-Paljutkin quantum group, which satisfies $|G|=8$, and is transitive. $\square$

Here is now a list of examples of transitive quantum groups, coming from the various constructions from the previous sections:

Theorem 14.4. *The following are transitive subgroups $G\subset S_N^+$:*

1. *The quantum permutation group S_N^+ itself.*
2. *The transitive subgroups $G\subset S_N$. These are the classical examples.*
3. *The subgroups $\widehat{G}\subset S_{|G|}$, with G abelian. These are the group dual examples.*
4. *The quantum groups $F\subset S_N^+$ which are finite, $|F|<\infty$, and transitive.*
5. *The quantum automorphism groups of transitive graphs $G^+(X)$, with $|X|=N$.*
6. *In particular, we have the hyperoctahedral quantum group $H_n^+\subset S_N^+$, with $N=2n$.*
7. *We have as well the twisted orthogonal group $O_n^{-1}\subset S_N^+$, with $N=2^n$.*

In addition, the class of transitive quantum permutation groups $\{G\subset S_N^+|N\in\mathbb N\}$ is stable under direct products $\times$, wreath products $\wr$ and free wreath products $\wr_*$.

Proof. All these assertions are well-known. In what follows we briefly describe the idea of each proof, and indicate a reference. We will be back to all these examples, gradually, in the context of certain matrix modelling questions, to be formulated later on.

(1) This comes from the fact that we have an inclusion $S_N\subset S_N^+$. Indeed, since S_N is transitive, so must be S_N^+, because its coordinates u_{ij} map to those of S_N. See [36].

(2) This is again trivial. Indeed, for a classical group $G\subset S_N$, the variables $u_{ij}=\chi(\sigma\in S_N|\sigma(j)=i)$ are all nonzero precisely when G is transitive. See [36].

(3) This follows from the general results of Bichon in [51], who classified there all the group dual subgroups $\widehat{\Gamma}\subset S_N^+$. For a discussion here, we refer to [36].

(4) Here we use the convention $|F|=\dim_{\mathbb C}C(F)$, and the statement itself is empty, and is there just for reminding us that these examples are to be investigated.

(5) This is trivial, because X being transitive means that $G(X)\curvearrowright X$ is transitive, and by definition of $G^+(X)$, we have $G(X)\subset G^+(X)$. See [7].

(6) This comes from a result from [21], stating that we have $H_n^+=G^+(I_n)$, where I_n is the graph formed by n segments, having $N=2n$ vertices.

(7) Once again this comes from a result from [21], stating that we have $O_n^{-1}=G^+(K_n)$, where K_n is the n-dimensional hypercube, having $N=2^n$ vertices.

Finally, the stability assertion is clear from the definition of the various products involved, from [48], [139]. This is well-known, and we will be back later on to this. $\square$

Summarizing, we have a substantial list of examples. We will see in the next sections that there are several other interesting examples, coming from the matrix models.

We will be back with more general theory at the end of this section.

Let us discuss now classification results at small values of N.

In order to discuss the case $N=4$, we will need a very precise result, stating that S_4^+ is a twist of SO_3. Let us start with the following definition, from [15]:

Definition 14.5. *$C(SO_3^{-1})$ is the universal C^*-algebra generated by the entries of a 3×3 orthogonal matrix $a=(a_{ij})$, with the following relations:*

1. *Skew-commutation: $a_{ij}a_{kl}=\pm a_{kl}a_{ij}$, with sign $+$ if $i\neq k,j\neq l$, and $-$ otherwise.*
2. *Twisted determinant condition: $\Sigma_{\sigma\in S_3}a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}=1$.*

Normally, our first task would be to prove that $C(SO_3^{-1})$ is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following result, from [15]:

Theorem 14.6. *We have an isomorphism of compact quantum groups*

$$S_4^+=SO_3^{-1}$$

given by the Fourier transform over the Klein group $K=\mathbb Z_2\times\mathbb Z_2$.

Proof. Consider indeed the following matrix, corresponding to the standard vector space action of SO_3^{-1} on $\mathbb C^4$:

$$a^+=\begin{pmatrix}1&0\\0&a\end{pmatrix}$$

We apply to this matrix the Fourier transform over the Klein group $K=\mathbb Z_2\times\mathbb Z_2$:

$$u=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over K converts the relations in Definition 14.5 into the magic relations. Thus, we obtain the identification from the statement. $\square$

We have the following classification result, also from [15]:

Theorem 14.7. *The closed subgroups of $S_4^+=SO_3^{-1}$ are as follows:*

1. *Infinite quantum groups: S_4^+, O_2^{-1}, $\widehat{D}_\infty$.*
2. *Finite groups: S_4, and its subgroups.*
3. *Finite group twists: S_4^{-1}, A_5^{-1}.*
4. *Series of twists: D_{2n}^{-1} $(n\geq 3)$, DC_{2n}^{-1} $(n\geq 2)$.*
5. *A group dual series: $\widehat{D}_n$, with $n\geq 3$.*

Moreover, these quantum groups are subject to an ADE classification result.

Proof. The idea here is that the classification result can be obtained by taking some inspiration from the McKay classification of the subgroups of SO_3. See [15]. $\square$

By restricting now the attention to the transitive case, we obtain:

Theorem 14.8. *The small order transitive quantum groups are as follows:*

(1) *At $N = 1, 2, 3$ we have $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, S_3.*
(2) *At $N = 4$ we have $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_4, A_4, S_4, O_2^{-1}, S_4^+$ and S_4^{-1}, A_5^{-1}.*

Proof. This follows from the above result, the idea being as follows:

(1) This follows from the fact that we have $S_N = S_N^+$ at $N \leq 3$, from [140].

(2) This follows from the ADE classification of the subgroups $G \subset S_4^+$, from [13], with all the twists appearing in the statement being standard twists. See [13]. □

As an interesting consequence of the above result, we have:

Proposition 14.9. *The inclusion of compact quantum groups*

$$S_4 \subset S_4^+$$

is maximal, in the sense that there is no quantum group in between.

Proof. This follows indeed from the above classification result. See [15]. □

It is conjectured in fact that $S_N \subset S_N^+$ should be maximal, for any $N \in \mathbb{N}$. We will be back to this.

Let us study now the quantum subgroups $G \subset S_5^+$. We first have the following elementary observations, regarding such subgroups:

Proposition 14.10. *We have the following examples of subgroups $G \subset S_5^+$:*

(1) *The classical subgroups, $G \subset S_5$. There are 16 such subgroups, having order $1, 2, 3, 4, 4, 5, 6, 6, 8, 10, 12, 12, 20, 24, 60, 120$.*
(2) *The group duals, $G = \widehat{\Gamma} \subset S_5^+$. These appear, via a Fourier transform construction, from the various quotients Γ of the groups $\mathbb{Z}_4, \mathbb{Z}_2 * \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_3$.*

In addition, we have as well all the ADE quantum groups $G \subset S_4^+ \subset S_5^+$ from Theorem 14.7 above, embedded via the 5 standard embeddings $S_4^+ \subset S_5^+$.

Proof. These results are well-known, the proof being as follows:

(1) This is a classical result, with the groups which appear being respectively the cyclic groups $\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathbb{Z}_5, \mathbb{Z}_6, S_3, D_4, D_5, A_4$, then a copy of $S_3 \rtimes \mathbb{Z}_2$, the general affine group $GA_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, and finally S_4, A_5, S_5.

(2) This follows from Bichon's result in [51], stating that the group dual subgroups $G = \widehat{\Gamma} \subset S_N^+$ appear from the various quotients $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N_1 + \ldots + N_k = N$. At $N = 5$ the partitions are $5 = 1 + 4, 1 + 2 + 2, 2 + 3$, and this gives the result. □

Summarizing, the classification of the subgroups $G\subset S_5^+$ is a particularly difficult task, the situation here being definitely much more complicated than at $N=4$.

Consider now an intermediate compact quantum group, as follows:

$$S_N\subset G\subset S_N^+$$

Then G must be transitive. Thus, we can restrict the attention to such quantum groups.

Regarding such quantum groups, we first have the following elementary result:

Proposition 14.11. *We have the following examples of transitive subgroups $G\subset S_5^+$:*

1. *The classical transitive subgroups $G\subset S_5$. There are only 5 such subgroups, namely $\mathbb Z_5,D_5,GA_1(5),A_5,S_5$.*
2. *The transitive group duals, $G=\widehat{\Gamma}\subset S_5^+$. There is only one example here, namely the dual of $\Gamma=\mathbb Z_5$, which is $\mathbb Z_5$, already appearing above.*

In addition, all the ADE quantum groups $G\subset S_4^+\subset S_5^+$ are not transitive.

Proof. This follows indeed by examining the lists in Proposition 14.10:

(1) The result here is well-known, and elementary. Observe that $GA_1(5)=\mathbb Z_5\rtimes\mathbb Z_4$, which is by definition the general affine group of $\mathbb F_5$, is indeed transitive.

(2) This follows from the results in [51], because with $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}\to\Gamma$ as in the proof of Proposition 14.10 (2), the orbit decomposition is precisely $N=N_1+\ldots+N_k$.

Finally, the last assertion is clear, because the embedding $S_4^+\subset S_5^+$ is obtained precisely by fixing a point. Thus S_4^+ and its subgroups are not transitive, as claimed. □

In order to prove the uniqueness result, we will use the recent progress in subfactor theory [108], concerning the classification of the small index subfactors.

For our purposes, the most convenient formulation of the result in [108] is:

Theorem 14.12. *The principal graphs of the irreducible index 5 subfactors are:*

1. *A_∞, and a non-extremal perturbation of $A_\infty^{(1)}$.*
2. *The McKay graphs of $\mathbb Z_5,D_5,GA_1(5),A_5,S_5$.*
3. *The twists of the McKay graphs of A_5,S_5.*

Proof. This is a heavy result, and we refer to [108] for the whole story. The above formulation is the one from [108], with the subgroup subfactors there replaced by fixed point subfactors [3], and with the cyclic groups denoted as usual by $\mathbb Z_N$. □

In the quantum permutation group setting, this result becomes:

Theorem 14.13. *The set of principal graphs of the transitive subgroups $G\subset S_5^+$ coincide with the set of principal graphs of the subgroups $\mathbb Z_5,D_5,GA_1(5),A_5,S_5,S_5^+$.*

Proof. We must take the list of graphs in Theorem 14.12, and exclude some of the graphs, on the grounds that the graph cannot be realized by a transitive subgroup $G \subset S_5^+$.

We have 3 cases here to be studied, as follows:

(1) The graph A_∞ corresponds to S_5^+ itself. As for the perturbation of $A_\infty^{(1)}$, this dissapears, because our notion of transitivity requires the subfactor extremality.

(2) For the McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$ there is nothing to be done, all these graphs being solutions to our problem.

(3) The possible twists of A_5, S_5, coming from the graphs in Theorem 14.12 (3) above, cannot contain S_5, because their cardinalities are smaller or equal than $|S_5| = 120$. □

In connection now with our maximality questions, we have:

Theorem 14.14. *The inclusion $S_5 \subset S_5^+$ is maximal.*

Proof. This follows indeed from Theorem 14.13, with the remark that S_5 being transitive, so must be any intermediate subgroup $S_5 \subset G \subset S_5^+$. □

With a little more work, the above considerations can give the full list of transitive subgroups $G \subset S_5^+$. To be more precise, we have here the various subgroups appearing in Theorem 14.13, plus some possible twists of A_5, S_5, which remain to be investigated.

In general, the maximality of $S_N \subset S_N^+$ is a difficult question. The only known general result here is in the easy case, as follows:

Theorem 14.15. *There is no intermediate easy quantum group $S_N \subset G \subset S_N^+$.*

Proof. This follows by doing some combinatorics. To be more precise, the idea is to show that any $\pi \in P - NC$ has the following property:

$$< \pi >= P$$

And, in order to establish this formula, the idea is to cap π with semicircles, as to preserve one crossing, chosen in advance, and to end up, by a recurrence procedure, with the standard crossing. We refer to [32] for full details here.

We can actually prove this at easiness level 2, as follows. Our first claim is that, assuming that $G \subset H$ comes from an inclusion of categories of partitions $D \subset E$, the maximality at order 2 is equivalent to the following condition, for any $\pi, \sigma \in E$, not both in D, and for any $\alpha, \beta \neq 0$:

$$< span(D), \alpha T_\pi + \beta T_\sigma >= span(E)$$

Consider indeed a category $span(D) \subset C \subset span(E)$, corresponding to a quantum group $G \subset K \subset H$ having order 2. The order 2 condition means that we have $C =< C \cap$

$span_2(P)>$, where $span_2$ denotes the space of linear combinations having 2 components. Since we have $span(E)\cap span_2(P)=span_2(E)$, the order 2 formula reads:

$$C=<C\cap span_2(E)>$$

Now observe that the category on the right is generated by the categories $C_{\pi\sigma}^{\alpha\beta}$ constructed in the statement. Thus, the order 2 condition reads:

$$C=\left\langle C_{\pi\sigma}^{\alpha\beta}\Big|\pi,\sigma\in E,\alpha,\beta\in\mathbb C\right\rangle$$

Now since the maximality at order 2 of the inclusion $G\subset H$ means that we have $C\in\{span(D),span(E)\}$, for any such C, we are led to the following condition:

$$C_{\pi\sigma}^{\alpha\beta}\in\{span(D),span(E)\}\quad,\quad\forall\pi,\sigma\in E,\alpha,\beta\in\mathbb C$$

Thus, we have proved our claim. In order to show now that $S_N\subset S_N^+$ is maximal at order 2, we can use the old "semicircle capping" method. That method shows that any $\pi\in P-NC$ has the property $<\pi>=P$, and in order to establish this formula, the idea is to cap π with semicircles, as to preserve one crossing, chosen in advance, and to end up, by a recurrence procedure, with the standard crossing.

In our present case now, at level 2, the statement that we have to prove is as follows: "for $\pi\in P-NC,\sigma\in P$ and $\alpha,\beta\neq 0$ we have $<\alpha T_\pi+\beta T_\sigma>=span(P)$".

In order to do this, our claim is that the same method as at level 1 applies, after some suitable modifications. We have indeed two cases, as follows:

(1) Assuming that π,σ have at least one different crossing, we can cap the partition π as to end up with the basic crossing, and σ becomes in this way an element of $P(2,2)$ different from this basic crossing, and so a noncrossing partition, from $NC(2,2)$. Now by substracting this noncrossing partition, which belongs to $C_{S_N^+}=span(NC)$, we obtain that the standard crossing belongs to $<\alpha T_\pi+\beta T_\sigma>$, and we are done.

(2) In the case where π,σ have exactly the same crossings, we can start our descent procedure by selecting one common crossing, and then two strings of π,σ which are different, and then joining the crossing to these two strings. We obtain in this way a certain linear combination $\alpha'T_{\pi'}+\beta'T_{\sigma'}\in<\alpha T_\pi+\beta T_\sigma>$ which satisfies the conditions in (1) above, and we can continuate as indicated there. $\square$

The corresponding orthogonal quantum group questions are somehow easier, and our purpose in what follows will be that of discussing all this. We first have:

Theorem 14.16. *There is only one proper intermediate easy quantum group*

$$O_N\subset G\subset O_N^+$$

namely the half-classical orthogonal group O_N^*.

Proof. We must compute here the categories of pairings $NC_2 \subset D \subset P_2$, and this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi \in P_2 - NC_2$, having $s \geq 4$ strings. Our claim is that:

– If $\pi \in P_2 - P_2^*$, there exists a semicircle capping $\pi' \in P_2 - P_2^*$.

– If $\pi \in P_2^* - NC_2$, there exists a semicircle capping $\pi' \in P_2^* - NC_2$.

Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi \in P_2(k,l) - NC_2(k,l)$. Our claim is that:

– If $\pi \in P_2(k,l) - P_2^*(k,l)$ then $< \pi >= P_2$.

– If $\pi \in P_2^*(k,l) - NC_2(k,l)$ then $< \pi >= P_2^*$.

This can be indeed proved by recurrence on the number of strings, $s = (k+l)/2$, by using (1), which provides us with a descent procedure $s \to s-1$, at any $s \geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N \subset G \subset O_N^+$, coming from certain sets of pairings $D(k,l) \subset P_2(k,l)$. We have three cases:

– If $D \not\subset P_2^*$, we obtain $G = O_N$.

– If $D \subset P_2, D \not\subset NC_2$, we obtain $G = O_N^*$.

– If $D \subset NC_2$, we obtain $G = O_N^+$.

Thus, we are led to the conclusion in the statement. $\square$

We have as well the following result, from [21], going beyond easiness:

Theorem 14.17. *The following inclusions are maximal:*

(1) $\mathbb{T}O_N \subset U_N$.
(2) $PO_N \subset PU_N$.
(3) $O_N \subset O_N^*$.

Proof. In order to prove these results, consider as well the group $\mathbb{T}SO_N$. Observe that we have $\mathbb{T}SO_N = \mathbb{T}O_N$ if N is odd. If N is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N, \mathfrak{u}_N$ the Lie algebras of SO_N, U_N. It is well-known that $\mathfrak{u}_N$ consists of the matrices $M \in M_N(\mathbb{C})$ satisfying $M^* = -M$, and that $\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$. Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N \oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N \geq 2$, the adjoint representation of SO_N on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X \in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = span\left\{UXU^t \middle| U \in SO_N\right\}$$

We first prove that V contains all the diagonal matrices of trace zero. Since we may diagonalize X by conjugating with an element of SO_N, our space V contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = diag(d_1, d_2, \ldots, d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&I_{N-2}\end{pmatrix}\in SO_N$$

We conclude that our space V contains as well the following matrix:

$$D' = diag(d_2, d_1, d_3, \ldots, d_N)$$

More generally, we see that for any $1 \leq i, j \leq N$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V. Now since S_N is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D. But it is well-known that this representation of S_N on the diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices, as claimed.

In order to conclude now, assume that Y is an arbitrary real symmetric matrix of trace zero. We can find then an element $U \in SO_N$ such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb{T}SO_N \subset U_N$ is maximal in the category of connected compact groups.

Let indeed G be a connected compact group satisfying $\mathbb{T}SO_N \subset G \subset U_N$. Then G is a Lie group. Let $\mathfrak{g}$ denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_N$$

Let ad_G be the action of G on $\mathfrak{g}$ obtained by differentiating the adjoint action of G on itself. This action turns $\mathfrak{g}$ into a G-module. Since $SO_N \subset G$, $\mathfrak{g}$ is also a SO_N-module.

Now if $G \neq \mathbb{T}SO_N$, then since G is connected we must have $\mathfrak{so}_N \oplus i\mathbb{R} \neq \mathfrak{g}$. It follows from the real vector space structure of the Lie algebras $\mathfrak{u}_N$ and $\mathfrak{so}_N$ that there exists a nonzero symmetric real matrix of trace zero X such that:

$$iX \in \mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of SO_N under the adjoint action. Thus $\mathfrak{g}$ must contain all such X, and hence $\mathfrak{g} = \mathfrak{u}_N$. But since U_N is connected, it follows that $G = U_N$.

Step 3. Our claim is that the commutant of SO_N in $M_N(\mathbb{C})$ is as follows:

(1) $SO_2' = \left\{\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\Big|\alpha,\beta\in\mathbb{C}\right\}$.

(2) If $N \geq 3$, $SO_N' = \{\alpha I_N | \alpha \in \mathbb{C}\}$.

Indeed, at $N=2$ this is a direct computation. At $N\geq 3$, an element in $X\in SO_N'$ commutes with any diagonal matrix having exactly $N-2$ entries equal to 1 and two entries equal to -1. Hence X is a diagonal matrix. Now since X commutes with any even permutation matrix and $N\geq 3$, it commutes in particular with the permutation matrix associated with the cycle (i,j,k) for any $1<i<j<k$, and hence all the entries of X are the same. We conclude that X is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in SO_N.

At $N=2$ this is clear, since the set of elements in SO_2 having a given trace is finite. So assume $N>2$, and let $T\in SO_N\simeq SO(\mathbb R^N)$ with $Tr(T)=0$. Let $E\subset\mathbb R^N$ be a 2-dimensional subspace preserved by T, such that $T_{|E}\in SO(E)$.

Let $\varepsilon>0$ and let $S_\varepsilon\in SO(E)$ with $||T_{|E}-S_\varepsilon||<\varepsilon$, and with $Tr(T_{|E})\neq Tr(S_\varepsilon)$, in the $N=2$ case. Now define $T_\varepsilon\in SO(\mathbb R^N)=SO_N$ by:

$$T_{\varepsilon|E}=S_\varepsilon\quad,\quad T_{\varepsilon|E^\perp}=T_{|E^\perp}$$

It is clear that $||T-T_\varepsilon||\leq||T_{|E}-S_\varepsilon||<\varepsilon$ and that:

$$Tr(T_\varepsilon)=Tr(S_\varepsilon)+Tr(T_{|E^\perp})\neq 0$$

Thus, we have proved our claim.

Step 5. Our claim is that $\mathbb TO_N$ is the normalizer of $\mathbb TSO_N$ in U_N, i.e. is the subgroup of U_N consisting of the unitaries U for which $U^{-1}XU\in\mathbb TSO_N$ for all $X\in\mathbb TSO_N$.

It is clear that the group $\mathbb TO_N$ normalizes $\mathbb TSO_N$, so in order to prove the result, we must show that if $U\in U_N$ normalizes $\mathbb TSO_N$ then $U\in\mathbb TO_N$.

First note that U normalizes SO_N. Indeed if $X\in SO_N$ then $U^{-1}XU\in\mathbb TSO_N$, so $U^{-1}XU=\lambda Y$ for some $\lambda\in\mathbb T$ and $Y\in SO_N$. If $Tr(X)\neq 0$, we have $\lambda\in\mathbb R$ and hence:

$$\lambda Y=U^{-1}XU\in SO_N$$

The set of matrices having nonzero trace being dense in SO_N, we conclude that $U^{-1}XU\in SO_N$ for all $X\in SO_N$. Thus, we have:

$$\begin{aligned}X\in SO_N&\implies (UXU^{-1})^t(UXU^{-1})=I_N\\&\implies X^tU^tUX=U^tU\\&\implies U^tU\in SO_N'\end{aligned}$$

It follows that at $N\geq 3$ we have $U^tU=\alpha I_N$, with $\alpha\in\mathbb T$, since U is unitary. Hence we have $U=\alpha^{1/2}(\alpha^{-1/2}U)$ with $\alpha^{-1/2}U\in O_N$, and $U\in\mathbb TO_N$. If $N=2$, $(U^tU)^t=U^tU$ gives again that $U^tU=\alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb TO_N\subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb TO_N\subset G\subset U_N$ is a compact group such that $G\neq U_N$. It is a well-known fact that the connected component of the identity in G is a normal subgroup, denoted G_0. Since we have $\mathbb{T}SO_N \subset G_0 \subset U_N$, we must have $G_0 = \mathbb{T}SO_N$. But since G_0 is normal in G, the group G normalizes $\mathbb{T}SO_N$, and hence $G \subset \mathbb{T}O_N$.

Step 7. Our claim is that the inclusion $PO_N \subset PU_N$ is maximal in the category of compact groups.

This follows from the above result. Indeed, if $PO_N \subset G \subset PU_N$ is a proper intermediate subgroup, then its preimage under the quotient map $U_N \to PU_N$ would be a proper intermediate subgroup of $\mathbb{T}O_N \subset U_N$, which is a contradiction.

Step 8. Our claim is that the inclusion $O_N \subset O_n^*$ is maximal in the category of compact compact groups.

Consider indeed a sequence of surjective Hopf $*$-algebra maps as follows, whose composition is the canonical surjection:

$$C(O_N^*) \xrightarrow{f} A \xrightarrow{g} C(O_N)$$

This produces a diagram of Hopf algebra maps with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, "f_|"] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PA \arrow[r] \arrow[d, "g_|"] & A \arrow[r] \arrow[d, "g"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PC(O_N) \arrow[r] & C(O_N) \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Consider now the following composition, with the isomorphism on the left being something well-known, coming from [54], that we will explain in section 16 below:

$$C(PU_N) \simeq C(PO_N^*) \xrightarrow{f_|} PA \xrightarrow{g_|} PC(O_N) \simeq C(PO_N)$$

This induces, at the group level, the embedding $PO_N \subset PU_N$. Thus $f_|$ or $g_|$ is an isomorphism. If $f_|$ is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, equal] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] & A \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Then f is an isomorphism. Similarly if $g_|$ is an isomorphism, then g is an isomorphism. For further details on all this, we refer to [21]. $\square$

Summarizing, we have many interesting questions here, and both the maximality of $S_N \subset S_N^+$ and of $O_N^* \subset O_N^+$ are main problems in the area.

Let us go back now to theoretical questions, in relation with the notion of transitivity. We will discuss in what follows some modifications of the usual notion of transitivity.

It is convenient to introduce a few more related objects, as follows:

Definition 14.18. *Associated to a quantum group $G \subset S_N^+$, producing the equivalence relation on $\{1,\ldots,N\}$ given by $i \sim j$ when $u_{ij} \neq 0$, are as well:*

1. *The partition $\pi \in P(N)$ having as blocks the equivalence classes under $\sim$.*
2. *The binary matrix $\varepsilon \in M_N(0,1)$ given by $\varepsilon_{ij} = \delta_{u_{ij},0}$.*

Observe that each of the objects $\sim, \pi, \varepsilon$ determines the other two ones.

We will often assume, without mentioning it, that the orbits of $G \subset S_N^+$ come in increasing order, in the sense that the corresponding partition is as follows:

$$\pi = \{1,\ldots,K_1\},\ldots,\{K_1+\ldots+K_{M-1}+1,\ldots,K_1+\ldots+K_M\}$$

Indeed, at least for the questions that we are interested in here, we can always assume that it is so, simply by conjugating everything by a suitable permutation $\sigma \in S_N$.

In terms of these objects, the notion of transitivity reformulates as follows:

Definition 14.19. *We call $G \subset S_N^+$ transitive when $u_{ij} \neq 0$ for any i,j. Equivalently:*

1. *$\sim$ must be trivial, $i \sim j$ for any i,j.*
2. *π must be the 1-block partition.*
3. *ε must be the all-1 matrix.*

Let us discuss now the quantum analogue of the fact that given a subgroup $G \subset S_N$, with orbits of lenghts $K_1,\ldots,K_M$, we have an inclusion as follows:

$$G \subset S_{K_1} \times \ldots \times S_{K_M}$$

Given two quantum permutation groups $G \subset S_K^+, H \subset S_L^+$, with magic corepresentations denoted u,v, we can consider the following algebra, and matrix:

$$A = C(G) * C(H) \quad , \quad w = diag(u,v)$$

The pair (A,w) satisfies Woronowicz's axioms, and since w is magic, we therefore obtain a quantum permutation group, denoted $G \,\hat{*}\, H \subset S_{K+L}^+$. See [139].

With this notion in hand, we have the following result:

Proposition 14.20. *Given a quantum group $G\subset S_N^+$, with associated orbit decomposition partition $\pi\in P(N)$, having blocks of length $K_1,\ldots,K_M$, we have an inclusion*
$$G\subset S_{K_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{K_M}^+$$
where the product on the right is constructed with respect to the blocks of π. In the classical case, $G\subset S_N$, we obtain in this way the usual inclusion $G\subset S_{K_1}\times\ldots\times S_{K_M}$.

Proof. Since the standard coordinates $u_{ij}\in C(G)$ satisfy $u_{ij}=0$ for $i\not\sim j$, the algebra $C(G)$ appears as quotient of the following algebra:
$$\begin{aligned}C(S_N^+)\Big/\left<u_{ij}=0,\forall i\not\sim j\right>&=C(S_{K_1}^+)*\ldots*C(S_{K_M}^+)\\&=C(S_{K_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{K_M}^+)\end{aligned}$$

Thus, we have an inclusion of quantum groups, as in the statement. Finally, observe that the classical version of the quantum group $S_{K_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{K_M}^+$ is given by:
$$\begin{aligned}(S_{K_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{K_M}^+)_{class}&=(S_{K_1}\times\ldots\times S_{K_M})_{class}\\&=S_{K_1}\times\ldots\times S_{K_M}\end{aligned}$$

Thus in the classical case we obtain $G\subset S_{K_1}\times\ldots\times S_{K_M}$, as claimed. □

Let us discuss now an extension of the notion of transitivity, from [36], as follows:

Definition 14.21. *A quantum permutation group $G\subset S_N^+$ is called quasi-transitive when all its orbits have the same size. Equivalently:*

(1) *$\sim$ has equivalence classes of same size.*
(2) *π has all the blocks of equal length.*
(3) *ε is block-diagonal with blocks the flat matrix of size K.*

As a first example, if G is transitive then it is quasi-transitive. In general now, if we denote by $K\in\mathbb N$ the common size of the blocks, and by $M\in\mathbb N$ their multiplicity, then we must have $N=KM$. We have the following result:

Proposition 14.22. *Assuming that $G\subset S_N^+$ is quasi-transitive, we must have*
$$G\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$$
where $K\in\mathbb N$ is the common size of the orbits, and $M\in\mathbb N$ is their number.

Proof. This follows indeed from definitions. □

Observe that in the classical case, we obtain in this way the usual embedding:
$$G\subset\underbrace{S_K\times\ldots\times S_K}_{M\ terms}$$

Let us discuss now the examples. Assume that we are in the following situation:

$$G\subset S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+$$

If u,v are the fundamental corepresentations of $C(S_N^+),C(S_K^+)$, consider the quotient map $\pi_i:C(S_N^+)\to C(S_K^+)$ constructed as follows:

$$u\to diag(1_K,\ldots,1_K,\underbrace{v}_{i-th\ term},1_K,\ldots,1_K)$$

We can then set $C(G_i)=\pi_i(C(G))$, and we have the following result:

Proposition 14.23. *If G_i is transitive for all i, then G is quasi-transitive.*

Proof. We know that we have embeddings as follows:

$$G_1\times\ldots\times G_M\subset G\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$$

It follows that the size of any orbit of G is at least K, because it contains $G_1\times\ldots\times G_M$, and at most K, because it is contained in $S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+$. Thus, G is quasi-transitive. □

We call the quasi-transitive subgroups appearing as above "of product type". There are quasi-transitive groups which are not of product type, as for instance:

$$G=S_2\subset S_2\times S_2\subset S_4$$

$$\sigma\to(\sigma,\sigma)$$

Indeed, the quasi-transitivity is clear, say by letting G act on the vertices of a square. On the other hand, since we have $G_1=G_2=\{1\}$, this group is not of product type.

In general, we can construct examples by using various product operations:

Proposition 14.24. *Given transitive subgroups $G_1,\ldots,G_M\subset S_K^+$, the following constructions produce quasi-transitive subgroups as follows, of product type:*

$$G\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$$

1. *The usual product: $G=G_1\times\ldots\times G_M$.*
2. *The dual free product: $G=G_1\,\hat{*}\,\ldots\,\hat{*}\,G_M$.*

Proof. All these assertions are clear from definitions, because in each case, the quantum groups $G_i\subset S_K^+$ constructed before are those in the statement. □

In the group dual case, we have the following result:

Proposition 14.25. *The group duals which are of product type*

$$\widehat{\Gamma}\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$$

are precisely those appearing from intermediate groups of the following type:

$$\underbrace{\mathbb{Z}_K*\ldots*\mathbb{Z}_K}_{M\ terms}\to\Gamma\to\underbrace{\mathbb{Z}_K\times\ldots\times\mathbb{Z}_K}_{M\ terms}$$

Proof. In one sense, this is clear. Conversely, consider a group dual $\widehat{\Gamma}\subset S_N^+$, coming from a quotient group, as follows:

$$\mathbb{Z}_K^{*M}\to\Gamma$$

The subgroups $G_i\subset\widehat{\Gamma}$ constructed above must be group duals as well, $G_i=\widehat{\Gamma}_i$, for certain quotient groups $\Gamma\to\Gamma_i$. Now if $\widehat{\Gamma}$ is of product type, $\widehat{\Gamma}_i\subset S_K^+$ must be transitive, and hence equal to $\widehat{\mathbb{Z}}_K$. Thus we have $\Gamma\to\mathbb{Z}_K^M$. □

In order to construct now some other classes of examples, we use the notion of normality for compact quantum groups. This notion, from [80], [141], is introduced as follows:

Definition 14.26. *Given a quantum subgroup $H\subset G$, coming from a quotient map $\pi:C(G)\to C(H)$, the following are equivalent:*

1. $A=\{a\in C(G)|(id\otimes\pi)\Delta(a)=a\otimes 1\}$ *satisfies* $\Delta(A)\subset A\otimes A$.
2. $B=\{a\in C(G)|(\pi\otimes id)\Delta(a)=1\otimes a\}$ *satisfies* $\Delta(B)\subset B\otimes B$.
3. *We have $A=B$, as subalgebras of $C(G)$.*

If these conditions are satisfied, we say that $H\subset G$ is a normal subgroup.

Now with this notion in hand, we have, following [36]:

Theorem 14.27. *Assuming that $G\subset S_N^+$ is transitive, and that $H\subset G$ is normal, $H\subset S_N^+$ follows to be quasi-transitive.*

Proof. Consider the quotient map $\pi:C(G)\to C(H)$, given at the level of standard coordinates by $u_{ij}\to v_{ij}$. Consider two orbits O_1,O_2 of H and set:

$$x_i=\sum_{j\in O_1}u_{ij}\quad,\quad y_i=\sum_{j\in O_2}u_{ij}$$

These two elements are orthogonal projections in $C(G)$ and they are nonzero, because they are sums of nonzero projections by transitivity of G. We have:

$$\begin{aligned}(id\otimes\pi)\Delta(x_i) &= \sum_k\sum_{j\in O_1}u_{ik}\otimes v_{kj}\\ &= \sum_{k\in O_1}\sum_{j\in O_1}u_{ik}\otimes v_{kj}\\ &= \sum_{k\in O_1}u_{ik}\otimes 1\\ &= x_i\otimes 1\end{aligned}$$

Thus by normality of H we have the following formula:

$$(\pi\otimes id)\Delta(x_i)=1\otimes x_i$$

On the other hand, assuming that we have $i\in O_2$, we obtain:

$$(\pi\otimes id)\Delta(x_i)=\sum_k\sum_{j\in O_1}v_{ik}\otimes u_{kj}=\sum_{k\in O_2}v_{ik}\otimes x_k$$

Multiplying this by $v_{ik}\otimes 1$ with $k\in O_2$ yields $v_{ik}\otimes x_k=v_{ik}\otimes x_i$, that is to say, $x_k=x_i$. In other words, x_i only depends on the orbit of i. The same is of course true for y_i. By using this observation, we can compute the following element:

$$z=\sum_{k\in O_2}\sum_{j\in O_1}u_{kj}=\sum_{k\in O_2}x_k=|O_2|x_i$$

On the other hand, by applying the antipode, we have as well:

$$S(z)=\sum_{k\in O_2}\sum_{j\in O_1}u_{jk}=\sum_{j\in O_1}y_j=|O_1|y_j$$

We therefore obtain the following formula:

$$S(x_i)=\frac{|O_1|}{|O_2|}y_j$$

Now since both x_i and y_j have norm one, we conclude that the two orbits have the same size, and this finishes the proof. □

Finally, let us discuss the notion of double transitivity. We have here:

Definition 14.28. *Let $G\subset S_N^+$ be a closed subgroup, with magic unitary $u=(u_{ij})$, and consider the equivalence relation on $\{1,\ldots,N\}^2$ given by $(i,k)\sim(j,l)\iff u_{ij}u_{kl}\neq 0$.*

(1) *The equivalence classes under $\sim$ are called orbitals of G.*
(2) *G is called doubly transitive when the action has two orbitals.*

In other words, we call $G\subset S_N^+$ doubly transitive when $u_{ij}u_{kl}\neq 0$, for any $i\neq k, j\neq l$.

To be more precise, it is clear from definitions that the diagonal $D\subset\{1,\ldots,N\}^2$ is an orbital, and that its complement D^c must be a union of orbitals. With this remark in hand, the meaning of (2) is that the orbitals must be D,D^c.

Among the results established in [115] is the fact that, with suitable definitions, the space $Fix(u^{\otimes 2})$ consists of the functions which are constant on the orbitals. We have:

Theorem 14.29. *For a doubly transitive subgroup $G\subset S_N^+$, we have:*

$$\int_G u_{ij}u_{kl}=\begin{cases}\frac{1}{N}&\text{if }i=k,j=l\\ 0&\text{if }i=k,j\neq l\text{ or }i\neq k,j=l\\ \frac{1}{N(N-1)}&\text{if }i\neq k,j\neq l\end{cases}$$

Moreover, this formula characterizes the double transitivity.

Proof. We use the standard fact, from [147], that the integrals in the statement form the projection onto $Fix(u^{\otimes 2})$. Now if we assume that G is doubly transitive, $Fix(u^{\otimes 2})$ has dimension 2, and therefore coincides with $Fix(u^{\otimes 2})$ for the usual symmetric group S_N. Thus the integrals in the statement coincide with those for the symmetric group S_N, which are given by the above formula. Finally, the converse is clear as well. □

15. Matrix models

One interesting method for the study of the subgroups $G \subset S_N^+$, that we have not tried yet, consists in modelling the coordinates $u_{ij} \in C(G)$ by concrete variables $U_{ij} \in B$. Indeed, assuming that the model is faithful in some suitable sense, that the algebra B is something quite familiar, and that the variables U_{ij} are not too complicated, all the questions about G would correspond in this way to routine questions inside B.

We discuss here these questions, first for the arbitrary quantum groups $G \subset U_N^+$, and then for the quantum permutation groups $G \subset S_N^+$. Regarding the choice of the target algebra B, some very convenient algebras are the random matrix ones, $B = M_K(C(T))$, with $K \in \mathbb{N}$, and T being a compact space. These algebras generalize indeed the most familiar algebras that we know, namely the matrix ones $M_K(\mathbb{C})$, and the commutative ones $C(T)$. We are led in this way to the following general definition:

Definition 15.1. *A matrix model for $G \subset U_N^+$ is a morphism of C^*-algebras*

$$\pi : C(G) \to M_K(C(T))$$

where T is a compact space, and $K \geq 1$ is an integer.

There are many examples of such models, and will discuss them later on. For the moment, let us develop some general theory. The question to be solved is that of understanding the suitable faithfulness assumptions needed on π, as for the model to "remind" the quantum group. As we will see, this is something quite tricky.

The simplest situation is when π is faithful in the usual sense. This is of course something quite restrictive, because the algebra $C(G)$ must be of type I in this case. However, there are many interesting examples here, and all this is worth a detailed look. Following [9], let us introduce the following notion, which is related to faithfulness:

Definition 15.2. *A matrix model $\pi : C(G) \to M_K(C(T))$ is called stationary when*

$$\int_G = \left(tr \otimes \int_T\right)\pi$$

where $\int_T$ is the integration with respect to a given probability measure on T.

Here the term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this.

As a first result now, which is something which is not exactly trivial, and whose proof requires some functional analysis, the stationarity property implies the faithfulness:

Theorem 15.3. *Assuming that a closed subgroup $G\subset U_N^+$ has a stationary model,*

$$\pi:C(G)\to M_K(C(T))$$

it follows that G must be coamenable, and that the model is faithful. Moreover, π extends into an embedding of von Neumann algebras, as follows,

$$L^\infty(G)\subset M_K(L^\infty(T))$$

which commutes the canonical integration functionals.

Proof. Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi:C(G)\to C(G)_{red}\subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, we have an embedding as follows:

$$C(G)\subset M_K(C(T))$$

Now observe that the GNS construction gives a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. This means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb{C})\,dx$$

In particular, we can see from this that $C(G)\subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of π consists of the identity, and of an inclusion. $\square$

We refer to [26], [36] for more on this. Summarizing, what we have is a slight strengthening of the notion of faithfulness. We will see later that are many interesting examples of such models, while remaining of course in the coamenable and type I setting.

Let us discuss now the general, non-coamenable case, with the aim of finding a weaker notion of faithfulness, which still does the job, of "reminding" the quantum group. The idea comes by looking at the group duals $G=\widehat{\Gamma}$. Consider indeed a general model for the associated algebra, which can be written as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

The point now is that such a representation of the group algebra must come by linearization from a unitary group representation, as follows:

$$\rho:\Gamma\to C(T,U_K)$$

Now observe that when ρ is faithful, the representation π is in general not faithful, for instance because when $T = \{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" Γ, so can be used in order to fully understand Γ.

Summarizing, we have an idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker. This weaker notion is called "inner faithfulness", and the general theory here, from [16], is as follows:

Definition 15.4. *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

(1) *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization as follows:*
$$\pi : C(G) \to C(H) \to M_K(C(T))$$
(2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

These constructions work in fact for any C^*-algebra representation $\pi : C(G) \to B$, but here we will be only interested in the random matrix case, $B = M_K(C(T))$. As a first example, motivated by the above discussion, in the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation, as follows:
$$\rho : \Gamma \to C(T, U_K)$$
Thus the minimal factorization in (1) is obtained by taking the image:
$$\rho : \Gamma \to \Lambda \subset C(T, U_K)$$
Thus, as a conclusion, π is inner faithful precisely when:
$$\Gamma \subset C(T, U_K)$$
Dually now, given a compact Lie group G, and elements $g_1, \ldots, g_K \in G$, we have a diagonal representation $\pi : C(G) \to M_K(\mathbb{C})$, appearing as follows:
$$f \to \begin{pmatrix} f(g_1) & & \\ & \ddots & \\ & & f(g_K) \end{pmatrix}$$
The minimal factorization of this representation π, as in (1) above, is then via the algebra $C(H)$, with H being the following closed subgroup of G:
$$H = \overline{< g_1, \ldots, g_K >}$$
Thus, as a conclusion, π is inner faithful precisely when we have:
$$G = H$$
In general, the existence and uniqueness of the Hopf image comes from dividing $C(G)$ by a suitable ideal. In Tannakian terms, as explained in [16], we have:

Theorem 15.5. *Assuming $G\subset U_N^+$, with fundamental corepresentation $u=(u_{ij})$, the Hopf image of a model $\pi:C(G)\to M_K(C(T))$ comes from the Tannakian category*

$$C_{kl}=Hom(U^{\otimes k},U^{\otimes l})$$

where $U_{ij}=\pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

Proof. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(U^{\otimes k},U^{\otimes l})$$

More generally, we have such inclusions when replacing (G,u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H,v) given by:

$$Hom(v^{\otimes k},v^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. □

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [35], [142], is as follows:

Theorem 15.6. *Given an inner faithful model $\pi:C(G)\to M_K(C(T))$, the Haar integration over G is given by*

$$\int_G=\lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k\int_G^r$$

with the truncations of the integration on the right being given by

$$\int_G^r=(\varphi\circ\pi)^{*r}$$

with $\phi*\psi=(\phi\otimes\psi)\Delta$, and with $\varphi=tr\otimes\int_T$ being the random matrix trace.

Proof. As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in section 1 above. In fact, the above result holds more generally for any model $\pi:C(G)\to B$, with $\varphi\in B^*$ being a faithful trace. With this picture in hand, the Woronowicz construction simply corresponds to the case $\pi=id$, and the result itself is therefore a generalization of Woronowicz's result.

In order to prove now the result, we can proceed as in section 1. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that we have:

$$\int_G'=\int_G$$

It is enough to check this on the coefficients of corepresentations, and if we let $v = u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id \otimes \int_G'\right) v = \left(id \otimes \int_G\right) v$$

We already know, from section 1 above, that the matrix on the right is the orthogonal projection onto $Fix(v)$:

$$\left(id \otimes \int_G\right) v = Proj[Fix(v)]$$

Regarding now the matrix on the left, the trick in [147] applied to the linear form $\varphi\pi$ tells us that this is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi\pi)v$:

$$\left(id \otimes \int_G'\right) v = Proj[1 \in (id \otimes \varphi\pi)v]$$

Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have the following formula:

$$(id \otimes \varphi\pi)v = (id \otimes \varphi)V$$

Thus, we can apply the trick in [147], or rather use the same computation as there, which is only based on the biunitarity condition, and we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 15.5, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. □

Regarding now the law of the main character, we have the following result:

Proposition 15.7. *Assume that a model $\pi : C(G) \to M_K(C(T))$ is inner faithful, let*

$$\mu = law(\chi)$$

and let μ^r be the law of χ with respect to $\int_G^r = (\varphi \circ \pi)^{*r}$, where $\varphi = tr \otimes \int_T$.

(1) *We have the following convergence formula, in moments:*

$$\mu = \lim_{k\to\infty} \frac{1}{k} \sum_{r=0}^{k} \mu^r$$

(2) *The moments of μ^r are the numbers $c_\varepsilon^r = Tr(T_\varepsilon^r)$, where:*

$$(T_\varepsilon)_{i_1\ldots i_p, j_1\ldots j_p} = \left(tr \otimes \int_T\right)(U_{i_1j_1}^{\varepsilon_1} \ldots U_{i_pj_p}^{\varepsilon_p})$$

Proof. These formulae are both elementary, by using the convergence result established in Theorem 15.6, the proof being as follows:

(1) This follows from the limiting formula in Theorem 15.6.

(2) This follows from the definition of T_e, by summing over equal indices, $i_r = j_r$. □

In order to detect the stationary models, we have the following criterion, from [17]:

Theorem 15.8. *For a model $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and the Haar integration on it is:*
$$\psi = \left(tr \otimes \int_T\right)\pi$$

(2) *The linear form $\psi = (tr \otimes \int_T)\pi$ satisfies the idempotent state property:*
$$\psi * \psi = \psi$$

(3) *We have $T_e^2 = T_e$, $\forall p \in \mathbb{N}$, $\forall e \in \{1, *\}^p$, where:*
$$(T_e)_{i_1\ldots i_p, j_1\ldots j_p} = \left(tr \otimes \int_T\right)(U_{i_1j_1}^{e_1} \ldots U_{i_pj_p}^{e_p})$$

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 15.4 above:
$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that (1,2,3) above depend only on the factorized representation:
$$\nu : C(H) \to M_K(C(T))$$

Thus, we can assume in practice that we have $G = H$, which means that we can assume that π is inner faithful. With this assumption made, the integration formula in Theorem 15.6 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:
$$\psi * \psi = \psi$$

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$, and Theorem 15.6 gives $\int_G = \psi$. By using now Theorem 15.3, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:
$$\psi^{*r}(u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}) = (T_e^r)_{i_1\ldots i_p, j_1\ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. $\square$

As a first illustration, we can apply the above criterion to certain models for O_N^*,U_N^*. We first have the following result, coming from the work in [18], [33], [54]:

Proposition 15.9. *We have a matrix model as follows,*

$$C(O_N^*)\to M_2(C(U_N))$$

$$u_{ij}\to\begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}$$

where v is the fundamental corepresentation of $C(U_N)$, as well as a model as follows,

$$C(U_N^*)\to M_2(C(U_N\times U_N))$$

$$u_{ij}\to\begin{pmatrix}0&v_{ij}\\ w_{ij}&0\end{pmatrix}$$

where v,w are the fundamental corepresentations of the two copies of $C(U_N)$.

Proof. It is routine to check that the matrices on the right are indeed biunitaries, and since the first matrix is also self-adjoint, we obtain in this way models as follows:

$$C(O_N^+)\to M_2(C(U_N))$$

$$C(U_N^+)\to M_2(C(U_N\times U_N))$$

Regarding now the half-commutation relations, this comes from something general, regarding the antidiagonal 2×2 matrices. Consider indeed matrices as follows:

$$X_i=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}$$

We have then the following computation:

$$X_iX_jX_k=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&x_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&x_k\\ y_k&0\end{pmatrix}=\begin{pmatrix}0&x_iy_jx_k\\ y_ix_jy_k&0\end{pmatrix}$$

Since this quantity is symmetric in i,k, we obtain $X_iX_jX_k=X_kX_jX_i$. Thus, the antidiagonal 2×2 matrices half-commute, and so our models factorize as claimed. $\square$

We can now formulate our first concrete modelling theorem, as follows:

Theorem 15.10. *The above antidiagonal models, namely*

$$C(O_N^*)\to M_2(C(U_N))$$

$$C(U_N^*)\to M_2(C(U_N\times U_N))$$

are both stationary, and in particular they are faithful.

Proof. Let us first discuss the case of O_N^*. We will use Theorem 15.8 (3). Since the fundamental representation is self-adjoint, the various matrices T_e with $e\in\{1,*\}^p$ are all equal. We denote this common matrix by T_p. We have, by definition:

$$(T_p)_{i_1\ldots i_p,j_1\ldots j_p}=\left(tr\otimes\int_H\right)\left[\begin{pmatrix}0&v_{i_1j_1}\\ \bar{v}_{i_1j_1}&0\end{pmatrix}\ldots\ldots\begin{pmatrix}0&v_{i_pj_p}\\ \bar{v}_{i_pj_p}&0\end{pmatrix}\right]$$

Since when multipliying an odd number of antidiagonal matrices we obtain an atidiagonal matrix, we have $T_p=0$ for p odd. Also, when p is even, we have:

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p}&=\left(tr\otimes\int_H\right)\begin{pmatrix}v_{i_1j_1}\ldots\bar{v}_{i_pj_p}&0\\ 0&\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\end{pmatrix}\\ &=\frac{1}{2}\left(\int_H v_{i_1j_1}\ldots\bar{v}_{i_pj_p}+\int_H\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\right)\\ &=\int_H Re(v_{i_1j_1}\ldots\bar{v}_{i_pj_p})\end{aligned}$$

We have $T_p^2=T_p=0$ when p is odd, so we are left with proving that for p even we have $T_p^2=T_p$. For this purpose, we use the following formula:

$$Re(x)Re(y)=\frac{1}{2}\left(Re(xy)+Re(x\bar{y})\right)$$

By using this identity for each of the terms which appear in the product, and multi-index notations in order to simplify the writing, we obtain:

$$\begin{aligned}&(T_p^2)_{ij}\\ =&\sum_{k_1\ldots k_p}(T_p)_{i_1\ldots i_p,k_1\ldots k_p}(T_p)_{k_1\ldots k_p,j_1\ldots j_p}\\ =&\int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}\ldots\bar{v}_{i_pk_p})Re(w_{k_1j_1}\ldots\bar{w}_{k_pj_p})dvdw\\ =&\frac{1}{2}\int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}w_{k_1j_1}\ldots\bar{v}_{i_pk_p}\bar{w}_{k_pj_p})+Re(v_{i_1k_1}\bar{w}_{k_1j_1}\ldots\bar{v}_{i_pk_p}w_{k_pj_p})dvdw\\ =&\frac{1}{2}\int_H\int_H Re((vw)_{i_1j_1}\ldots(\bar{v}\bar{w})_{i_pj_p})+Re((v\bar{w})_{i_1j_1}\ldots(\bar{v}w)_{i_pj_p})dvdw\end{aligned}$$

Now since $vw\in H$ is uniformly distributed when $v,w\in H$ are uniformly distributed, the quantity on the left integrates up to $(T_p)_{ij}$. Also, since H is conjugation-stable, $\bar{w}\in H$ is uniformly distributed when $w\in H$ is uniformly distributed, so the quantity on the right integrates up to the same quantity, namely $(T_p)_{ij}$. Thus, we have:

$$(T_p^2)_{ij}=\frac{1}{2}\Big((T_p)_{ij}+(T_p)_{ij}\Big)=(T_p)_{ij}$$

Summarizing, we have obtained that for any p, we have $T_p^2 = T_p$. Thus Theorem 15.8 applies, and shows that our model is stationary, as claimed. As for the proof of the stationarity for the model for U_N^*, this is similar. See [9], [18]. □

As a second illustration, regarding H_N^*, K_N^*, we have:

Theorem 15.11. *We have a stationary matrix model as follows,*

$$C(H_N^*) \to M_2(C(K_N))$$

$$u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

where v is the fundamental corepresentation of $C(K_N)$, as well as a stationary model

$$C(K_N^*) \to M_2(C(K_N \times K_N))$$

$$u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}$$

where v, w are the fundamental corepresentations of the two copies of $C(K_N)$.

Proof. This follows by adapting the proof of Proposition 15.9 and Theorem 15.10 above, by adding there the H_N^+, K_N^+ relations. All this is in fact part of a more general phenomenon, concerning half-liberation in general, and we refer here to [18], [53], [54]. □

Let us go back now to the general problem of modelling a given quantum permutation group $G \subset S_N^+$. The "simplest" matrix models that we can use are as follows:

Definition 15.12. *Given a subgroup $G \subset S_N^+$, a random matrix model of type*

$$\pi : C(G) \to M_K(C(T))$$

is called flat when the fibers $P_{ij}^x = \pi(u_{ij})(x)$ all have rank 1.

Observe that we must have $N = K$ in this case. Also, the quantum permutation group $G \subset S_N^+$ to be modelled must be transitive, in order for such a model to exist.

Following [36], let us formulate as well a second definition, which is a bit more general, covering many interesting examples of quantum permutation groups $G \subset S_N^+$ which are not transitive, such as the quasi-transitive ones discussed in the previous section:

Definition 15.13. *Given a subgroup $G \subset S_N^+$, a random matrix model of type*

$$\pi : C(G) \to M_K(C(T))$$

is called quasi-flat when the fibers $P_{ij}^x = \pi(u_{ij})(x)$ all have rank ≤ 1.

Observe that the functions $x \mapsto r_{ij}^x = rank(P_{ij}^x)$ are locally constant over T, so they are constant over the connected components of X. Thus, when T is connected, our assumption is that we have $r_{ij}^x = r_{ij} \in \{0, 1\}$, for any $x \in T$, and any i, j.

As a first result now, regarding the quasi-flat models, we have:

Proposition 15.14. *Assume that we have a quasi-flat model $\pi : C(G) \to M_K(C(T))$, mapping $u_{ij} \to P_{ij}$, and consider the matrix $r_{ij} = rank(P_{ij})$.*
1. *r is bistochastic, with sums K.*
2. *We have $r_{ij} \leq \varepsilon_{ij}$, for any i,j.*
3. *If G is quasi-transitive, with orbits of size K, then $r_{ij} = \varepsilon_{ij}$ for any i,j.*
4. *If π is assumed to be flat, then G must be transitive.*

Proof. These results are all elementary, the proof being as follows:

(1) This is clear from the fact that each $P^x = (P^x_{ij})$ is bistochastic, with sums 1.

(2) This simply comes from $u_{ij} = 0 \implies P_{ij} = 0$.

(3) The matrices $r = (r_{ij})$ and $\varepsilon = (\varepsilon_{ij})$ are both bistochastic, with sums K, and they satisfy $r_{ij} \leq \varepsilon_{ij}$, for any i,j. Thus, these matrices must be equal, as stated.

(4) This is clear, because $rank(P_{ij}) = 1$ implies $u_{ij} \neq 0$, for any i,j. □

In order to construct now universal quasi-flat models, it is convenient to identify the rank one projections in $M_N(\mathbb{C})$ with the elements of the complex projective space $P^{N-1}_{\mathbb{C}}$. We first have the following observation, which goes back to [39]:

Proposition 15.15. *The algebra $C(S_N^+)$ has a universal flat model, given by*

$$\pi_N : C(S_N^+) \to M_N(C(X_N))$$
$$\pi_N(u_{ij}) = [P \to P_{ij}]$$

where X_N is the set of matrices $P \in M_N(P^{N-1}_{\mathbb{C}})$ which are bistochastic with sums 1.

Proof. This is clear from definitions, because any flat model $C(S_N^+) \to M_N(\mathbb{C})$ must map the magic corepresentation $u = (u_{ij})$ into a matrix $P = (P_{ij})$ belonging to X_N. □

Regarding now the general quasi-transitive case, we have here:

Theorem 15.16. *Given a quasi-transitive subgroup $G \subset S_N^+$, with orbits of size K, we have a universal quasi-flat model $\pi : C(G) \to M_K(C(X))$, constructed as follows:*
1. *For $G = \underbrace{S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+}_{M\ terms}$ with $N = KM$, the model space is $X_{N,K} = \underbrace{X_K \times \ldots \times X_K}_{M\ terms}$, and with $u = diag(u^1, \ldots, u^M)$ the modelling map is:*
$$\pi_{N,K}(u^r_{ij}) = [(P^1, \ldots, P^M) \to P^r_{ij}]$$
2. *In general, the model space is the submanifold $X_G \subset X_{N,K}$ obtained via the Tannakian relations defining G.*

Proof. This is standard by using Tannakian duality, as follows:

(1) This follows from Tannakian duality, by using Proposition 15.14 (3), which tells us that the 0 entries of the model must appear exactly where $u = (u_{ij})$ has 0 entries.

(2) Assume that $G\subset S_N^+$ is quasi-transitive, with orbits of size K. We have then an inclusion $G\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$, and in order to construct the universal quasi-flat model for $C(G)$, we need a universal solution to the following factorization problem:

$$\begin{array}{ccc}C(\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms})&\to&M_K(C(X_{N,K}))\\ \\ \downarrow&&\downarrow\\ \\ C(G)&\to&M_K(C(X_G))\end{array}$$

But, the solution to this latter question is given by the following construction, with the Hom-spaces at left being taken as usual in a formal sense:

$$C(X_G)=C(X_{N,K})\Big/\Big(T\in Hom(P^{\otimes k},P^{\otimes l}),\forall k,l\in\mathbb{N},\forall T\in Hom(u^{\otimes k},u^{\otimes l})\Big)$$

With this result in hand, the Gelfand spectrum of the algebra on the left is then an algebraic submanifold $X_G\subset X_{N,K}$, having the desired universality property. $\square$

As an illustration, let us discuss now the classical case. With the convention that we identify the rank one projections in $M_K(\mathbb{C})$ with the corresponding elements of the complex projective space $P^{K-1}_{\mathbb{C}}$, we have the following result, from [36]:

Theorem 15.17. *Given a quasi-transitive group $G\subset S_N$, with orbits having size K, the associated universal quasi-flat model space is $X_G=E_K\times L_{N,K}^G$, where:*

$$E_K=\left\{P_1,\ldots,P_K\in P^{K-1}_{\mathbb{C}}\Big|P_i\perp P_j,\forall i,j\right\}$$

$$L_{N,K}^G=\left\{\sigma_1,\ldots,\sigma_K\in G\Big|\sigma_1(i),\ldots,\sigma_K(i)\text{ distinct},\forall i\in\{1,\ldots,N\}\right\}$$

In addition, assuming that we have $L_{N,K}^G\neq\emptyset$, the universal quasi-flat model is stationary, with respect to the Haar measure on E_K times the discrete measure on $L_{N,K}^G$.

Proof. This result is from [36], the idea being as follows:

(1) Let us call "sparse Latin square" any matrix $L\in M_N(*,1,\ldots,K)$ whose rows and columns consists of a permutation of the numbers $1,\ldots,K$, completed with $*$ entries.

(2) Our claim is that the quasi-flat representations $\pi:C(S_N)\to M_K(\mathbb{C})$ appear as follows, where $P_1,\ldots,P_K\in M_K(\mathbb{C})$ are rank 1 projections, summing up to 1, and where $L\in M_N(*,1,\ldots,K)$ is a sparse Latin square, with the convention $P_*=0$:

$$u_{ij}\to P_{L_{ij}}$$

Indeed, assuming that $\pi:C(S_N)\to M_K(\mathbb{C})$ is quasi-flat, the elements $P_{ij}=\pi(u_{ij})$ are projections of rank ≤ 1, which pairwise commute, and form a magic unitary.

Let $P_1,\ldots,P_K\in M_K(\mathbb C)$ be the rank one projections appearing in the first row of $P=(P_{ij})$. Since these projections form a partition of unity with rank one projections, any rank one projection $Q\in M_K(\mathbb C)$ commuting with all of them satisfies:

$$Q\in\{P_1,\ldots,P_K\}$$

In particular we have $P_{ij}\in\{P_1,\ldots,P_K\}$ for any i,j such that $P_{ij}\neq 0$. Thus we can write $u_{ij}\to P_{L_{ij}}$, for a certain matrix $L\in M_N(*,1,\ldots,K)$, with the convention $P_*=0$.

In order to finish, the remark is that $u_{ij}\to P_{L_{ij}}$ defines a representation $\pi:C(S_N)\to M_K(\mathbb C)$ precisely when the matrix $P=(P_{L_{ij}})_{ij}$ is magic. But this condition tells us precisely that L must be a sparse Latin square, as desired.

(3) In order to finish, we must compute the Hopf image. Given a sparse Latin square $L\in M_N(*,1,\ldots,K)$, consider the permutations $\sigma_1,\ldots,\sigma_K\in S_N$ given by:

$$\sigma_x(j)=i\iff L_{ij}=x$$

Our claim is that the Hopf image associated to a representation $\pi:C(S_N)\to M_K(\mathbb C)$, $u_{ij}\to P_{L_{ij}}$ as above is then the algebra $C(G_L)$, where:

$$G_L=<\sigma_1,\ldots,\sigma_K>\subset S_N$$

Indeed, the image of π being generated by $P_1,\ldots,P_K$, we have an isomorphism of algebras $\alpha:Im(\pi)\simeq C(1,\ldots,K)$ given by $P_i\to\delta_i$. Consider the following diagram:

$$\begin{tikzcd}
C(S_N) \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Im(\pi) \arrow[r] \arrow[d, "\alpha"] & M_K(\mathbb C) \\
 & C(1,\ldots,K) &
\end{tikzcd}$$

Here the map on the right is the canonical inclusion and $\varphi=\alpha\pi$. Since the Hopf image of π coincides with the one of φ, it is enough to compute the latter. We know that φ is given by $\varphi(u_{ij})=\delta_{L_{ij}}$, with the convention $\delta_*=0$. By Gelfand duality, φ must come from a certain map $\sigma:\{1,\ldots,K\}\to S_N$, via the transposition formula:

$$\varphi(f)(x)=f(\sigma_x)$$

With the choice $f=u_{ij}$, we obtain $\delta_{L_{ij}}(x)=u_{ij}(\sigma_x)$. Now observe that:

$$\delta_{L_{ij}}(x)=\begin{cases}1&\text{if }L_{ij}=x\\ 0&\text{otherwise}\end{cases}$$

We have as well the following formula:

$$u_{ij}(\sigma_x)=\begin{cases}1&\text{if }\sigma_x(j)=i\\ 0&\text{otherwise}\end{cases}$$

We conclude that σ_x is the permutation in the statement. Summarizing, we have shown that φ comes by transposing the map $x \to \sigma_x$, with σ_x being as in the statement. Thus the Hopf image of φ is the algebra $C(G_L)$, with:

$$G_L =< \sigma_1, \ldots, \sigma_K >$$

Thus, we are led to the conclusion in the statement. □

There are many other explicit computations in the quasi-flat case, especially in the group dual case, and we refer here to [25], [36].

Following [25], [39] and related papers, we have the following result:

Proposition 15.18. *Assuming that a model $\pi : C(G) \to M_K(C(X))$ is inner faithful and quasi-flat, mapping $u_{ij} \to Proj(\xi_{ij}^x)$, with $||\xi_{ij}^x|| \in \{0, 1\}$, we have*

$$T_p = \int_X T_p(\xi^x) dx$$

where the matrix $T_p(\xi) \in M_{N^p}(\mathbb{C})$, associated to an array $\xi \in M_N(\mathbb{C}^K)$ is given by

$$T_p(\xi)_{i_1 \ldots i_p, j_1 \ldots j_p} = \frac{1}{K} < \xi_{i_1 j_1}, \xi_{i_2 j_2} >< \xi_{i_2 j_2}, \xi_{i_3 j_3} > \ldots \ldots < \xi_{i_p j_p}, \xi_{i_1 j_1} >$$

with the scalar product being the usual one on $\mathbb{C}^K$, taken linear at right.

Proof. We have the following well-known computation, valid for any vectors $\xi_1, \ldots, \xi_p$ having norms $||\xi_i|| \in \{0, 1\}$, with the scalar product being linear at right:

$$\begin{aligned}
& Proj(\xi_i)x =< \xi_i, x > \xi_i, \forall i \\
\implies\ & Proj(\xi_1) \ldots Proj(\xi_p)(x) =< \xi_1, \xi_2 > \ldots \ldots < \xi_{p-1}, \xi_p >< \xi_p, x > \xi_1 \\
\implies\ & Tr(Proj(\xi_1) \ldots Proj(\xi_p)) =< \xi_1, \xi_2 > \ldots \ldots < \xi_{p-1}, \xi_p >< \xi_p, \xi_1 >
\end{aligned}$$

Thus, the matrices T_p can be computed as follows:

$$\begin{aligned}
(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} &= \int_X tr\left(Proj(\xi_{i_1 j_1}^x) Proj(\xi_{i_2 j_2}^x) \ldots Proj(\xi_{i_p j_p}^x) \right) dx \\
&= \frac{1}{K} \int_X < \xi_{i_1 j_1}^x, \xi_{i_2 j_2}^x >< \xi_{i_2 j_2}^x, \xi_{i_3 j_3}^x > \ldots \ldots < \xi_{i_p j_p}^x, \xi_{i_1 j_1}^x > dx \\
&= \int_X (T_p(\xi^x))_{i_1 \ldots i_p, j_1 \ldots j_p} dx
\end{aligned}$$

We therefore obtain the formula in the statement. See [25], [39]. □

An even more conceptual result, from [17], [25], [39], is as follows:

Theorem 15.19. *Given an inner faithful quasi-flat model*

$$\pi : C(G) \to M_K(C(X))$$
$$u_{ij} \to Proj(\xi_{ij}^x)$$

with $||\xi_{ij}^x|| \in \{0,1\}$*, the law of the normalized character* χ/K *with respect to the truncated integral* $\int_G^r$ *coincides with that of the Gram matrix of the vectors*

$$\xi_{i_1\ldots i_r}^x = \frac{1}{\sqrt{K}} \cdot \xi_{i_1i_2}^{x_1} \otimes \xi_{i_2i_3}^{x_2} \otimes \ldots \otimes \xi_{i_ri_1}^{x_r}$$

with respect to the normalized matrix trace, and to the integration functional on X^r*.*

Proof. The moments C_p of the measure that we are interested in are given by:

$$\begin{aligned} C_p &= \frac{1}{K^p}\int_G^r\left(\sum_i u_{ii}\right)^p \\ &= \frac{1}{K^p}\sum_{i_1\ldots i_p}(T_p^r)_{i_1\ldots i_p,i_1\ldots i_p} \\ &= \frac{1}{K^p}\cdot Tr(T_p^r) \end{aligned}$$

The trace on the right is given by the following formula:

$$Tr(T_p^r) = \sum_{i_1^1\ldots i_p^r}(T_p)_{i_1^1\ldots i_p^1,i_1^2\ldots i_p^2}\ldots\ldots(T_p)_{i_1^r\ldots i_p^r,i_1^1\ldots i_p^1}$$

In view of the formula in Proposition 15.18, this quantity will expand in terms of the matrices $T_p(\xi)$ constructed there. To be more precise, we have:

$$Tr(T_p^r) = \int_{X^r}\sum_{i_1^1\ldots i_p^r}T_p(\xi^{x_1})_{i_1^1\ldots i_p^1,i_1^2\ldots i_p^2}\ldots\ldots T_p(\xi^{x_r})_{i_1^r\ldots i_p^r,i_1^1\ldots i_p^1}\,dx$$

By using now the explicit formula of each $T_p(\xi)$, from Proposition 15.18, we have:

$$\begin{aligned} Tr(T_p^r) &= \frac{1}{K^r}\int_{X^r}\sum_{i_1^1\ldots i_p^r} < \xi_{i_1^1i_1^2}^{x_1},\xi_{i_2^1i_2^2}^{x_1} > \ldots\ldots < \xi_{i_p^1i_p^2}^{x_1},\xi_{i_1^1i_1^2}^{x_1} > \\ &\qquad\qquad\qquad\qquad\qquad\ldots \\ &\qquad < \xi_{i_1^ri_1^1}^{x_r},\xi_{i_2^ri_2^1}^{x_r} > \ldots\ldots < \xi_{i_p^ri_p^1}^{x_r},\xi_{i_1^ri_1^1}^{x_r} > dx \end{aligned}$$

By changing the order of the summation, we can write this formula as:

$$\begin{aligned} Tr(T_p^r) &= \frac{1}{K^r}\int_{X^r}\sum_{i_1^1\ldots i_p^r} < \xi_{i_1^1i_1^2}^{x_1},\xi_{i_2^1i_2^2}^{x_1} > \ldots\ldots < \xi_{i_1^ri_1^1}^{x_r},\xi_{i_2^ri_2^1}^{x_r} > \\ &\qquad\qquad\qquad\qquad\qquad\ldots \\ &\qquad < \xi_{i_p^1i_p^2}^{x_1},\xi_{i_1^1i_1^2}^{x_1} > \ldots\ldots < \xi_{i_p^ri_p^1}^{x_r},\xi_{i_1^ri_1^1}^{x_r} > dx \end{aligned}$$

But this latter formula can be written as follows:

$$\begin{aligned} Tr(T_p^r) &= K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}\frac{1}{K}<\xi^{x_1}_{i_1^1i_1^2}\otimes\ldots\otimes\xi^{x_r}_{i_1^ri_1^1}\ ,\ \xi^{x_1}_{i_2^1i_2^2}\otimes\ldots\otimes\xi^{x_r}_{i_2^ri_2^1}> \\ &\qquad\qquad\ldots \\ &\qquad\qquad\frac{1}{K}<\xi^{x_1}_{i_p^1i_p^2}\otimes\ldots\otimes\xi^{x_r}_{i_p^ri_p^1}\ ,\ \xi^{x_1}_{i_1^1i_1^2}\otimes\ldots\otimes\xi^{x_r}_{i_1^ri_1^1}>dx \end{aligned}$$

In terms of the vectors in the statement, and of their Gram matrix G_r^x, we obtain:

$$\begin{aligned} Tr(T_p^r) &= K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}<\xi^x_{i_1^1\ldots i_1^r},\xi^x_{i_2^1\ldots i_2^r}>\ldots\ldots<\xi^x_{i_p^1\ldots i_p^r},\xi^x_{i_1^1\ldots i_1^r}>dx \\ &= K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}(G_r^x)_{i_1^1\ldots i_1^r,i_2^1\ldots i_2^r}\ldots\ldots(G_r^x)_{i_p^1\ldots i_p^r,i_1^1\ldots i_1^r}\,dx \\ &= K^{p-r}\int_{X^r}Tr((G_r^x)^p)dx \end{aligned}$$

Summarizing, the moments of the measure in the statement are given by:

$$\begin{aligned} C_p &= \frac{1}{K^r}\int_{X^r}Tr((G_r^x)^p)dx \\ &= \left(tr\otimes\int_{X^r}\right)(G_r^p) \end{aligned}$$

This gives the formula in the statement of the theorem. □

Following [39], let us study now the universal flat model for $C(S_N^+)$. Given a flat magic unitary, we can write it, in a non-unique way, as $u_{ij}=Proj(\xi_{ij})$. The array $\xi=(\xi_{ij})$ is then a "magic basis", in the sense that each of its rows and columns is an orthonormal basis of $\mathbb{C}^N$. We are therefore led to two spaces, as follows:

Definition 15.20. *Associated to any $N\in\mathbb{N}$ are the following spaces:*

1. *X_N, the space of all $N\times N$ flat magic unitaries $u=(u_{ij})$.*
2. *K_N, the space of all $N\times N$ magic bases $\xi=(\xi_{ij})$.*

Let us recall now that the rank 1 projections $p\in M_N(\mathbb{C})$ can be identified with the corresponding 1-dimensional subspaces $E\subset\mathbb{C}^N$, which are by definition the elements of the complex projective space $P^{N-1}_\mathbb{C}$. In addition, if we consider the complex sphere, $S^{N-1}_\mathbb{C}=\{z\in\mathbb{C}^N|\sum_i|z_i|^2=1\}$, we have a quotient map $\pi:S^{N-1}_\mathbb{C}\to P^{N-1}_\mathbb{C}$ given by $z\to Proj(z)$. Observe that $\pi(z)=\pi(z')$ precisely when $z'=wz$, for some $w\in\mathbb{T}$.

Consider as well the embedding $U_N\subset(S^{N-1}_\mathbb{C})^N$ given by $x\to(x_1,\ldots,x_N)$, where $x_1,\ldots,x_N$ are the rows of x. Finally, let us call an abstract matrix stochastic/bistochastic when the entries on each row/each row and column sum up to 1.

With these notations, the abstract model spaces X_N, K_N that we are interested in, and some related spaces, are as follows:

Proposition 15.21. *We have inclusions and surjections as follows,*

$$\begin{array}{ccccc}K_N&\subset&U_N^N&\subset&M_N(S^{N-1}_\mathbb{C})\\ \\ \downarrow&&\downarrow&&\downarrow\\ \\ X_N&\subset&Y_N&\subset&M_N(P^{N-1}_\mathbb{C})\end{array}$$

where X_N, Y_N consist of bistochastic/stochastic matrices, and K_N is the lift of X_N.

Proof. This follows from the above discussion. Indeed, the quotient map $S^{N-1}_\mathbb{C}\to P^{N-1}_\mathbb{C}$ induces the quotient map $M_N(S^{N-1}_\mathbb{C})\to M_N(P^{N-1}_\mathbb{C})$ at right, and the lift of the space of stochastic matrices $Y_N\subset M_N(P^{N-1}_\mathbb{C})$ is then the rescaled group U_N^N, as claimed. □

In order to get some insight into the structure of X_N, K_N, we can use inspiration from the Sinkhorn algorithm. This algorithm starts with a $N\times N$ matrix having positive entries and produces, via successive averagings over rows/columns, a bistochastic matrix.

In our situation, we would like to have an "averaging" map $Y_N\to Y_N$, whose infinite iteration lands in the model space X_N. Equivalently, we would like to have an "averaging" map $U_N^N\to U_N^N$, whose infinite iteration lands in K_N.

In order to construct such averaging maps, we use the orthogonalization procedure coming from the polar decomposition. First, we have the following result:

Proposition 15.22. *We have orthogonalization maps as follows,*

$$\begin{array}{ccc}(S^{N-1}_\mathbb{C})^N&\xrightarrow{\alpha}&(S^{N-1}_\mathbb{C})^N\\ \downarrow&&\downarrow\\ (P^{N-1}_\mathbb{C})^N&\xrightarrow{\beta}&(P^{N-1}_\mathbb{C})^N\end{array}$$

where $\alpha(x)_i=Pol([(x_i)_j]_{ij})$, and $\beta(p)=(P^{-1/2}p_iP^{-1/2})_i$, with $P=\sum_ip_i$.

Proof. Our first claim is that we have a factorization as in the statement. Indeed, pick $p_1,\ldots,p_N\in P^{N-1}_\mathbb{C}$, and write $p_i=Proj(x_i)$, with $||x_i||=1$. We can then apply α, as to obtain a vector $\alpha(x)=(x_i')_i$, and then set $\beta(p)=(p_i')$, where $p_i'=Proj(x_i')$.

Our first task is to prove that β is well-defined. Consider indeed vectors $\tilde{x}_i$, satisfying $Proj(\tilde{x}_i)=Proj(x_i)$. We have then $\widetilde{x_i}=\lambda_ix_i$, for certain scalars $\lambda_i\in\mathbb{T}$, and so the matrix formed by these vectors is $\widetilde{M}=\Lambda M$, with $\Lambda=diag(\lambda_i)$. It follows that $Pol(\widetilde{M})=\Lambda Pol(M)$, and so $\tilde{x}_i'=\lambda_ix_i$, and finally $Proj(\tilde{x}_i')=Proj(x_i')$, as desired.

It remains to prove that β is given by the formula in the statement. For this purpose, observe first that, given $x_1,\ldots,x_N\in S^{N-1}_\mathbb{C}$, with $p_i=Proj(x_i)$ we have:

$$\begin{aligned}\sum_i p_i &= \sum_i[(\bar{x}_i)_k(x_i)_l]_{kl}\\ &= \sum_i(\bar{M}_{ik}M_{il})_{kl}\\ &= ((M^*M)_{kl})_{kl}\\ &= M^*M\end{aligned}$$

We can now compute the projections $p_i'=Proj(x_i')$. Indeed, the coefficients of these projections are given by $(p_i')_{kl}=\bar{U}_{ik}U_{il}$ with $U=MP^{-1/2}$, and we obtain, as desired:

$$\begin{aligned}(p_i')_{kl} &= \sum_{ab}\bar{M}_{ia}P_{ak}^{-1/2}M_{ib}P_{bl}^{-1/2}\\ &= \sum_{ab}P_{ka}^{-1/2}\bar{M}_{ia}M_{ib}P_{bl}^{-1/2}\\ &= \sum_{ab}P_{ka}^{-1/2}(p_i)_{ab}P_{bl}^{-1/2}\\ &= (P^{-1/2}p_iP^{-1/2})_{kl}\end{aligned}$$

An alternative proof uses the fact that the elements $p_i'=P^{-1/2}p_iP^{-1/2}$ are self-adjoint, and sum up to 1. The fact that these elements are indeed idempotents can be checked directly, via $p_iP^{-1}p_i=p_i$, because this equality holds on $\ker p_i$, and also on x_i. □

As an illustration, here is how the orthogonalization works at $N=2$:

Proposition 15.23. *At $N=2$ the orthogonalization procedure for $(Proj(x),Proj(y))$ amounts in considering the vectors $(x\pm y)/\sqrt{2}$, and then rotating by 45°.*

Proof. By performing a rotation, we can restrict attention to the case $x=(\cos t,\sin t)$ and $y=(\cos t,-\sin t)$, with $t\in(0,\pi/2)$. Here the computations are as follows:

$$\begin{aligned}M=\begin{pmatrix}\cos t&\sin t\\ \cos t&-\sin t\end{pmatrix} &\implies P=M^*M=\begin{pmatrix}2\cos^2 t&0\\ 0&2\sin^2 t\end{pmatrix}\\ &\implies P^{-1/2}=|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\cos t}&0\\ 0&\frac{1}{\sin t}\end{pmatrix}\\ &\implies U=M|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\ 1&-1\end{pmatrix}\end{aligned}$$

Thus the orthogonalization procedure replaces $(Proj(x),Proj(y))$ by the orthogonal projections on the vectors $(\frac{1}{\sqrt{2}}(1,1),\frac{1}{\sqrt{2}}(-1,1))$, and this gives the result. □

With these preliminaries in hand, let us discuss now the version that we need of the Sinkhorn algorithm. The orthogonalization procedure is as follows:

Theorem 15.24. *The orthogonalization maps α,β induce maps as follows,*

$$\begin{array}{ccc} U_N^N & \xrightarrow{\Phi} & U_N^N \\ \downarrow & & \downarrow \\ Y_N & \xrightarrow{\Psi} & Y_N \end{array}$$

which are the transposition maps on K_N,X_N, and which are projections at $N=2$.

Proof. It follows from definitions that $\Phi(x)$ is obtained by putting the components of $x=(x_i)$ in a row, then picking the j-th column vectors of each x_i, calling M_j this matrix, then taking the polar part $x_j'=Pol(M_j)$, and finally setting $\Phi(x)=x'$. Thus:

$$\Phi(x)=Pol((x_{ij})_i)_j\quad,\quad\Psi(u)=(P_i^{-1/2}u_{ji}P_i^{-1/2})_{ij}$$

Thus, the first assertion is clear, and the second assertion is clear too. □

Our claim is that the algorithm converges, as follows:

Conjecture 15.25. *The above maps Φ,Ψ increase the volume,*

$$vol:U_N^N\to Y_N\to[0,1],\quad vol(u)=\prod_j|\det((u_{ij})_i)|$$

and respectively land, after an infinite number of steps, in K_N/X_N.

As a main application of the above conjecture, the infinite iteration $(\Phi^2)^\infty:U_N^N\to K_N$ would provide us with an integration on K_N, and hence on the quotient space $K_N\to X_N$ as well, by taking the push-forward measures, coming from the Haar measure on U_N^N. In relation now with the matrix model problematics, we have:

Conjecture 15.26. *The universal $N\times N$ flat matrix representation*

$$\pi_N:C(S_N^+)\to M_N(C(X_N)),\quad \pi_N(w_{ij})=(u\to u_{ij})$$

is faithful at $N=4$, and is inner faithful at any $N\geq 5$.

We refer to [39] and related papers for further details regarding this conjecture, and also for other applications of the Sinkhorn algorithm philosophy to modelling questions for the quantum permutation groups.

16. Weyl and Fourier

Following [28], [39], let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

Definition 16.1. *Given a finite abelian group H, the associated Weyl matrices are*

$$W_{ia}:e_b\to<i,b>e_{a+b}$$

where $i\in H$, $a,b\in\widehat{H}$, and where $(i,b)\to<i,b>$ is the Fourier coupling $H\times\widehat{H}\to\mathbb{T}$.

As a basic example, consider the cyclic group $H=\mathbb{Z}_2=\{0,1\}$. Here the Fourier coupling is given by $<i,b>=(-1)^{ib}$, and so the Weyl matrices act via:

$$W_{00}:e_b\to e_b\quad,\quad W_{10}:e_b\to(-1)^be_b$$

$$W_{11}:e_b\to(-1)^be_{b+1}\quad,\quad W_{01}:e_b\to e_{b+1}$$

Thus, we have the following formulae:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad,\quad W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad,\quad W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices. Now back to the general case, we have the following well-known result:

Proposition 16.2. *The Weyl matrices are unitaries, and satisfy:*

1. $W_{ia}^*=<i,a>W_{-i,-a}$.
2. $W_{ia}W_{jb}=<i,b>W_{i+j,a+b}$.
3. $W_{ia}W_{jb}^*=<j-i,b>W_{i-j,a-b}$.
4. $W_{ia}^*W_{jb}=<i,a-b>W_{j-i,b-a}$.

Proof. The unitarity follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned}W_{ia}^*&=\left(\sum_b<i,b>E_{a+b,b}\right)^*\\&=\sum_b<-i,b>E_{b,a+b}\\&=\sum_b<-i,b-a>E_{b-a,b}\\&=<i,a>W_{-i,-a}\end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned}
W_{ia}W_{jb} &= \left(\sum_d < i,b+d > E_{a+b+d,b+d}\right)\left(\sum_d < j,d > E_{b+d,d}\right)\\
&= \sum_d < i,b >< i+j,d > E_{a+b+d,d}\\
&= < i,b > W_{i+j,a+b}
\end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned}
W_{ia}W_{jb}^* &= < j,b > W_{ia}W_{-j,-b}\\
&= < j,b >< i,-b > W_{i-j,a-b}
\end{aligned}$$

We obtain as well the following formula:

$$\begin{aligned}
W_{ia}^*W_{jb} &= < i,a > W_{-i,-a}W_{jb}\\
&= < i,a >< -i,b > W_{j-i,b-a}
\end{aligned}$$

But this gives the formulae in the statement, and we are done. □

With $n=|H|$, we can use an isomorphism $l^2(\widehat{H})\simeq\mathbb{C}^n$ as to view each W_{ia} as a usual matrix, $W_{ia}\in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia}\in U_n$. Also, given a vector ξ, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$. Following [39], we have:

Proposition 16.3. *Given a closed subgroup $E\subset U_n$, we have a representation*

$$\pi_H:C(S_N^+)\to M_N(C(E))$$

$$w_{ia,jb}\to[U\to Proj(W_{ia}UW_{jb}^*)]$$

where $n=|H|,N=n^2$, and where W_{ia} are the Weyl matrices associated to H.

Proof. The Weyl matrices being given by $W_{ia}:e_b\to< i,b > e_{a+b}$, we have:

$$tr(W_{ia})=\begin{cases}1&\text{if }(i,a)=(0,0)\\0&\text{if }(i,a)\neq(0,0)\end{cases}$$

Together with the formulae in Proposition 16.2, this shows that the Weyl matrices are pairwise orthogonal with respect to the following scalar product on $M_n(\mathbb{C})$:

$$< x,y >= tr(x^*y)$$

Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W=\left\{W_{ia}\Big|i\in H,a\in\widehat{H}\right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb}=W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. □

We will need the following well-known result:

Proposition 16.4. *With $T=Proj(x_1)\ldots Proj(x_p)$ and $||x_i||=1$ we have*
$$<\xi,T\eta>=<\xi,x_1><x_1,x_2>\ldots<x_{p-1},x_p><x_p,\eta>$$
for any ξ,η, with the scalar product being linear at right. In particular, we have:
$$Tr(T)=<x_1,x_2><x_2,x_3>\ldots<x_p,x_1>$$

Proof. We have indeed the following computation:
$$\begin{aligned}
T\eta &= Proj(x_1)\ldots Proj(x_p)\eta\\
&= Proj(x_1)\ldots Proj(x_{p-1})x_p<x_p,\eta>\\
&= Proj(x_1)\ldots Proj(x_{p-2})x_{p-2}<x_{p-1},x_p><x_p,\eta>\\
&\ \vdots\\
&= x_1<x_1,x_2>\ldots<x_{p-1},x_p><x_p,\eta>
\end{aligned}$$

Now by taking the scalar product with ξ, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi=\eta=e_i$. □

Now back to the Weyl matrix models, let us first compute T_p. We have:

Proposition 16.5. *We have the formula*
$$\begin{aligned}
&(T_p)_{ia,jb}\\
=\ &\frac{1}{N}<i_1,a_1-a_2>\ldots<i_p,a_p-a_1><j_2,b_2-b_1>\ldots<j_1,b_1-b_p>\\
&\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{j_1-j_2,b_1-b_2}U^*)\ldots tr(W_{i_1-i_p,a_1-a_p}UW_{j_p-j_1,b_p-b_1}U^*)dU
\end{aligned}$$
with all the indices varying in a cyclic way.

Proof. By using the trace formula in Proposition 16.4 above, we obtain:
$$\begin{aligned}
&(T_p)_{ia,jb}\\
=\ &\left(tr\otimes\int_E\right)\left(Proj(W_{i_1a_1}UW_{j_1b_1}^*)\ldots Proj(W_{i_pa_p}UW_{j_pb_p}^*)\right)\\
=\ &\frac{1}{N}\int_E<W_{i_1a_1}UW_{j_1b_1}^*,W_{i_2a_2}UW_{j_2b_2}^*>\ldots<W_{i_pa_p}UW_{j_pb_p}^*,W_{i_1a_1}UW_{j_1b_1}^*>dU
\end{aligned}$$

In order to compute now the scalar products, observe that we have:
$$\begin{aligned}
&<W_{ia}UW_{jb}^*,W_{kc}UW_{ld}^*>\\
=\ &tr(W_{jb}U^*W_{ia}^*W_{kc}UW_{ld}^*)\\
=\ &tr(W_{ia}^*W_{kc}UW_{ld}^*W_{jb}U^*)\\
=\ &<i,a-c><l,d-b>tr(W_{k-i,c-a}UW_{j-l,b-d}U^*)
\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. □

Consider now the Weyl group $W = \{W_{ia}\} \subset U_n$, that we already met in the proof of Proposition 16.3 above. We have the following result, from [39]:

Theorem 16.6. *For any compact group $W \subset E \subset U_n$, the model*
$$\pi_H : C(S_N^+) \to M_N(C(E))$$
$$w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$
constructed above is stationary on its image.

Proof. We must prove that we have $T_p^2 = T_p$. We have:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
= &\sum_{kc}(T_p)_{ia,kc}(T_p)_{kc,jb}\\
= &\frac{1}{N^2}\sum_{kc} < i_1, a_1 - a_2 > \ldots < i_p, a_p - a_1 >< k_2, c_2 - c_1 > \ldots < k_1, c_1 - c_p >\\
&< k_1, c_1 - c_2 > \ldots < k_p, c_p - c_1 >< j_2, b_2 - b_1 > \ldots < j_1, b_1 - b_p >\\
&\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2}U^*) \ldots tr(W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1}U^*)dU\\
&\int_E tr(W_{k_2-k_1,c_2-c_1}VW_{j_1-j_2,b_1-b_2}V^*) \ldots tr(W_{k_1-k_p,c_1-c_p}VW_{j_p-j_1,b_p-b_1}V^*)dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
= &\frac{1}{N^2} < i_1, a_1 - a_2 > \ldots < i_p, a_p - a_1 >< j_2, b_2 - b_1 > \ldots < j_1, b_1 - b_p >\\
&\int_E\int_E\sum_{kc} < k_1 - k_2, c_1 - c_2 > \ldots < k_p - k_1, c_p - c_1 >\\
&tr(W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2}U^*)tr(W_{k_2-k_1,c_2-c_1}VW_{j_1-j_2,b_1-b_2}V^*)\\
&\ldots\ldots\\
&tr(W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1}U^*)tr(W_{k_1-k_p,c_1-c_p}VW_{j_p-j_1,b_p-b_1}V^*)dUdV
\end{aligned}$$

Let us denote by I the above double integral. By using $W_{kc}^* =< k, c > W_{-k,-c}$ for each of the couplings, and by moving as well all the U^* variables to the left, we obtain:

$$\begin{aligned}
&I\\
= &\int_E\int_E\sum_{kc} tr(U^*W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2})tr(W_{k_1-k_2,c_1-c_2}^*VW_{j_1-j_2,b_1-b_2}V^*)\\
&\ldots\ldots\\
&tr(U^*W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1})tr(W_{k_p-k_1,c_p-c_1}^*VW_{j_p-j_1,b_p-b_1}V^*)dUdV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}
tr(AW_{kc})tr(W_{kc}^*B) &= \frac{1}{N}\sum_{qrst} A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st}B_{ts}\\
&= \frac{1}{N}\sum_{qrst} A_{qr} < k,q > \delta_{r-q,c} < k,-s > \delta_{t-s,c}B_{ts}\\
&= \frac{1}{N}\sum_{qs} < k,q-s > A_{q,q+c}B_{s+c,s}
\end{aligned}$$

If we denote by A_x, B_x the variables which appear in the formula of I, we have:

$$\begin{aligned}
& I\\
= & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1-k_2, q_1-s_1 > \ldots < k_p-k_1, q_p-s_p >\\
& (A_1)_{q_1,q_1+c_1-c_2}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{q_p,q_p+c_p-c_1}(B_p)_{s_p+c_p-c_1,s_p}\\
= & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1, q_1-s_1-q_p+s_p > \ldots < k_p, q_p-s_p-q_{p-1}+s_{p-1} >\\
& (A_1)_{q_1,q_1+c_1-c_2}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{q_p,q_p+c_p-c_1}(B_p)_{s_p+c_p-c_1,s_p}
\end{aligned}$$

Now observe that we can perform the sums over $k_1,\ldots,k_p$. We obtain in this way a multiplicative factor n^p, along with the condition $q_1-s_1=\ldots=q_p-s_p$. Thus we must have $q_x=s_x+a$ for a certain a, and the above formula becomes:

$$I=\frac{1}{n^p}\int_E\int_E\sum_{csa}(A_1)_{s_1+a,s_1+c_1-c_2+a}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{s_p+a,s_p+c_p-c_1+a}(B_p)_{s_p+c_p-c_1,s_p}$$

Consider now the variables $r_x=c_x-c_{x+1}$, which altogether range over the set Z of multi-indices having sum 0. By replacing the sum over c_x with the sum over r_x, which creates a multiplicative n factor, we obtain the following formula:

$$I=\frac{1}{n^{p-1}}\int_E\int_E\sum_{r\in Z}\sum_{sa}(A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\ldots(A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}$$

Since for an arbitrary multi-index r we have $\delta_{\sum_i r_i,0}=\frac{1}{n}\sum_i < i,r_1 > \ldots < i,r_p >$, we can replace the sum over $r\in Z$ by a full sum, as follows:

$$\begin{aligned}
I &= \frac{1}{n^p}\int_E\int_E\sum_{rsia} < i,r_1 > (A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\\
&\qquad\qquad\ldots\ldots\\
&\qquad\qquad < i,r_p > (A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}
\end{aligned}$$

In order to "absorb" now the indices i,a, we can use the following formula:

$$\begin{aligned}
&W_{ia}^*AW_{ia}\\
&= \left(\sum_b < i,-b > E_{b,a+b}\right)\left(\sum_{bc}E_{a+b,a+c}A_{a+b,a+c}\right)\left(\sum_c < i,c > E_{a+c,c}\right)\\
&= \sum_{bc} < i,c-b > E_{bc}A_{a+b,a+c}
\end{aligned}$$

Thus we have $(W_{ia}^*AW_{ia})_{bc} = < i,c-b > A_{a+b,a+c}$, and our formula becomes:

$$\begin{aligned}
&I\\
&= \frac{1}{n^p}\int_E\int_E\sum_{rsia}(W_{ia}^*A_1W_{ia})_{s_1,s_1+r_1}(B_1)_{s_1+r_1,s_1}\ldots(W_{ia}^*A_pW_{ia})_{s_p,s_p+r_p}(B_p)_{s_p+r_p,s_p}\\
&= \int_E\int_E\sum_{ia}tr(W_{ia}^*A_1W_{ia}B_1)\ldots\ldots tr(W_{ia}^*A_pW_{ia}B_p)
\end{aligned}$$

Now by replacing A_x,B_x with their respective values, we obtain:

$$\begin{aligned}
&I\\
&= \int_E\int_E\sum_{ia}tr(W_{ia}^*U^*W_{i_2-i_1,a_2-a_1}UW_{ia}VW_{j_1-j_2,b_1-b_2}V^*)\\
&\qquad\ldots\ldots\\
&\qquad tr(W_{ia}^*U^*W_{i_1-i_p,a_1-a_p}UW_{ia}VW_{j_p-j_1,b_p-b_1}V^*)dUdV
\end{aligned}$$

By moving the $W_{ia}^*U^*$ variables at right, we obtain, with $S_{ia}=UW_{ia}V$:

$$\begin{aligned}
I &= \sum_{ia}\int_E\int_E tr(W_{i_2-i_1,a_2-a_1}S_{ia}W_{j_1-j_2,b_1-b_2}S_{ia}^*)\\
&\qquad\ldots\ldots\\
&\qquad tr(W_{i_1-i_p,a_1-a_p}S_{ia}W_{j_p-j_1,b_p-b_1}S_{ia}^*)dUdV
\end{aligned}$$

Now since S_{ia} is Haar distributed when U,V are Haar distributed, we obtain:

$$I = N\int_E\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{j_1-j_2,b_1-b_2}U^*)\ldots tr(W_{i_1-i_p,a_1-a_p}UW_{j_p-j_1,b_p-b_1}U^*)dU$$

But this is exactly N times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 16.5 above.

Since the N factor cancels with one of the two N factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. $\square$

As an illustration for the above result, going back to [28], we have:

Theorem 16.7. *We have a stationary matrix model*

$$\pi : C(S_4^+) \subset M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

Proof. As already explained in the comments following Definition 16.1, the Pauli matrices appear as particular cases of the Weyl matrices. By working out the details, we conclude that Theorem 16.6 produces in this case the model in the statement. $\square$

Observe that, since $Proj(c_i x c_j)$ depends only on the image of x in the quotient $SU_2 \to SO_3$, we can replace the model space SU_2 by the smaller space SO_3. This is something that can be used in conjunction with the isomorphism $S_4^+ \simeq SO_3^{-1}$, and as explained in [15], our model becomes in this way something more conceptual, as follows:

$$\pi : C(SO_3^{-1}) \subset M_4(C(SO_3))$$

As a somewhat philosophical conclusion, to this and to some previous findings as well, no matter what we do, we always end up getting back to SU_2, SO_3. Thus, we are probably doing some physics here. This is indeed the case, the above computations being closely related to the standard computations for the Ising and Potts models. The general relation, however, between quantum permutations and lattice models, is not axiomatixed yet.

Let us discuss now some generalizations of the Weyl matrix models. We will need:

Definition 16.8. *A 2-cocycle on a group G is a function $\sigma : G \times G \to \mathbb{T}$ satisfying:*

$$\sigma(gh, k)\sigma(g, h) = \sigma(g, hk)\sigma(h, k) \quad , \quad \sigma(g, 1) = \sigma(1, g) = 1$$

The algebra $C^*(G)$, with multiplication given by $g \cdot h = \sigma(g, h)gh$, and with the involution making the standard generators $g \in C^*_\sigma(G)$ unitaries, is denoted $C^*_\sigma(G)$.

As explained in [39], we have the following general construction:

Proposition 16.9. *Given a finite group $G = \{g_1, \ldots, g_N\}$ and a 2-cocycle $\sigma : G \times G \to \mathbb{T}$ we have a matrix model as follows,*

$$\pi : C(S_N^+) \to M_N(C(E)) \quad : \quad w_{ij} \to [x \to Proj(g_i x g_j^*)]$$

for any closed subgroup $E \subset U_A$, where $A = C^*_\sigma(G)$.

Proof. This is indeed clear from definitions, because the standard generators $\{g_1, \ldots, g_N\}$ are pairwise orthogonal with respect to the canonical trace of A. See [39]. $\square$

In order to investigate the stationarity of π, we use:

Proposition 16.10. *We have the formula*

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p} &= \overline{\sigma(i_1,i_1^{-1}i_2)}\ldots\overline{\sigma(i_p,i_p^{-1}i_1)}\cdot\overline{\sigma(j_2,j_2^{-1}j_1)}\ldots\overline{\sigma(j_1,j_1^{-1}j_p)}\\ &\quad\frac{1}{N}\int_E tr(g_{i_1^{-1}i_2}xg_{j_2^{-1}j_1}x^*)\ldots\ldots tr(g_{i_p^{-1}i_1}xg_{j_1^{-1}j_p}x^*)dx\end{aligned}$$

with all the indices varying in a cyclic way.

Proof. According to the definition of T_p, we have the following formula:

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p} &= \left(tr\otimes\int_E\right)\Big(Proj(g_{i_1}xg_{j_1}^*)\ldots Proj(g_{i_p}xg_{j_p}^*)\Big)dx\\ &= \frac{1}{N}\int_E < g_{i_1}xg_{j_1}^*,g_{i_2}xg_{j_2}^* > \ldots < g_{i_p}xg_{j_p}^*,g_{i_1}xg_{j_1}^* > dx\end{aligned}$$

Since we have $g_ig_{i^{-1}k}=\sigma(i,i^{-1}k)g_k$, and so $g_i^*g_k=\overline{\sigma(i,i^{-1}k)}g_{i^{-1}k}$, we obtain:

$$\begin{aligned}< g_ixg_j^*,g_kxg_l^* > &= tr(g_jx^*g_i^*g_kxg_l^*)\\ &= tr(g_i^*g_kxg_l^*g_jx^*)\\ &= \overline{\sigma(i,i^{-1}k)}\cdot\overline{\sigma(l,l^{-1}j)}\cdot tr(g_{i^{-1}k}xg_{l^{-1}j}x^*)\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. □

We have the following result, which generalizes some previous computations:

Theorem 16.11. *For any intermediate closed subgroup $G\subset E\subset U_A$, the matrix model $\pi:C(S_N^+)\to M_N(C(E))$ constructed above is stationary on its image.*

Proof. We use the formula in Proposition 16.10. Let us write $(T_p)_{ij}=\rho(i,j)(T_p^\circ)_{ij}$, where $\rho(i,j)$ is the product of σ terms appearing there. We have:

$$(T_p^2)_{ij}=\sum_k(T_p)_{ik}(T_p)_{kj}=\sum_k\rho(i,k)\rho(k,j)(T_p^\circ)_{ik}(T_p^\circ)_{kj}$$

Let us first compute the ρ term. We have:

$$\begin{aligned}\rho(i,k)\rho(k,j) &= \overline{\sigma(i_1,i_1^{-1}i_2)}\ldots\overline{\sigma(i_p,i_p^{-1}i_1)}\cdot\overline{\sigma(k_2,k_2^{-1}k_1)}\ldots\overline{\sigma(k_1,k_1^{-1}k_p)}\\ &\quad\overline{\sigma(k_1,k_1^{-1}k_2)}\ldots\overline{\sigma(k_p,k_p^{-1}k_1)}\cdot\overline{\sigma(j_2,j_2^{-1}j_1)}\ldots\overline{\sigma(j_1,j_1^{-1}j_p)}\\ &= \sigma(i,j)\cdot\overline{\sigma(k_2,k_2^{-1}k_1)}\cdot\overline{\sigma(k_1,k_1^{-1}k_2)}\ldots\ldots\overline{\sigma(k_1,k_1^{-1}k_p)}\cdot\overline{\sigma(k_p,k_p^{-1}k_1)}\end{aligned}$$

Now observe that by multiplying $\sigma(i,i^{-1}k)g_i^*g_k=g_{i^{-1}k}$ and $\sigma(k,k^{-1}i)g_k^*g_i=g_{k^{-1}i}$ we obtain $\sigma(i,i^{-1}k)\sigma(k,k^{-1}i)=\sigma(i^{-1}k,k^{-1}i)$. Thus, our expression further simplifies:

$$\rho(i,k)\rho(k,j)=\sigma(i,j)\cdot\overline{\sigma(k_2^{-1}k_1,k_1^{-1}k_2)}\ldots\ldots\overline{\sigma(k_1^{-1}k_p,k_p^{-1}k_1)}$$

On the other hand, the T° term can be written as follows:

$$\begin{aligned}(T_p^\circ)_{ik}(T_p^\circ)_{kj} &= \frac{1}{N^2}\int_E\int_E tr(g_{i_1^{-1}i_2}xg_{k_2^{-1}k_1}x^*)tr(g_{k_1^{-1}k_2}yg_{j_2^{-1}j_1}y^*)\\ &\qquad\qquad\ldots\ldots\\ &\qquad tr(g_{i_p^{-1}i_1}xg_{k_1^{-1}k_p}x^*)tr(g_{k_p^{-1}k_1}yg_{j_1^{-1}j_p}y^*)dxdy\end{aligned}$$

We therefore conclude that we have the following formula:

$$\begin{aligned}(T_p^2)_{ij} &= \frac{\sigma(i,j)}{N^2}\int_E\int_E\sum_{k_1\ldots k_p}\overline{\sigma(k_2^{-1}k_1,k_1^{-1}k_2)}tr(g_{i_1^{-1}i_2}xg_{k_2^{-1}k_1}x^*)tr(g_{k_1^{-1}k_2}yg_{j_2^{-1}j_1}y^*)\\ &\qquad\qquad\ldots\ldots\\ &\qquad\overline{\sigma(k_1^{-1}k_p,k_p^{-1}k_1)}tr(g_{i_p^{-1}i_1}xg_{k_1^{-1}k_p}x^*)tr(g_{k_p^{-1}k_1}yg_{j_1^{-1}j_p}y^*)dxdy\end{aligned}$$

By using now $g_i^*=\overline{\sigma(i,i^{-1})}g_{i^{-1}}$, and moving as well the x^* variables at left, we obtain:

$$\begin{aligned}(T_p^2)_{ij} &= \frac{\sigma(i,j)}{N^2}\int_E\int_E\sum_{k_1\ldots k_p}tr(x^*g_{i_1^{-1}i_2}xg_{k_2^{-1}k_1})tr(g^*_{k_2^{-1}k_1}yg_{j_2^{-1}j_1}y^*)\\ &\qquad\qquad\ldots\ldots\\ &\qquad tr(x^*g_{i_p^{-1}i_1}xg_{k_1^{-1}k_p})tr(g^*_{k_1^{-1}k_p}yg_{j_1^{-1}j_p}y^*)dxdy\end{aligned}$$

We can compute the products of traces by using the following formula:

$$\begin{aligned}tr(Ag_k)tr(g_k^*B) &= \sum_{qs}< g_q,Ag_k >< g_s,g_k^*B >\\ &= \sum_{qs}tr(g_q^*Ag_k)tr(g_s^*g_k^*B)\end{aligned}$$

Thus are left with an integral involving the variable $z=xy$, which gives T_p°. □

Let us discuss now the relationship with the Weyl matrices. We have:

Proposition 16.12. *Given a finite abelian group H, consider the product $G=H\times\widehat{H}$, and endow it with its standard Fourier cocycle.*

1. *With $E=U_n$, where $n=|H|$, the model $\pi:C(S_N^+)\to M_N(C(U_n))$ constructed above, where $N=n^2$, is the Weyl matrix model associated to H.*
2. *When assuming in addition that H is cyclic, $H=\mathbb{Z}_n$, we obtain in this way the matrix model for $C(S_N^+)$ coming from the usual Weyl matrices.*
3. *In the particular case $H=\mathbb{Z}_2$, the model $\pi:C(S_4^+)\to M_N(C(U_2))$ constructed above is the matrix model for $C(S_4^+)$ coming from the Pauli matrices.*

Proof. All this is well-known. The general construction in Proposition 16.9 above came in fact by successively generalizing (3) → (2) → (1), and then by performing one more generalization, with $G=H\times\widehat{H}$ with its standard Fourier cocycle being replaced by an arbitrary finite group G, with a 2-cocycle on it. For full details here, see [39]. □

Regarding now the associated quantum permutation groups, in the general context of Proposition 16.9, we have the following result:

Theorem 16.13. *For a generalized Weyl matrix model, as in Proposition 16.9 above, the moments of the main character of the associated quantum group are*

$$c_p=\frac{1}{N}\sum_{j_1\ldots j_p}^{\circ}\int_E tr(g_{j_1}xg_{j_1}^*x^*)\ldots tr(g_{j_p}xg_{j_p}^*x^*)dx$$

where $\circ$ means that the indices are subject to the condition $j_1\ldots j_p=1$.

Proof. According to Proposition 16.9 and to Proposition 16.10 above, the moments of the main character are the following numbers:

$$\begin{aligned}c_p &= \frac{1}{N}\sum_{i_1\ldots i_p}\overline{\sigma(i_1,i_1^{-1}i_2)}\ldots\overline{\sigma(i_p,i_p^{-1}i_1)}\cdot\overline{\sigma(i_2,i_2^{-1}i_1)}\ldots\overline{\sigma(i_1,i_1^{-1}i_p)}\\ &\qquad\int_E tr(g_{i_1^{-1}i_2}xg_{i_2^{-1}i_1}x^*)\ldots\ldots tr(g_{i_p^{-1}i_1}xg_{i_1^{-1}i_p}x^*)dx\end{aligned}$$

We can compact the cocycle part by using the following formulae:

$$\begin{aligned}\sigma(i_p,i_p^{-1}i_{p+1})\sigma(i_{p+1},i_{p+1}^{-1}i_p) &= \sigma(i_{p+1},i_{p+1}^{-1}i_p\cdot i_p^{-1}i_{p+1})\sigma(i_{p+1}^{-1}i_p,i_p^{-1}i_{p+1})\\ &= \sigma(i_{p+1},1)\sigma(i_{p+1}^{-1}i_p,i_p^{-1}i_{p+1})\\ &= \sigma(i_{p+1}^{-1}i_p,i_p^{-1}i_{p+1})\end{aligned}$$

Thus, in terms of the indices $j_1=i_1^{-1}i_2,\ldots,j_p=i_p^{-1}i_1$, which are subject to the condition $j_1\ldots j_p=1$, we have the following formula:

$$c_p=\frac{1}{N}\sum_{j_1\ldots j_p}^{\circ}\overline{\sigma(j_1^{-1},j_1)}\ldots\overline{\sigma(j_p^{-1},j_p)}\int_E tr(g_{j_1}xg_{j_1^{-1}}x^*)\ldots tr(g_{j_p}xg_{j_p^{-1}}x^*)dx$$

Here the $\circ$ symbol above the sum is there for reminding us that the indices are subject to the condition $j_1\ldots j_p=1$. By using now $g_j^*=\overline{\sigma(j^{-1},j)}g_{j^{-1}}$, we obtain:

$$c_p=\frac{1}{N}\sum_{j_1\ldots j_p}^{\circ}\int_E tr(g_{j_1}xg_{j_1}^*x^*)\ldots tr(g_{j_p}xg_{j_p}^*x^*)dx$$

Thus, we have obtained the formula in the statement. $\square$

It is quite unclear whether the above formula further simplifies, in general. In the context of the Fourier cocycles, as in Proposition 16.12, it is possible to pass to a plain sum, by inserting a certain product of multiplicative factors $c(j_1)\ldots c(j_p)$, which equals 1 when $j_1\ldots j_p=1$, and the computation can be finished as follows:

$$\begin{aligned} c_p &= \frac{1}{N}\int_E\left(\sum_j c(j)tr(g_jxg_j^*x^*)\right)^p dx\\ &= \frac{1}{N}\int_E tr(xx^*)dx \end{aligned}$$

Thus, the law of the main character of the corresponding quantum group coincides with the law of the main character of PE. All this suggests that the quantum group associated to a Weyl matrix model, as above, should appear as a suitable twist of PE. In addition, we believe that in the case where E is easy these examples should be covered by a suitable projective extension of the Schur-Weyl twisting procedure.

Following [8], [17], [22], [52], [40] and related papers, let us discuss now the Hadamard matrix models, which are of particular importance as well. Let us start with:

Definition 16.14. *A complex Hadamard matrix is a square matrix*

$$H\in M_N(\mathbb C)$$

whose entries are on the unit circle, and whose rows are pairwise orthogonal.

Observe that the orthogonality condition tells us that the rescaled matrix $U=H/\sqrt N$ must be unitary. Thus, these matrices form a real algebraic manifold, given by:

$$X_N=M_N(\mathbb T)\cap\sqrt N U_N$$

The basic example is the Fourier matrix, $F_N=(w^{ij})$ with $w=e^{2\pi i/N}$. More generally, we have as example the Fourier coupling of any finite abelian group G, regarded via the isomorphism $G\simeq\widehat G$ as a square matrix, $F_G\in M_G(\mathbb C)$:

$$F_G=<i,j>_{i\in G,j\in\widehat G}$$

Observe that for the cyclic group $G=\mathbb Z_N$ we obtain in this way the above standard Fourier matrix F_N. In general, we obtain a tensor product of Fourier matrices F_N. There are many other examples of Hadamard matrices, some being elementary, some other fairly exotic, appearing in various branches of mathematics and physics. The idea is that the complex Hadamard matrices can be though of as being “generalized Fourier matrices”, and this is where the interest in these matrices comes from. We refer here to [131].

In relation with the quantum groups, the starting observation is as follows:

Proposition 16.15. *If $H\in M_N(\mathbb C)$ is Hadamard, the rank one projections*

$$P_{ij}=Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1,\ldots,H_N\in\mathbb T^N$ are the rows of H, form a magic unitary.

Proof. This is clear, the verification for the rows being as follows:

$$\left\langle\frac{H_i}{H_j},\frac{H_i}{H_k}\right\rangle=\sum_l\frac{H_{il}}{H_{jl}}\cdot\frac{H_{kl}}{H_{il}}=\sum_l\frac{H_{kl}}{H_{jl}}=N\delta_{jk}$$

The verification for the columns is similar, as follows:

$$\left\langle\frac{H_i}{H_j},\frac{H_k}{H_j}\right\rangle=\sum_l\frac{H_{il}}{H_{jl}}\cdot\frac{H_{jl}}{H_{kl}}=\sum_l\frac{H_{il}}{H_{kl}}=N\delta_{ik}$$

Thus, we have indeed a magic unitary, as claimed. □

We can proceed now in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction:

Definition 16.16. *To any Hadamard matrix $H\in M_N(\mathbb{C})$ we associate the quantum permutation group $G\subset S_N^+$ given by the fact that $C(G)$ is the Hopf image of*

$$\pi:C(S_N^+)\to M_N(\mathbb{C})$$

$$u_{ij}\to Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1,\ldots,H_N\in\mathbb{T}^N$ are the rows of H.

Summarizing, we have a construction $H\to G$, and our claim is that this construction is something really useful, with G encoding the combinatorics of H. To be more precise, our claim is that "H can be thought of as being a kind of Fourier matrix for G".

This is of course quite interesting, philosophically speaking. There are several results supporting this, with the main evidence coming from the following result, coming from [22], [40], which collects the basic known results regarding the construction:

Theorem 16.17. *The construction $H\to G$ has the following properties:*

1. *For a Fourier matrix $H=F_G$ we obtain the group G itself, acting on itself.*
2. *For $H\notin\{F_G\}$, the quantum group G is not classical, nor a group dual.*
3. *For a tensor product $H=H'\otimes H''$ we obtain a product, $G=G'\times G''$.*

Proof. All this material is standard, and elementary, as follows:

(1) Let us first discuss the cyclic group case, where our Hadamard matrix is a usual Fourier matrix, $H=F_N$. Here the rows of H are given by $H_i=\rho^i$, where:

$$\rho=(1,w,w^2,\ldots,w^{N-1})$$

Thus, we have the following formula, for the associated magic basis:

$$\frac{H_i}{H_j}=\rho^{i-j}$$

It follows that the corresponding rank 1 projections $P_{ij} = Proj(H_i/H_j)$ form a circulant matrix, all whose entries commute. Since the entries commute, the corresponding quantum group must satisfy $G \subset S_N$. Now by taking into account the circulant property of $P = (P_{ij})$ as well, we are led to the conclusion that we have:

$$G = \mathbb{Z}_N$$

In the general case now, where $H = F_G$, with G being an arbitrary finite abelian group, the result can be proved either by extending the above proof, of by decomposing $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ and using (3) below, whose proof is independent from the rest.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction $H \to G$, we refer to [40].

(3) Assume that we have a tensor product $H = H' \otimes H''$, and let G, G', G'' be the associated quantum permutation groups. We have then a diagram as follows:

$$\begin{tikzcd}
C(S_{N'}^+) \otimes C(S_{N''}^+) \arrow[r] & C(G') \otimes C(G'') \arrow[r] & M_{N'}(\mathbb{C}) \otimes M_{N''}(\mathbb{C}) \arrow[d] \\
C(S_N^+) \arrow[u] \arrow[r] & C(G) \arrow[r] & M_N(\mathbb{C})
\end{tikzcd}$$

Here all the maps are the canonical ones, with those on the left and on the right coming from $N = N'N''$. At the level of standard generators, the diagram is as follows:

$$\begin{tikzcd}
u'_{ij} \otimes u''_{ab} \arrow[r] & w'_{ij} \otimes w''_{ab} \arrow[r] & P'_{ij} \otimes P''_{ab} \arrow[d] \\
u_{ia,jb} \arrow[u] \arrow[r] & w_{ia,jb} \arrow[r] & P_{ia,jb}
\end{tikzcd}$$

Now observe that this diagram commutes. We conclude that the representation associated to H factorizes indeed through $C(G') \otimes C(G'')$, and this gives the result. $\square$

Generally speaking, going beyond the above result, with explicit computations of quantum permutation groups associated to explicit complex Hadamard matrices, is a quite difficult task. The main results which are known so far concern the deformations of the Fourier matrices, and we refer here to [8], [17], [52] and related papers.

At the general level now, we have the following result, from [22]:

Theorem 16.18. *The Tannakian category of the quantum group $G\subset S_N^+$ associated to a complex Hadamard matrix $H\in M_N(\mathbb C)$ is given by*

$$T\in Hom(u^{\otimes k},u^{\otimes l})\iff T^\circ G^{k+2}=G^{l+2}T^\circ$$

where the objects on the right are constructed as follows:

(1) $T^\circ=id\otimes T\otimes id$.
(2) $G_{ia}^{jb}=\sum_kH_{ik}\bar{H}_{jk}\bar{H}_{ak}H_{bk}$.
(3) $G^k_{i_1\ldots i_k,j_1\ldots j_k}=G_{i_ki_{k-1}}^{j_kj_{k-1}}\ldots G_{i_2i_1}^{j_2j_1}$.

Proof. We use the Tannakian result for the Hopf image of a representation, discussed in section 15 above. With the notations here, we have the following formula:

$$Hom(u^{\otimes k},u^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

The vector space on the right consists by definition of the complex $N^l\times N^k$ matrices T, satisfying the following relation:

$$TU^{\otimes k}=U^{\otimes l}T$$

If we denote this equality by $L=R$, the left term L is given by:

$$\begin{aligned}L_{ij}&=(TU^{\otimes k})_{ij}\\&=\sum_aT_{ia}U^{\otimes k}_{aj}\\&=\sum_aT_{ia}U_{a_1j_1}\ldots U_{a_kj_k}\end{aligned}$$

As for the right term R, this is given by:

$$\begin{aligned}R_{ij}&=(U^{\otimes l}T)_{ij}\\&=\sum_bU^{\otimes l}_{ib}T_{bj}\\&=\sum_bU_{i_1b_1}\ldots U_{i_lb_l}T_{bj}\end{aligned}$$

Consider now the vectors $\xi_{ij}=H_i/H_j$. Since these vectors span the ambient Hilbert space, the equality $L=R$ is equivalent to the following equality:

$$<L_{ij}\xi_{pq},\xi_{rs}>=<R_{ij}\xi_{pq},\xi_{rs}>$$

We use now the following well-known formula, expressing a product of rank one projections $P_1,\ldots,P_k$ in terms of the corresponding image vectors $\xi_1,\ldots,\xi_k$:

$$<P_1\ldots P_kx,y>=<x,\xi_k><\xi_k,\xi_{k-1}>\ldots\ldots<\xi_2,\xi_1><\xi_1,y>$$

This gives the following formula for L:

$$\begin{aligned}
< L_{ij}\xi_{pq},\xi_{rs} > &= \sum_a T_{ia} < P_{a_1j_1}\ldots P_{a_kj_k}\xi_{pq},\xi_{rs} > \\
&= \sum_a T_{ia} < \xi_{pq},\xi_{a_kj_k} > \ldots < \xi_{a_1j_1},\xi_{rs} > \\
&= \sum_a T_{ia} G_{pa_k}^{qj_k} G_{a_ka_{k-1}}^{j_kj_{k-1}} \ldots G_{a_2a_1}^{j_2j_1} G_{a_1r}^{j_1s} \\
&= \sum_a T_{ia} G_{rap,sjq}^{k+2} \\
&= (T^\circ G^{k+2})_{rip,sjq}
\end{aligned}$$

As for the right term R, this is given by:

$$\begin{aligned}
< R_{ij}\xi_{pq},\xi_{rs} > &= \sum_b < P_{i_1b_1}\ldots P_{i_lb_l}\xi_{pq},\xi_{rs} > T_{bj} \\
&= \sum_b < \xi_{pq},\xi_{i_lb_l} > \ldots < \xi_{i_1b_1},\xi_{rs} > T_{bj} \\
&= \sum_b G_{pi_l}^{qb_l} G_{i_li_{l-1}}^{b_lb_{l-1}} \ldots G_{i_2i_1}^{b_2b_1} G_{i_1r}^{b_1s} T_{bj} \\
&= \sum_b G_{rip,sbq}^{l+2} T_{bj} \\
&= (G^{l+2}T^\circ)_{rip,sjq}
\end{aligned}$$

Thus, we obtain the formula in the statement. See [22]. □

Let us discuss now the relation with subfactor theory, and with planar algebras. As a starting point, we have the following basic observation of Popa [124]:

Proposition 16.19. *Up to a conjugation by a unitary, the pairs of orthogonal MASA in the simplest factor, namely the matrix algebra $M_N(\mathbb{C})$, are as follows,*

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta \subset M_N(\mathbb{C})$ being the diagonal matrices, and with $H \in M_N(\mathbb{C})$ being Hadamard.

Proof. Any maximal abelian subalgebra (MASA) in $M_N(\mathbb{C})$ being conjugated to Δ, we can assume, up to conjugation by a unitary, that we have, with $U \in U_N$:

$$A = \Delta \quad , \quad B = U\Delta U^*$$

Now observe that given two diagonal matrices $D,E\in\Delta$, we have:

$$\begin{aligned}tr(D\cdot UEU^*)&=&\frac{1}{N}\sum_i(DUEU^*)_{ii}\\&=&\frac{1}{N}\sum_{ij}D_{ii}U_{ij}E_{jj}\bar{U}_{ij}\\&=&\frac{1}{N}\sum_{ij}D_{ii}E_{jj}|U_{ij}|^2\end{aligned}$$

Thus, the orthogonality condition $A\perp B$ reformulates as follows:

$$\frac{1}{N}\sum_{ij}D_{ii}E_{jj}|U_{ij}|^2=\frac{1}{N^2}\sum_{ij}D_{ii}E_{jj}$$

But this tells us $H=\sqrt{N}U$ must be Hadamard, as claimed. □

Along the same lines, but at a more advanced level, we have:

Theorem 16.20. *Given a complex Hadamard matrix $H\in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal MASA, namely*

$$\begin{tikzcd}\Delta \arrow[r] & M_N(\mathbb{C})\\ \mathbb{C} \arrow[u] \arrow[r] & H\Delta H^* \arrow[u]\end{tikzcd}$$

is a commuting square in the sense of subfactor theory, in the sense that the expectations onto $\Delta,H\Delta H^*$ commute, and their product is the expectation onto $\mathbb{C}$.

Proof. The expectation $E_\Delta:M_N(\mathbb{C})\to\Delta$ is the operation $M\to M_\Delta$ which consists in keeping the diagonal, and erasing the rest. Consider now the other expectation:

$$E_{H\Delta H^*}:M_N(\mathbb{C})\to H\Delta H^*$$

It is better to identify this with the following expectation, with $U=H/\sqrt{N}$:

$$E_{U\Delta U^*}:M_N(\mathbb{C})\to U\Delta U^*$$

This latter expectation must be of the form $M\to UX_\Delta U^*$, with X satisfying:

$$<M,UDU^*>=<UX_\Delta U^*,UDU^*>\quad,\quad\forall D\in\Delta$$

The scalar products being given by $<a,b>=tr(ab^*)$, this condition reads:

$$tr(MUD^*U^*)=tr(X_\Delta D^*)\quad,\quad\forall D\in\Delta$$

Thus $X = U^*MU$, and the formulae of our two expectations are as follows:

$$\begin{aligned} E_\Delta(M) &= M_\Delta \\ E_{U\Delta U^*}(M) &= U(U^*MU)_\Delta U^* \end{aligned}$$

With these formulae in hand, an elementary computation gives the result. □

The point now is that any commuting square C produces a subfactor of the Murray-von Neumann hyperfinite II_1 factor R. Indeed, under suitable assumptions on the inclusions $C_{00} \subset C_{10}, C_{01} \subset C_{11}$, we can perform the basic construction for them, in finite dimensions, and we obtain a whole array of commuting squares, as follows:

$$\begin{tikzcd} A_0 & A_1 & A_2 & \\ C_{02} \arrow[r] \arrow[u, dotted] & C_{12} \arrow[r] \arrow[u, dotted] & C_{22} \arrow[r, dotted] \arrow[u, dotted] & B_2 \\ C_{01} \arrow[r] \arrow[u] & C_{11} \arrow[r] \arrow[u] & C_{21} \arrow[r, dotted] \arrow[u] & B_1 \\ C_{00} \arrow[r] \arrow[u] & C_{10} \arrow[r] \arrow[u] & C_{20} \arrow[r, dotted] \arrow[u] & B_0 \end{tikzcd}$$

Here the various A, B letters stand for the von Neumann algebras obtained in the limit, which are all isomorphic to the hyperfinite II_1 factor R, and we have:

Theorem 16.21. *In the context of the above diagram, the following happen:*

(1) *$A_0 \subset A_1$ is a subfactor, and $\{A_i\}$ is the Jones tower for it.*
(2) *The corresponding planar algebra is given by $A_0' \cap A_k = C_{01}' \cap C_{k0}$.*
(3) *A similar result holds for the "horizontal" subfactor $B_0 \subset B_1$.*

Proof. This is something very standard, the idea being as follows:

(1) This is something quite routine.

(2) This is a subtle result, called Ocneanu compactness theorem.

(3) This follows from (1,2), by flipping the diagram. □

Getting back now to the Hadamard matrices, we can extend our lineup of results, namely Proposition 16.19 and Theorem 16.20, with an advanced result, as follows:

Theorem 16.22. *Given a complex Hadamard matrix $H \in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal MASA, namely*

$$\begin{array}{ccc}\Delta & \longrightarrow & M_N(\mathbb{C})\\ \uparrow & & \uparrow\\ \mathbb{C} & \longrightarrow & H\Delta H^*\end{array}$$

is a commuting square in the sense of subfactor theory, and the planar algebra of the corresponding subfactor can be explicitly computed in terms of H.

Proof. The fact that we have indeed a commuting square follows from the above, the computation of the standard invariant is possible due to a result of Ocneanu, and the planar algebra formulation is due to Jones. For the precise formula, we refer to [105]. □

In relation now with our quantum group construction, we have:

Theorem 16.23. *The subfactor associated to $H \in M_N(\mathbb{C})$ is of the form*

$$R^G \subset (R \otimes \mathbb{C}^N)^G$$

where $G \subset S_N^+$ is the associated quantum permutation group, and its planar algebra is

$$P_k = End(u^{\otimes k})$$

having as Poincaré series the moment generating function of $\chi = \sum_i u_{ii}$.

Proof. There is a long story here, the idea being that the planar algebra formula from [105], mentioned in the proof of Theorem 16.22, coincides in fact with the quantum group formula from Theorem 16.18. Thus, we obtain the second assertion, and the first assertion can be proved as well, by using standard quantum group techniques. In fact, the correspondence $G \leftrightarrow P$ is part of the general correspondence between closed subgroups $G \subset S_N^+$, and subalgebras of the spin planar algebra, discussed in section 3 above. □

Summarizing, we have some interesting mathematics here. In practice now, a first problem is that of getting beyond Theorem 16.17, with explicit computations, and we refer here to [8], [17], [52] and related papers. Another problem is that of unifying all this with the Weyl matrix models, and we refer here to [28], [39] and related papers. Finally, in relation with the work of Jones [101], [102], [103], [104], [105], and of Connes as well [73], [74], [75], we have the question of understanding the physical meaning of all this.

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