Math: an Invention or a Discovery?

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Abstract

Math is a process in which both invention and discovery are essential ingredients in its development. But which one comes first?

1. Illustration

Before delving into this question, let's start with two cases that will bring out the point of view of this paper.

Case (i): Odd and even numbers.

Before we can discover these two properties of numbers, we need to invent a number system, which is written as the set of {1, 2, 3… n}, and the operation of division. With that in hand, we can explore and discover any number k as being divisible by 2, meaning the result j must be in that set. If k is divisible by two, it is an even number, if not, it is an odd number.

Case (ii): primary numbers.

This is an extension of case (i) by making the requirement that p be <u>not</u> divisible by the numbers 2, 3… up to n-1. If that is the case, p is a prime number; if it is divisible by any of those numbers, it isn't. By this process – an invention – we discover that 3, 5, 7, 11 etc. are prime numbers.

2. Math, Language, and Mappings

Here are two unique abilities of the human mind: (i) every thought the human mind can hold in his or her brain can be put as a set; (ii) the human brain can form mappings between various sets.

Humans have been doing this since the beginning of time but only became aware of this process not that long ago, relatively speaking.

Here is what can be said: Both math and language are the results of this unique ability.

At the base of this structure, our mind does the following function:

Real world → a mental construct

For instance,

Bird (observation in the real world) → the word "bird" (in the English language)

First we observe in the real world certain animals that have common characteristics – vertebrate, warm-blooded, egg-laying with feathers, wings, and a beak and typically able to fly - that we put in a set. Then we map:

The sound "bird" → the written word "bird"

The whole process can be symbolized as:

Observations in the real world → sets → sounds → symbols (written words)

Caution: Not to confuse the above mapping with its reverse:

A mental construct → real world

Which would mean the following: one invents a concept then one looks for something in the real world that would correspond to that concept. Think of creatures such as leprechauns, fire-breathing dragons, and werewolves.

Imaginary creatures (mental construct) → real world

We have to insist on the "imaginary" part of this mapping until evidence shows to the contrary.

For the moment, it's just sufficient to say that we will examine mappings going in this particular direction,

Real world → a mental construct

That is, the process we are alluding to initiates from the real world and it is then mapped out in our brain as a mental construct. This is how we teach language to our young ones: by pointing to an object "bird" and repeating the word/sound "bird". It's the most basic thing the human brain can do – a mapping between the real world and a symbol that our brain can grapple with, remember, assimilate and reproduce at some later times - a joy expressed by parents eager to recount their child's first spoken word.

Once that process has taken place, nothing forbids the child's versatile brain to quickly jumpstart to a realm of the imagination, which is fine as we often need creativity more often than reason in order to invent new mappings, new mental constructs, and so on.

When one of the side of a mapping is made up of elements of the real world, such as a bird, which we can point to with our fingers, this is the closest we can get to a true statement. Should we be pointing to something other than a bird, we would know that the statement, "This is a bird", is false.

Such mappings with one side of the mapping pertaining to the real world are of the 1st degree of abstraction. Another example of a mapping of the 1St degree of abstraction, taking cues from physics, would be:

Motion (observed in the real world) → vector representation (mental construct)

When the two sides of the mapping are themselves abstractions, we are at a higher level of abstraction (2nd degree). For instance,

A quantum state (mental construct) → a vector in a Hilbert space (mental construct)

Here "vector in a Hilbert space" can take any dimension from one to infinity as opposed to 3 dimensions for motion, and do not necessarily represent an object in the real world with a sense of direction like motion (a quantum state is such an object). These vectors in a Hilbert space obey certain mathematical rules, which are the same mathematical operations of ordinary vectors for motion. Such mappings of 2nd degree of abstraction are harder to establish and can be difficult to deal with. But in our quest to understand the universe, they are inescapable.

3. Math

What we are proposing is the following: From an observation in the real world we "invent" a mathematical concept, then go on an exploration of that concept, and then "discover" some qualities initially unsuspected in our invention. For instance, as we stated in section 1, after we've invented a number system, we discovered that some numbers are divisible by the number two, and there are those numbers that aren't. We then invented the concept of "even numbers" for the first case, and the "odd numbers" for the second case, which can lead to further discoveries and inventions, such as the concept of prime numbers.

Math is a powerful tool mainly because it is self-referential: it is based on the law of identity, of which mapping is the most powerful.

Consider these two descriptions of an observation in the real world: (a) we see a sheep in the field in front of us, we see another sheep, and then another one; or (b) we see three sheep in the field in front of us.

They both describe the same reality.

Notice that description (b) is shorter and easier to handle. Were we to count an additional sheep, description (b) is easily updated by saying: now we see four sheep in the field in front of us. This particular description lends itself to mathematical manipulations and the richness it entails.

Nevertheless numbers just like words are descriptions. Now a number system is only possible if we have already drawn a set in our mind. Before we count the windows in this room, we need to form the set of "all the windows in this room". Otherwise we might be counting the bricks in the wall, the doors, the frames on the wall, etc. In the above example, the set was "the sheep in the field in front of us". We're not counting the sheep in someone else's field. If we did, we would have to enlarge to the set of "all the fields surrounding us" for instance. Although not mentioned in case (i) of section 1, we needed to have invented the concept of a set, that is what we did when we wrote down {1, 2, 3… n}, in order for that illustration to be worked out. In case (ii), also not mentioned is the invention of the operation "minus" in "n-1".

Necessarily the number system, just like the alphabet system, is a mental construct. A very important development happened when we put numbers to a stick, and call that "measuring the length of the stick "– we had just invented, mentally speaking, a space connected with geometry (the stick is a line). When we extended that concept to areas – two sticks on the horizontal and two more sticks on the vertical - this completed the "square" and in so doing we also discovered the irrational number $\sqrt{2}$. The imaginary number $\mathrm{i} = \sqrt{-1}$ was initially invented to provide a solution to the equation $x^2 + 1 = 0$. We then discover that rotations and waves are perfectly fitted to be represented by the invention of these imaginary numbers.

So now we will define the word "mapping" in its narrow mathematical sense (an invention). Math is then the manipulation of mappings (a discovery).

Consider any two sets M and N, and a mapping f:

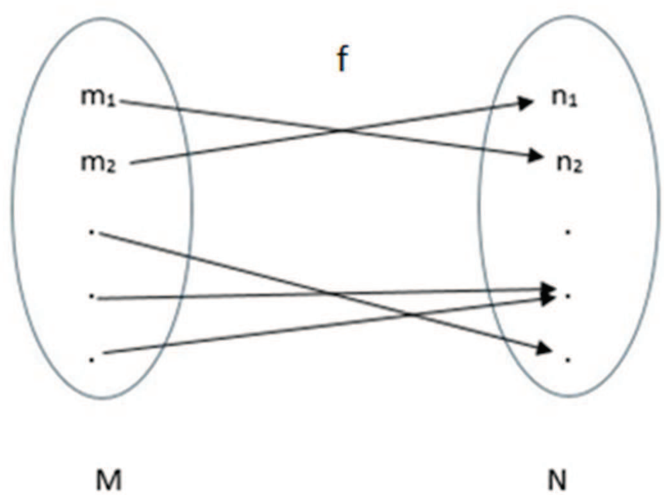

Fig. 1

Symbolically, this is written as:

$$f: M \rightarrow N$$

This means that each element of the set M is mapped onto an element of the set N. Note that not all the elements of N have a corresponding element in M.

One can choose an infinite number of sets like M, an infinite number of sets like N, and an infinite number of mappings like f between them. The above illustration is one example of that infinite soup. Now, out of this infinite soup, there is a lot of junk. Mathematicians are in the business of filtering out the good stuff from that junk. They identify different types of mappings (one-to-one, many-to-one, "onto", "into", injective, surjective, and so on). They also identify different sets (natural numbers, integers, rational numbers, etc.). They also narrow down certain sets to yield topologies, manifolds, bundle fibers, etc. They further narrow down certain mappings to give smooth manifolds, linearity, differentiability, etc. In this infinite soup that we've created, we can discover a number system that deals with the square root of minus one, as mentioned before, and discover its properties. Math is no doubt a very creative process: within its logical construction, it knows no limit. Only our imagination can stump us. These manipulations of mappings between all kinds of sets are just a series of steps one must follow like a recipe to produce a delicious cake.

Here's a mapping that has characterized our present civilization:

Switches on/off (the real world) → base two numbers (0,1) (a mental construct) → Alphabet/number on a screen (real world)

And that advancement led us to build computers and the internet.

As we have mentioned above, take the irrational numbers: they arise because our minds invented the concept of area. When in doubt, ask yourself: does the universe care that we, humans, define an area of a square of size "*b*" as – "*b* times *b*", or "b^2"? Absolutely not, the concept of an area is our invention. Well, when we draw a square if one unit, the diagonal of that square is inevitably root 2, or written as $\sqrt{2}$ or $2^{½}$. Irrational numbers pop out of that geometric situation – this is a discovery that logically came out of our invention of an area, which is an abstract mental construct that has come from our creative mind.

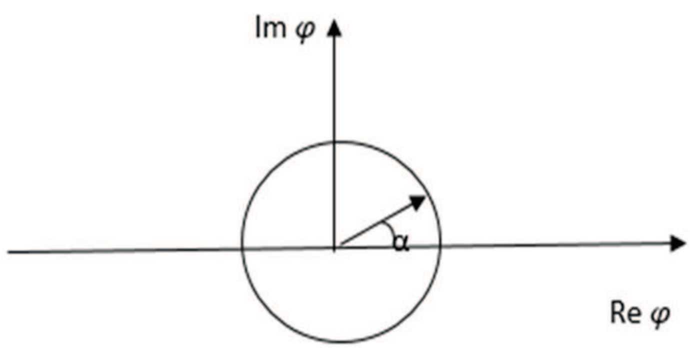

Fig. 2

On an imaginary axis (Fig. 2) $e^{i\alpha}$ describes a circle. Waves are described by functions of the type, $e^{i(kx - wt)}$, complimentary gift from Fourier who established the important idea, which roughly states: given any frequency of an oscillatory motion, as crazy as that motion can be, one can represent that as a series of sine and cosine functions, which is what $e^{i(kx - wt)}$ is. Those sine/cosine functions are perfect for ideal waves, as they have well-defined wavelengths and frequencies. I doubt very much they exist in nature – likewise for the perfect circle or the perfect square.

Another mental construct are coordinates: you look around, and they don't exist except as a mental construct. Similarly, time doesn't exist, except as a mental construct [1]. And therefore a space-time construct is more involved: we need a ruler to establish the x, y, z coordinates for the spatial axes, but then we need a clock for the temporal axis. However here's a secret: in reality, the so-called "time" axis in Special Relativity (SR) is "ct", where c is the speed of light, making that "temporal" axis in actuality a spatial axis as "ct" is the distance travelled by light.

4. Conclusion

Like language, some math can be useful in describing the real world, which scientists have exploited fruitfully, and there are other parts of the math world having no counterpart in the real world. But to reiterate, math is in the category of a mental construct in which we invent concepts and then discover the properties of our inventions. There is an interplay between invention and discovery, but the first spark comes from our creativity.

1. https://vixra.org/abs/2004.0278