Haga's theorems in paper folding and related theorems in Wasan geometry Part 2

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Abstract. We generalize problems in Wasan geometry which involve no folded figures but are related to Haga's fold in origamics. Using the tangent circles appeared in those problems with division by zero, we give a parametric representation of the generalized Haga's fold given in the first part of these two-part papers.

Keywords. Haga's fold, generalized Haga's fold, division by zero, golden mean, silver mean, Steiner chain, parametric representation, inverse of Haga's fold.

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

In the first part of these two-part papers, we have considered some geometric properties of the generalized Haga's fold [9]. Meanwhile there are several problems in Wasan geometry, which do not involve folded figures but are closely related Haga's fold. In this second part we consider those problems in a general way. Using tangent circles appeared in those problems, we give a parametric representation of the generalized Haga's fold with division by zero [7].

2. Related problems in Wasan geometry

In this section we consider several problems in Wasan geometry closely related to Haga's fold, though they are not involving folded figures. A general solution of the problems is given in the next section. We start with two similar problems. The following problem can be found in [1, 16, 20, 27, 29] (see Figure 1). A generalization of the problem can be found in [14].

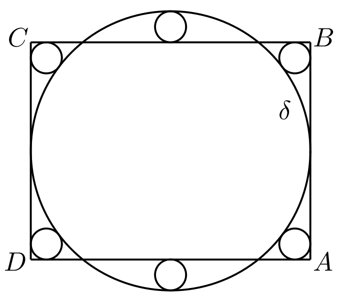

Figure 1.

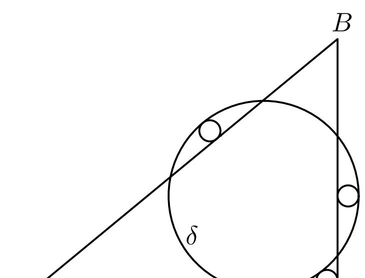

Figure 2.

Problem 2.1. Let δ be a circle of radius s with a rectangle $ABCD$ sharing its center with δ, where the side AB touches δ and the side BC intersect δ in two points. The inradius of the curvilinear triangle made by AB, BC and δ is r and the circle touching BC at its midpoint and touching the minor arc of δ cut by BC also has radius r. Find s in terms of r.

The next sangaku problem can be found in [2] (see Figure 2).

Problem 2.2. Let δ be a circle of radius s and let ABC be a right triangle with right angle at A. The side CA touches δ, and each of the sides AB and BC intersects δ in two points. The inradius of the curvilinear triangle made by CA, AB and δ equals r. The maximal circle touching AB from the side opposite to C and touching δ internally, and the maximal circle touching BC from the side opposite to A and touching δ internally have radius r. Find r in terms of s.

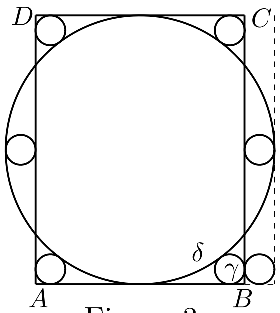

Figure 3.

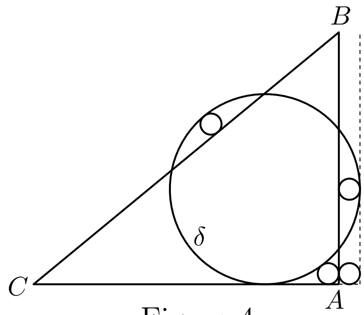

Figure 4.

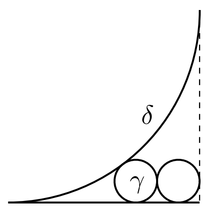

Figure 5.

We show that the two problems are essentially the same. Let γ be the incircle of the curvilinear triangle made by AB, BC and δ in Problem 2.1 (see Figure 3). If we draw the line parallel to BC touching δ and the reflection of γ in the line BC, extend the side AB, and remove the segment BC in the figure of Problems 2.1, we get Figure 5. We can also get the same figure from Figure 2 in a similar way (see Figure 4). Therefore the two problems are essentially the same. Problems considering Figure 5 can also be found in [3], [4], [16, 20], [24], [25], [26], [28] and [30]. We gave a generalization of Problem 2.1 in [14].

We state Problems 2.1 and 2.2 so that the body text gives enough information without the figures. However the most informations of the problems in Wasan geometry are given by the figures, thereby the body texts play only subsidiary roles. The next sangaku problem is stated in such a way [2]:

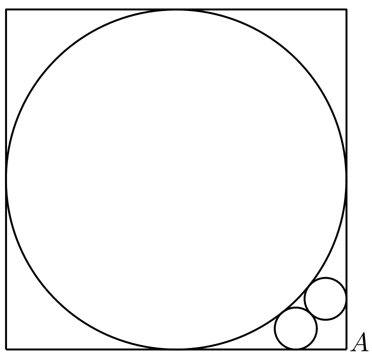

Figure 6.

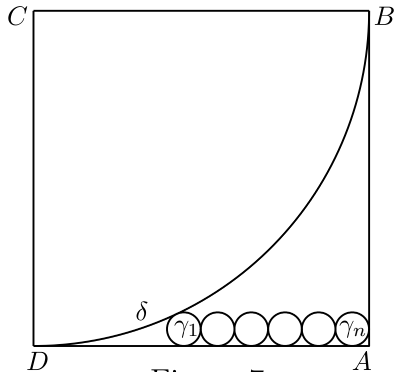

Figure 7.

Problem 2.3. There are a large circle of radius s and two small circles of radius r in a square as in Figure 6. Show $s = 9r$.

We show that the problem is incorrect by the next proposition.

Proposition 2.1. *Assume that an external common tangent of two circles of radii r and s touches the circles at points P and Q. Then the two circles touch externally if and only if $|PQ| = 2\sqrt{rs}$. In this event, the internal common tangent of the two circles passes through the midpoint of PQ.*

We have $s = 2\sqrt{rs} + \sqrt{2}r + r$ by the proposition. Solving the equation for s, we get $s = \left(3 + \sqrt{2} + 2\sqrt{2 + \sqrt{2}}\right) r \approx 8.11r$. Therefore the problem is incorrect. We guess that the two small circles were described as in Figure 5 in the original problem, however Figure 6 was used by transcription error. A general case of Problems 2.1 and 2.2 was considered by Toyoyoshi (see Figure 7):

Problem 2.4 ([17])**.** Let δ be a circle of radius s with center C passing through B for a square $ABCD$. Let γ_1, γ_2, $\cdots$, γ_n be congruent circles of radius r lying inside of the curvilinear triangle made by DA, AB and δ and touching DA such that γ_1 touches δ, γ_1 and γ_2 touch, γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} at the farthest point on γ_{i-1} from the center of γ_{i-2}, and γ_n touches AB. Show r in terms of s and n.

3. Generalized figure

We consider the figure of Problems 2.4 in a general way. For perpendicular lines k and l intersecting in a point A, let δ_1 and δ_2 be circles of radii s_1 and s_2 $(0 \le s_2 \le s_1)$, respectively, touching k and l from the same side. Let γ be a circle of radius r touching δ_1 and δ_2 externally and k at a point K. We denote the figure consisting of γ, δ_1, δ_2, k and l by $\mathcal{T}$. Identifying similar figures, $\mathcal{T}$ is uniquely determined by s_1/s_2. It is also uniquely determined by the real number

$$n = \frac{\tau|AK| + r}{r}, \tag{1}$$

where $\tau = 1$ if δ_1 and K lies on the same side of l otherwise $\tau = -1$ (see Figures 8 and 9). Then we explicitly denote the circle γ and the figure $\mathcal{T}$ by $\gamma(n)$ and $\mathcal{T}(n)$, respectively, The value n equals the ratio of the distance from l to the farthest point on γ from l to the radius of γ. If γ touches k at A, we consider that δ_2 degenerates to the point A and $s_2 = 0$. The figure is denoted by $\mathcal{T}(1)$ (see Figure 10). We also consider the case in which γ degenerates to a point $K \ne A$ on k. In this case we consider that δ_1 and δ_2 coincide and touch k at K (see Figure 11). However there is no real number satisfying (1) in this case. Therefore we introduce a new symbols $\overline{0}$, and denote the point circle K and the figure $\mathcal{T}$ by $\gamma(\overline{0})$ and $\mathcal{T}(\overline{0})$, respectively. In $\mathcal{T}(0)$, δ_1 and δ_2 coincide and γ is the reflection of δ_1 in l (see Figure 12). Notice that δ_1 and δ_2 coincide if and only if $n = 0$ or $n = \overline{0}$.

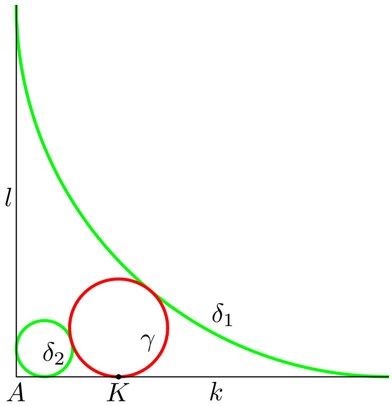

Figure 8: $\tau = 1$ $(1 < n)$.

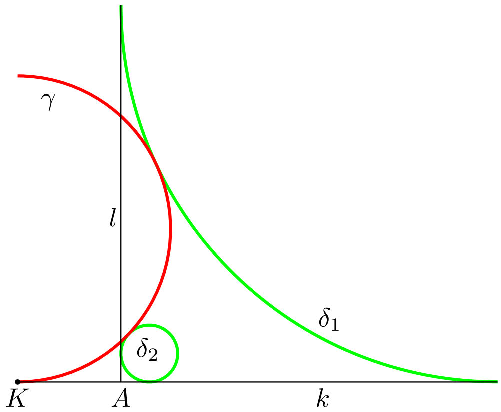

Figure 9: $\tau = -1$ $(0 < n < 1)$.

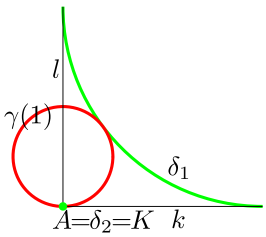

Figure 10: $\mathcal{T}(1)$.

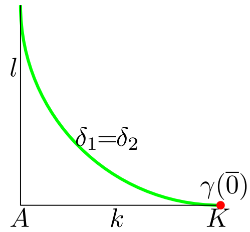

Figure 11: $\mathcal{T}(\overline{0})$.

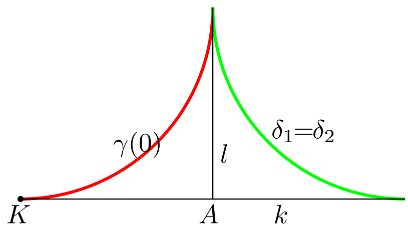

Figure 12: $\mathcal{T}(0)$.

Our definition of $\mathcal{T}(n)$ implies $0 \le n$, and $1 < n$ or $0 < n < 1$ according as K and δ_1 lie on the same side of l or not. If $n/2$ is a natural number, there are circles $\gamma_1, \gamma_2, \cdots, \gamma_{n/2}$ of radius r lying inside of the curvilinear triangle made by k, l and δ_1 and touching k such that l is the external common tangent of γ_1 and δ_1, γ_1 and γ_2 touch, γ_i $(i = 3, 4, \cdots, n/2)$ touches γ_{i-1} at the farthest point on γ_{i-1} from the center of γ_{i-2}, $\gamma_{n/2} = \gamma$. This is the case considered by Toyoyoshi stated as Problem 2.4. If we add the reflection of δ_1 and γ_i $(i = 1, 2, \cdots, n/2)$ in l and remove δ_2 and l, the resulting figure is the configuration $\mathcal{B}(n)$ in [10]. Therefore $\mathcal{T}(n)$ is a generalization of $\mathcal{B}(n)$ in this sense. If $n = 4$, the circles γ_1 and δ_2 coincide (see Figure 36, where regard that $\delta_1 = \delta$, k and l are the lines AB and DA, respectively, and $\gamma_=\gamma(4)$ in the figure). The relation between s_1 and r in (i) in the next theorem gives a solution of Problem 2.4.

Theorem 3.1. *The following statements are true for* $\mathcal{T}(n)$.
(i) *If* $n \ne \overline{0}$, $\sqrt{s_1} = (\sqrt{n} + 1)\sqrt{r}$ *and* $\sqrt{s_2} = |\sqrt{n} - 1|\sqrt{r}$.
(ii) $|AK| = \sqrt{s_1 s_2}$.
(iii) $2\sqrt{r} = \sqrt{s_1} + \sqrt{s_2}$ *if* $0 \le n \le 1$, *and* $2\sqrt{r} = \sqrt{s_1} - \sqrt{s_2}$ *if* $1 < n$.

Proof. By Proposition 2.1 we have $s_1 = \tau|AK| + 2\sqrt{s_1 r} = (n-1)r + 2\sqrt{s_1 r}$, which yields $\sqrt{s_1} = (\sqrt{n}+1)\sqrt{r}$. If $n > 1$, we have $s_2 = \tau|AK| - 2\sqrt{rs_2}$ by the same proposition, which yields $s_2 = (\sqrt{n}-1)^2 r$. If $0 \le n \le 1$, we have $s_2 = \tau|AK| + 2\sqrt{rs_2}$, which also yields $s_2 = (\sqrt{n}-1)^2 r$. Therefore we have $\sqrt{s_2} = |\sqrt{n}-1|\sqrt{r}$ in any case. The part (ii) follows from (i), since $|AK| = |n-1|r$. Eliminating n from the two equations in (i) we get (iii). □

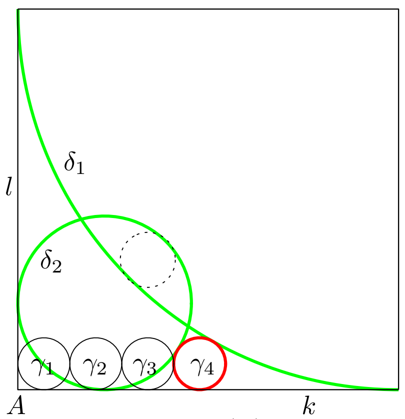

Figure 13: $\mathcal{T}(n)$, $n = 8$.

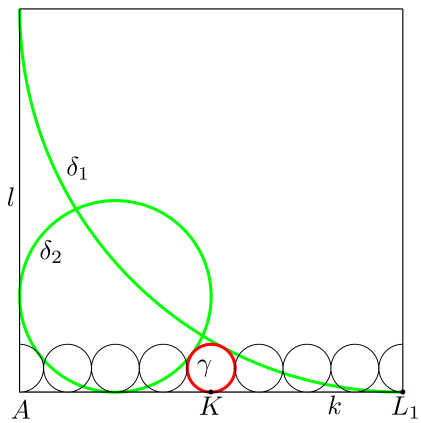

Figure 14: $\mathcal{T}(9)$.

If $n = 8$, then δ_1 and δ_2 intersect and the maximal circle touching δ_1 and δ_2 from inside of them has radius r, which is obtained by translating γ_3 parallel to l through distance $4r$ (see Figure 13). Let L_i be the point of tangency of δ_i and k. If $n = 9$, then $s_1 = 4s_2 = 16r$ by Theorem 3.1(i) and K is the midpoint of AL_1 (see Figure 14). Problems considering this case with the circle δ_2 can be found in [18, 19], [22, 23] and [24]. However the circle δ_2 seems to be ignored for $\mathcal{T}(n)$ in most cases except this case in Wasan geometry.

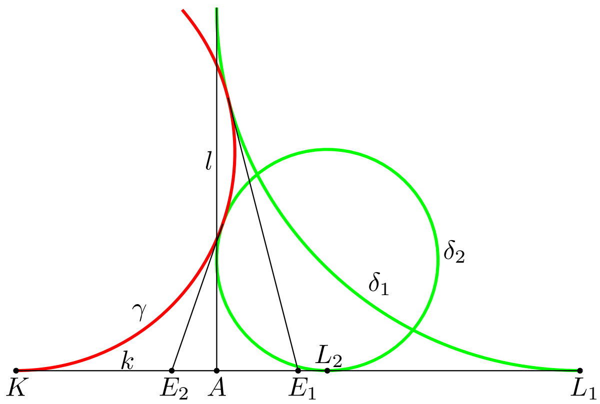

Figure 15: $\mathcal{T}(n)$, $(0 < n < 1)$.

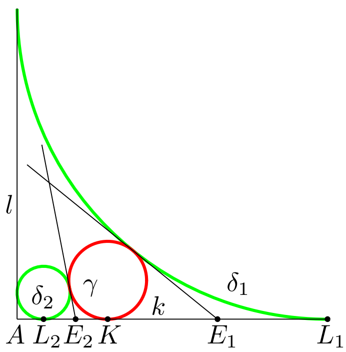

Figure 16: $\mathcal{T}(n)$, $(1 < n)$.

Let E_i be the point of intersection of k and the internal common tangent of δ_i and γ for $\mathcal{T}$, if δ_i and γ are proper circles (see Figures 15 and 16). Notice that E_i is the midpoint of the segment KL_i. If $\gamma = \gamma(\overline{0})$, then $K = L_1 = L_2$. Therefore we can consider that the point E_i coincides with L_i in this case. Hence we define $E_1 = E_2 = L_1$ for $\mathcal{T}(\overline{0})$ (see Figure 17). Similarly we define $E_2 = L_2 = A$ for $\mathcal{T}(1)$ (see Figure 18).

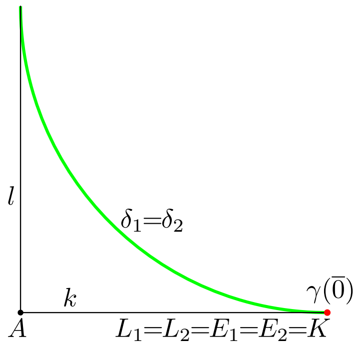

Figure 17: $\mathcal{T}(\overline{0})$.

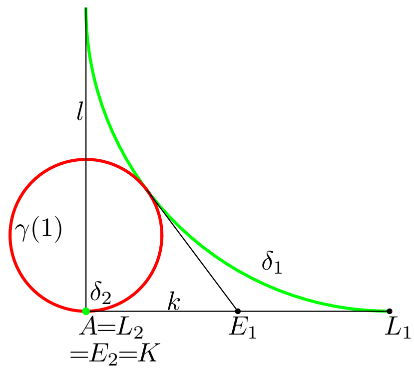

Figure 18: $\mathcal{T}(1)$.

Theorem 3.2. *If $n \neq \overline{0}$, then $\mathcal{T} = \mathcal{T}(n)$ if and only if the following relation holds:*

$$|AE_i| = \sqrt{n}|E_iL_i| \quad for \quad i = 1, 2. \tag{2}$$

Proof. Let $n \neq \overline{0}$. We assume $\mathcal{T} = \mathcal{T}(n)$. Then (2) holds if $n = 0$ since $E_i = A$. Also (2) holds if $n = 1$. Let $n \neq 0, 1$. Since E_i is the midpoint of the segment L_iK, $|E_1L_1| = \sqrt{s_1r} = (\sqrt{n}+1)r$ and $|E_2L_2| = \sqrt{s_2r} = |\sqrt{n}-1|r$ by Proposition 2.1 and Theorem 3.1(i). On the other hand,

$$|AE_1| = s_1 - |E_1L_1| = (\sqrt{n}+1)^2r - (\sqrt{n}+1)r = \sqrt{n}(\sqrt{n}+1)r = \sqrt{n}|E_1L_1|.$$

Therefore we get (2) for $i = 1$. If $0 < n < 1$, the internal common tangent of γ and δ_2 is obtained by rotating l about the center of δ_2 so that the point of intersection of the image of l and k moves from A to K (see Figure 15). Therefore E_2 lies between A and K in this case. Also E_2 lies between A and K in the case $1 < n$. Therefore in any case, we get

$$|AE_2| = |AK| - \frac{|L_2K|}{2} = |n-1|r - \sqrt{s_2r} = |n-1|r - |\sqrt{n}-1|r = \sqrt{n}|\sqrt{n}-1|r.$$

Hence we also get (2) for $i = 2$. Therefore $\mathcal{T} = \mathcal{T}(n)$ implies (2).

Conversely we assume (2) and $\mathcal{T} = \mathcal{T}(m)$ for a real number m. If $|E_1L_1| = 0$, then $|AE_1| = 0$ by (2), i.e., $L_1 = E_1 = A$, a contradiction. Hence $|E_1L_1| \neq 0$. With this fact and $|AE_1| = \sqrt{m}|E_1L_1|$ as proved just above, we get $\sqrt{n}|E_1L_1| = \sqrt{m}|E_1L_1|$. Therefore $m = n$, i.e., $\mathcal{T} = \mathcal{T}(n)$.

□

4. ANOTHER TOUCHING CIRCLE

There are two circles touching the circles δ_1 and δ_2 externally and k in general for the figure $\mathcal{T}$. However we have considered only one circle in the previous section. In this section we consider the figure together with the remaining touching circle. Let $\gamma_i = \gamma(n_i)$ $(i = 1, 2)$ be the circle of radius r_i such that $0 < r_2 \leq r_1$ touching δ_1 and δ_2 externally and k from the same side as δ_1. We denote the figure consisting of γ_i, δ_i, k and l by $\mathcal{U}$.

Theorem 4.1. *The following relations hold for $\mathcal{U}$:*

$$n_1 = \frac{1}{n_2} = \frac{r_2}{r_1}. \tag{3}$$

Proof. Since $0 \le n_1 \le 1$ and $1 \le n_2$, $\sqrt{s_1}$ and $\sqrt{s_2}$ equal $(\sqrt{n_1}+1)\sqrt{r_1} = (\sqrt{n_2}+1)\sqrt{r_2}$ and $(1-\sqrt{n_1})\sqrt{r_1} = (\sqrt{n_2}-1)\sqrt{r_2}$, respectively by Theorem 3.1(i). Solving the two equations for n_1 and n_2, we get $n_1 = r_2/r_1$ and $n_2 = r_1/r_2$. □

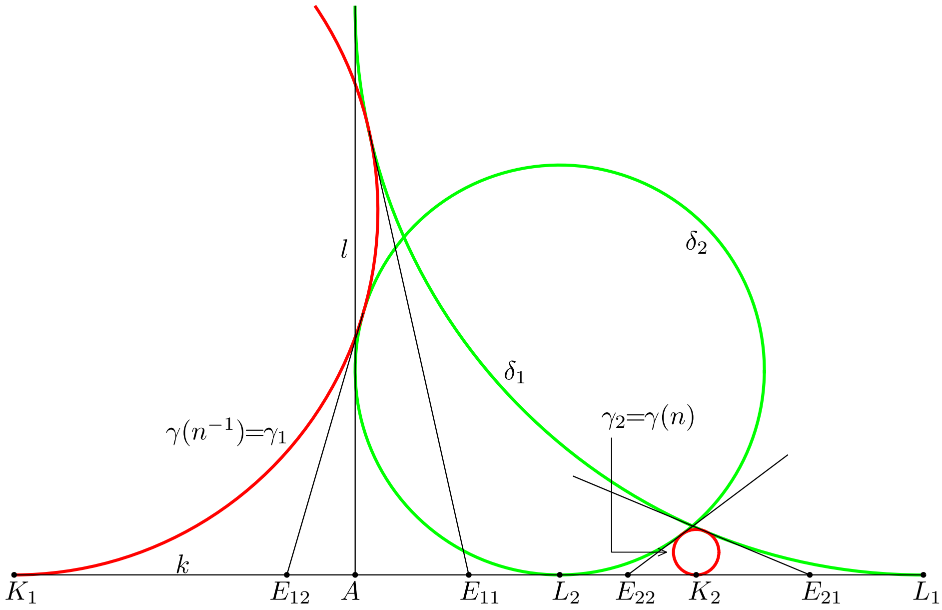

Figure 19: $\mathcal{U}(n)$ $(n = 16)$.

We now explicitly denote the figure $\mathcal{U}$ by $\mathcal{U}(n)$ if $\gamma_2 = \gamma(n)$, or equivalently $\gamma_1 = \gamma(n^{-1})$, which coincides with $\mathcal{T}(n) \cup \mathcal{T}(n^{-1})$ for a real number $n \ge 1$ (see Figure 19). We also denote the figure $\mathcal{T}(0) \cup \mathcal{T}(\overline{0})$ by $\mathcal{U}(0)$ (see Figure 20). Notice that $n = 0$ or $1 \le n$ by the definition for $\mathcal{U}(n)$. The point of tangency of γ_i and k is denoted by K_i. Let t_{ij} be the internal common tangent of the proper circles γ_i and δ_j. The point of intersection of t_{ij} and k is denoted by E_{ij}. We also define $E_{2i} = L_1$ for $\mathcal{U}(0)$ (see Figure 20), and $E_{i2} = A$ for $\mathcal{U}(1)$ (see Figure 21). The next theorem follows from Theorem 3.2.

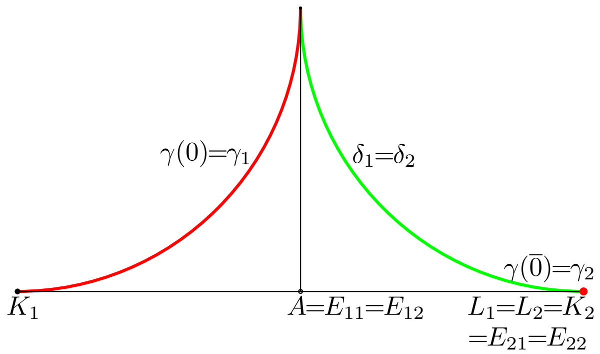

Figure 20: $\mathcal{U}(0)$.

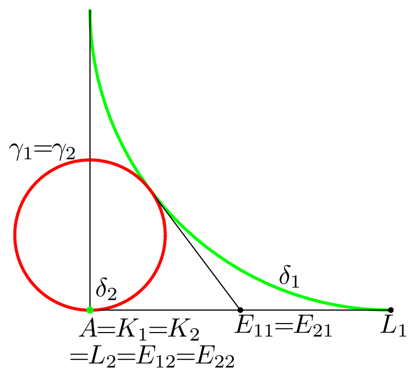

Figure 21: $\mathcal{U}(1)$.

Theorem 4.2. *If $n \ne 0$, the following statements are equivalent.*
(i) $\mathcal{U} = \mathcal{U}(n)$. (ii) $\dfrac{|AE_{2i}|}{\sqrt{n}} = |E_{2i}L_i|$. (iii) $|AE_{1i}| = \dfrac{|E_{1i}L_i|}{\sqrt{n}}$.

Theorem 4.3. *The following relations hold for $\mathcal{U}(n)$.*
(i) $|AK_1| = |AK_2|$.
(ii) $\sqrt{r_1} = \dfrac{\sqrt{s_1}+\sqrt{s_2}}{2}$ *and* $\sqrt{r_2} = \dfrac{\sqrt{s_1}-\sqrt{s_2}}{2}$.

(iii) $\sqrt{s_1} = \sqrt{r_1} + \sqrt{r_2}$ *and* $\sqrt{s_2} = \sqrt{r_1} - \sqrt{r_2}$.
(iv) $|AE_{1i}| = |E_{2i}L_i|$ *and* $|AE_{2i}| = |E_{1i}L_i|$.

Proof. The part (i) follows from Theorem 3.1(ii). The part (ii) holds by Theorem 3.1(iii). The part (iii) follows from (ii). By Theorem 4.2, we have $|AE_{1i}| = |E_{1i}L_i|/\sqrt{n} = \sqrt{r_1 s_i}/\sqrt{n} = \sqrt{nr_2 s_i}/\sqrt{n} = \sqrt{r_2 s_i} = |E_{2i}L_i|$. The rest of (iv) is proved similarly. □

Theorem 4.4. *The radical axis of the circles γ_1 and γ_2 passes through the point A and the farthest point on δ_i from k for the figure $\mathcal{U}$.*

Proof. We use a rectangular coordinate system with origin A such that B has coordinates $(s, 0)$. Then the circles γ_1 and γ_2 are expressed by the equations $c_1 = (x + |AK_1|))^2 + (y - r_1)^2 - r_1^2$ and $c_2 = (x - |AK_2|))^2 + (y - r_2)^2 - r_2^2$, respectively. This implies $c_1 - c_2 = 2\sqrt{s_1 s_2}(2x - y)$ by Theorem 3.1(ii) and Theorem 4.3(i). Therefore the radical has an equation $y = 2x$. □

5. Special cases, golden mean and silver mean

In this section we consider special cases for the figure $\mathcal{U}(n)$, and show unexpected facts that the golden mean and the silver mean appear when certain circles of $\mathcal{U}(n)$ touch.

5.1. **Golden mean.** Two quantities are said to be in the golden mean or in the golden ratio if the ratio of those quantities equals $1 : \phi$, where $\phi = (1 + \sqrt{5})/2$. The next theorem shows that the golden mean appears if the circles γ_1 and γ_2 touch for $\mathcal{U}(n)$ (see Figure 22). Let I_i be the farthest point on γ_i from k.

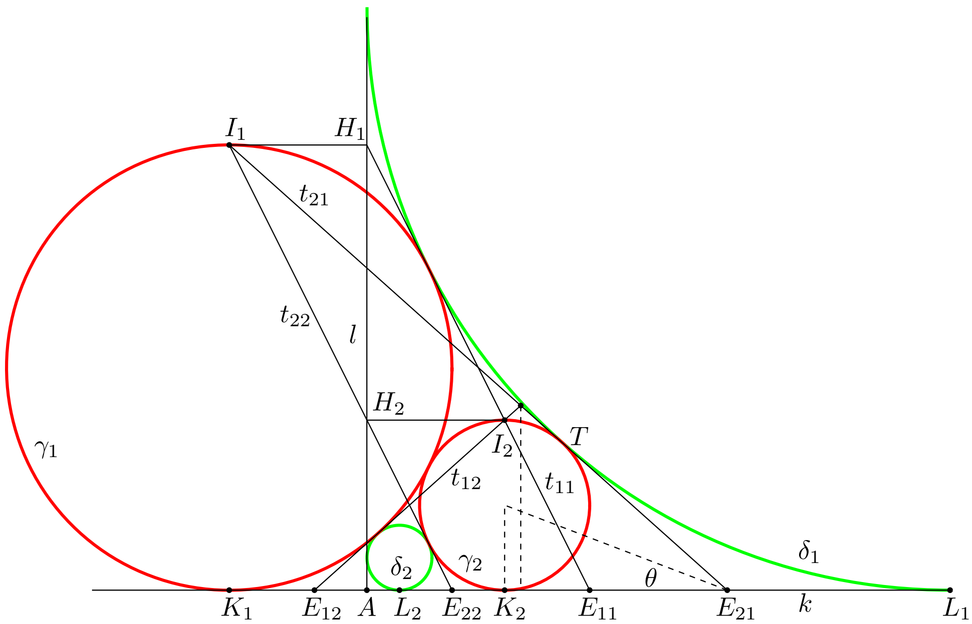

Figure 22: $\mathcal{U}(\phi^2)$.

Theorem 5.1. *The following statements are equivalent for $\mathcal{U}(n)$.*
(i) *The circles γ_1 and γ_2 touch.* (ii) $\sqrt{n} = \phi$.
(iii) $|AE_{2i}| = \phi|E_{2i}L_i|$. (iv) $|AE_{1i}| = \phi^{-1}|E_{1i}L_i|$.

(v) *The line* t_{1i} *(resp.* t_{2i}*) passes through the point* I_2 *(resp.* I_1*).*
(vi) *There is a similar transformation* f *such that* $f(\gamma_1) = \delta_1$ *and* $f(\delta_2) = \gamma_2$*.*

Proof. The statement (i) is equivalent to $|K_1K_2| = 2\sqrt{r_1r_2}$ by Proposition 2.1, while $|K_1K_2| = 2\sqrt{r_1s_2}+2\sqrt{r_2s_2}$. Therefore (i) is equivalent to $2\sqrt{r_1s_2}+2\sqrt{r_2s_2} = 2\sqrt{r_1r_2}$. While by Theorem 4.3(iii) and $r_2 = r_1/n$, we have

$$2\sqrt{r_1s_2} + 2\sqrt{r_2s_2} - 2\sqrt{r_1r_2} = 2(\sqrt{n} - \phi)(\sqrt{n} + \phi^{-1})r_2.$$

Therefore (i) and (ii) are equivalent. The equivalence of (ii), (iii) and (iv) follows from Theorem 4.2.

We prove the equivalence of (ii) and (v). Let T be the point of tangency of γ_2 and δ_1. Let $2\theta = \angle TE_{21}A$. Then (v) is equivalent to $\tan 2\theta = \tan\angle I_1E_{21}A$. While $\tan\theta = r_2/|E_{21}K_2| = r_2/\sqrt{r_2s_1} = \sqrt{r_2/s_1}$ by Proposition 2.1. Therefore

$$\tan 2\theta = \frac{2\sqrt{r_2s_1}}{s_1 - r_2} = \frac{2(\sqrt{n}+1)}{n+2\sqrt{n}}. \tag{4}$$

While by Proposition 2.1, Theorem 4.3(iii) and $r_1 = nr_2$, we also have

$$\tan\angle I_1E_{21}A = \frac{2r_1}{|K_1L_1| - |E_{21}L_1|} = \frac{2r_1}{2\sqrt{r_1s_1} - \sqrt{r_2s_1}} = \frac{2n}{2n+\sqrt{n}-1}.$$

Therefore

$$\tan 2\theta - \tan\angle I_1E_1A = -\frac{2(n-\phi^2)(n-\phi^{-2})}{(n+2\sqrt{n})(2n+\sqrt{n}-1)}.$$

The last equation shows that (ii) is equivalent to that the line t_{i1} passes through the point I_1 for $i = 2$, since $1 < n$. The case $i = 1$ is proved similarly. The equivalence of (ii) and the rest of (v) are proved in a similar way. The part (vi) is equivalent to (ii) since

$$\frac{s_1}{r_1} - \frac{r_2}{s_2} = \frac{(n-\phi^2)(n-\phi^{-2})}{n(\sqrt{n}-1)^2}.$$

□

We assume (i). Then t_{11} and t_{22} are parallel, since (v) holds. While $|AE_{11}| - |K_1E_{22}| = (\sqrt{r_1s_1} - |AK_1|) - (|AK_1| + s_2 + \sqrt{r_2s_2}) = 0$, i.e., $|AE_{11}| = |K_1E_{22}|$ by Theorem 3.1(ii) and Theorem 4.3(iii). Therefore if H_i is the point of intersection of the lines t_{ii} and l, then $H_1I_1E_{22}E_{11}$ is a parallelogram. Similarly $I_2H_2E_{22}E_{11}$ is a parallelogram. Since $s_1/r_1 = r_2/s_2$ is equivalent to $s_1/r_2 = r_1/s_2$, there is a similar transformation g such that $g(s_1) = r_2$ and $g(r_1) = s_2$ since (vi) holds. Therefore the internal common tangent of δ_1 and γ_2 and the internal common tangent of γ_1 and δ_2 are symmetric about the perpendicular from their point of intersection to k. The internal common tangent of γ_1 and γ_2 passes through the point A by Proposition 2.1 and Theorem 4.3(i).

5.2. **Silver mean.** Two quantities are said to be in the silver mean or the silver ratio if the ratio of those quantities equals $1 : \rho$, where $\rho = 1 + \sqrt{2}$. Let J_i be the farthest point on δ_i from k. The next theorem shows that the silver mean appears when the circles δ_1 and δ_2 touch (see Figure 23).

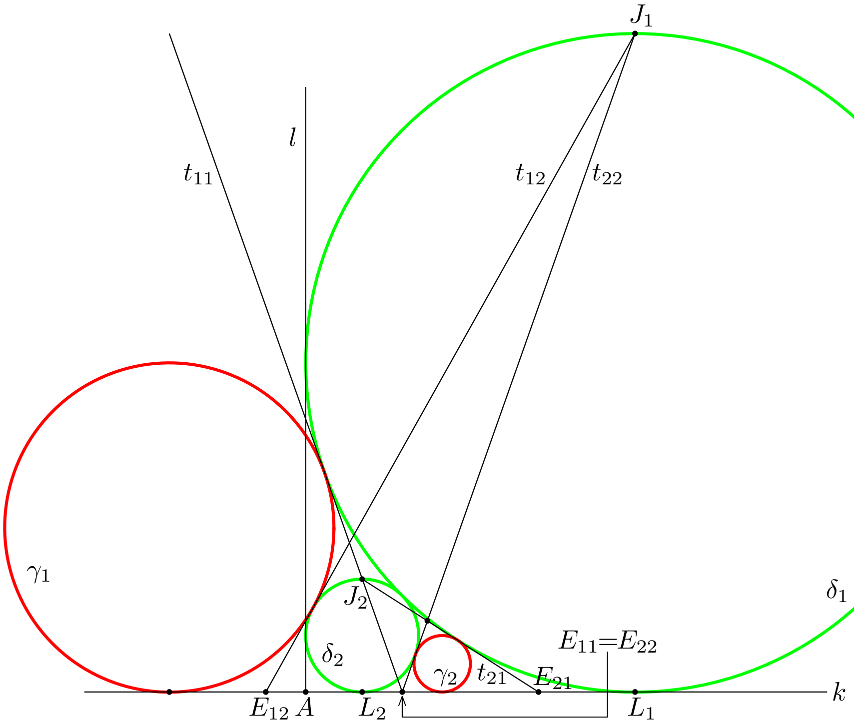

Figure 23: $\mathcal{U}(\rho^2)$.

Theorem 5.2. *The following statements are equivalent for* $\mathcal{U}(n)$.
(i) *The circles* δ_1 *and* δ_2 *touch.* (ii) $\sqrt{s_1} = \rho\sqrt{s_2}$.
(iii) $\sqrt{n} = \rho$. (iv) $|AE_{2i}| = \rho|E_{2i}L_i|$.
(v) $|AE_{1i}| = \rho^{-1}|E_{1i}L_i|$. (vi) *The points* E_{11} *and* E_{22} *coincide.*
(vii) *The line* t_{i1} (*resp.* t_{i2}) *passes through the point* J_2 (*resp.* J_1).
(viii) *There is a similar transformation* f *such that* $f(\gamma_1) = \delta_1$ *and* $f(\gamma_2) = \delta_2$.

Proof. The statement (i) is equivalent to that δ_2 is the incircle of the curvilinear triangle made by δ_1, k and l, which is equivalent to $|L_1A| = 2\sqrt{s_1s_2} + s_2 = s_1$. While by Theorem 4.3(iii) and $r_2 = r_1/n$, we have

$$2\sqrt{s_1s_2} + s_2 - s_1 = (\rho\sqrt{s_2} - \sqrt{s_1})(\rho^{-1}\sqrt{s_2} + \sqrt{s_1}) = 2(\sqrt{n} - \rho)(\sqrt{n} + \rho^{-1})r_2.$$

Therefore (i), (ii), (iii) are equivalent. The equivalence of (iii) (iv) and (v) follows from Theorem 4.2. Since $|AE_{22}| = |AL_2| + |L_2E_{22}| = s_2 + \sqrt{r_2s_2}$ and $|AE_{11}| = |AL_1| - |L_1E_{11}| = s_1 - \sqrt{r_1s_1}$, we get

$$|AE_{22}| - |AE_{11}| = (\sqrt{n} - \rho)(\sqrt{n} + \rho^{-1})r_2.$$

Hence (iii) and (vi) are equivalent. We prove the equivalence of (iii) and (vii). Let T be the point of tangency of γ_2 and δ_1 and $2\theta = \angle TE_{21}A$. Then (4) holds. While we have

$$\tan\angle JE_{21}A = \frac{2s_2}{|L_2K_2| + |K_2E_{21}|} = \frac{2s_2}{2\sqrt{r_2s_2} + \sqrt{r_2s_1}} = \frac{2(\sqrt{n}-1)^2}{3\sqrt{n}-1}.$$

Therefore

$$\tan 2\theta - \tan JE_1A = \frac{-2(n-\rho^2)(n-\rho^{-2})}{\sqrt{n}(3\sqrt{n}-1)(\sqrt{n}+2)}.$$

Therefore (iii) is equivalent to that t_{i1} passes through the farthest point on δ_2 from k for $i = 2$, since $1 < n$. The case $i = 1$ can be proved similarly. The equivalence of (iii) and the rest of (vii) is proved in a similar way. The equivalence of (iii) and (viii)follows from

$$\frac{s_1}{r_1} - \frac{s_2}{r_2} = -\frac{(n+1)(\sqrt{n}-\rho)(\sqrt{n}+\rho^{-1})}{n}.$$

□

For the figure $\mathcal{U}(\rho^2)$, the equivalence of (iii) and (viii) shows that t_{11} and t_{22} are symmetric about the perpendicular to k at the point $E_{11} = E_{22}$. Also Theorem 4.1 and Theorem 4.3(iii) shows $\sqrt{s_1} = \sqrt{r_1} + \sqrt{r_2} = \sqrt{r_1}(1 + 1/\rho) = \sqrt{2r_1}$. Hence we have $s_1 = 2r_1$. This also implies $s_2 = 2r_2$ by (viii).

5.3. **Steiner chain.** We consider the case in which there is a circle touching γ_i and δ_i externally for $\mathcal{U}(n)$ (see Figure 24). In this case γ_1, δ_1, γ_2, δ_2 form a Steiner chain touching this circle and k. It was known that if C_i $(i = 1, 2, 3, 4)$ form a Steiner chain and v_i is the curvature of C_i, then $v_1 + v_3 = v_2 + v_4$ holds [21]. While

$$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{s_1} - \frac{1}{s_2} = \frac{(n+1)(n-(2+\sqrt{3}))(n-(2-\sqrt{3}))}{(n-1)^2 r_1}$$

by Theorem 4.3(iii) and $r_1 = nr_2$. Therefore we get $n = 2+\sqrt{3}$ in this case. Let ε and e be the circle touching γ_i and δ_i and its radius. Considering another Steiner chain touching ε and k symmetric about the perpendicular from the center of ε to k, we see that the distance from the center of ε to k equals $3e$. Since γ_1, ε, γ_2 and k also form a Steiner chain touching δ_1 and δ_2, we have

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2}.$$

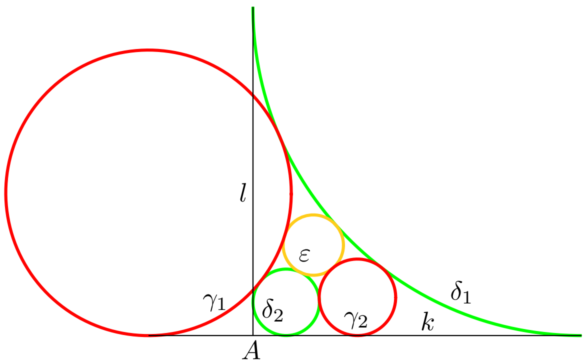

Figure 24: $\mathcal{U}(2+\sqrt{3})$.

6. THE CASE $n = 0, \overline{0}$ WITH DIVISION BY ZERO

From now on we assume that the symbol $\overline{0}$ has value 0, i.e., $\overline{0} = 0$ as a number, though $\overline{0}$ and 0 are different as symbols. From now on we also assume the definition of the division by zero in [7]:

$$\frac{n}{0} = 0 \quad \textit{for any real number } n. \tag{5}$$

Notice that reduction for fractions of zero denominator can not be done with this definition, i.e., $c = 0$ implies

$$\frac{ac}{bc} \neq \frac{a}{b}$$

in general. For the left side always equals $0/0 = 0 \neq a/b$ by (5).

We consider Theorem 3.2 in the case $n = \overline{0}$. By the definition of the value of $\overline{0}$, (2) does not hold if $n = \overline{0}$, since $|E_iL_i| = 0$ and $|AE_i| \neq 0$ for $\mathcal{T}(\overline{0})$. But if we state the relation in the following form, it still holds in the case $n = \overline{0}$ since both sides equal 0:

$$\frac{|AE_i|}{\sqrt{n}} = |E_iL_i| \quad for \quad i = 1, 2. \tag{6}$$

Conversely, if (6) holds for $n = \overline{0}$, then we get $|E_iL_i| = 0$, i.e., $E_i = L_i$. Hence we get $\mathcal{T} = \mathcal{T}(\overline{0})$.

If $\mathcal{U} = \mathcal{U}(0)$, then $n_1 = 0$, $n_2 = \overline{0}$ and $r_2 = 0$. Hence Theorem 4.1 holds in this case. Theorem 4.2 also holds in the case $n = 0$, since $\mathcal{U} = \mathcal{U}(0)$ is equivalent to $|E_{2i}L_i| = |AE_{1i}| = 0$.

7. Parametric representation of the generalized Haga's fold

We now consider Haga's fold considered in [9, 13]. Let $ABCD$ be a square. For a point E on the line DA, let m be the perpendicular bisector of the segment CE. The figure consisting of $ABCD$ and the reflection of $ABCD$ in the line m is called the figure made by the generalized Haga's fold determined by E or simply called the figure determined by E and denoted by $\mathcal{H}(E)$. We call m the crease line of $\mathcal{H}(E)$. In this figure the reflections of A B and D in m are not so important and we do not refer to them in most cases. Identifying similar figures, $\mathcal{H}(E)$ is determined uniquely by the square $ABCD$ and the point E. Ordinary Haga's fold is obtained if E lies between D and A (see Figures 27 and 28). Let δ be the circle of radius $s = |AB|$ and center C. In this section we give a parametric representation of $\mathcal{H}(E)$ using circles touching the line AB and the circle δ externally.

7.1. Parametric representation.

Let T be the point of tangency of δ and the remaining tangent of δ from E for $\mathcal{H}(E)$. Let γ be the circle touching δ externally at T and the line AB. Then $\gamma \longmapsto \mathcal{H}(E)$ is a bijection from the set of the circles touching δ externally and the line AB from the same side as δ to the set of the figures determined by E, where we consider that the point B is a member of the former set as a point circle, which corresponds to the figure made by E in the case $E = B$ (see Figure 30).

For two points P and Q on the line AB, $P < Q$ denotes that $\overrightarrow{PQ}$ has the same direction as $\overrightarrow{AB}$, and $P \leq Q$ denotes $P < Q$ or $P = Q$. Let K be the point of tangency of γ and the line AB and let r be the radius of γ. We define

$$n = \frac{\sigma(\tau|AK| + r)}{r}, \tag{7}$$

where $\sigma = 1$ if T lies inside of $ABCD$ or on the perimeter of $ABCD$ otherwise $\sigma = -1$ and $\tau = 1$ if $A \leq K$ otherwise $\tau = -1$. If $E = B$, the points K and T coincide with D (see Figure 30). In this case we use the symbol $\overline{0}$, and consider $n = \overline{0}$. We now explicitly denote the circle γ by $\gamma(n)$. The point circle B is also denoted by $\gamma\left(\overline{0}\right)$. Now any circle touching δ externally and the line DA can be expressed by $\gamma(n)$ for a real number n together with $\overline{0}$, and we also explicitly denote the figure $\mathcal{H}(E)$ by $\mathcal{H}(n)$[*1].

[*1] $\mathcal{H}\left(\overline{0}\right)$ is denoted by $\mathcal{H}(\infty)$ in [9]

7.2. **Seven cases.** We consider the value of n for $\mathcal{H}(n)$ as a function of the point E, which moves on the line AB with moving direction same as to $\overrightarrow{AB}$. In this case T moves on δ counterclockwise. Let M be the midpoint of AB, and let F be the point of intersection of the line DA and the reflection of the line CD in m if they meet. We consider the following seven cases:

1. $E < A$ (see Figure 25).
2. $E = A$ (see Figure 26).
3. $A < E < M$ (see Figure 27).
4. $E = M$ (see Figure 28).
5. $M < E < B$ (see Figure 29).
6. $E = B$ (see Figure 30).
7. $B < E$ (see Figure 31).

Assume $E < A$ (see Figure 25). Then $\sigma = \tau = -1$. Hence we have

$$n = \frac{-(-|AK| + r)}{r} = \frac{|BK| - |AB| - r}{r} = \frac{2\sqrt{sr} - s - r}{r} = -\left(\sqrt{\frac{s}{r}} - 1\right)^2. \tag{8}$$

While $s < r$, i.e., $0 < \sqrt{s/r} < 1$. Therefore we get $-1 < n < 0$ and n increases and approaches to 0 when E approaches to A. If $E = A$, then $n = 0$ (see Figure 26).

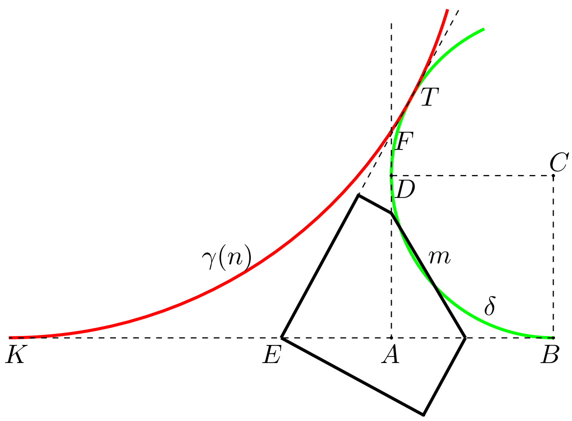

Figure 25: $-1 < n < 0$, $E < A$.

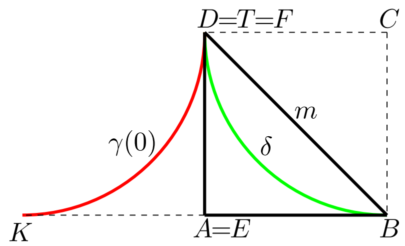

Figure 26: $\mathcal{H}(0)$, $E = A$.

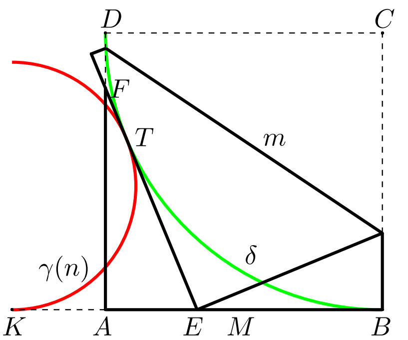

Figure 27: $0 < n < 1$, $A < E < M$.

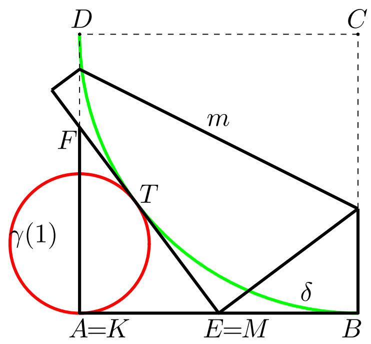

Figure 28: $\mathcal{H}(1)$, $E = M$.

If $E = M$, we get $n = 1$ by Theorem 3.2 (see Figure 28). Therefore $|AK| = 0$, i.e., $K = A$ in this case. Also we get $s = 4r$ by Theorem 4.3(iii). Assume $A < E < M$. Then $\sigma = 1$ and $\tau = -1$ (see Figure 27). Hence

$$n = \frac{-|AK| + r}{r} = \frac{|AB| - |BK| + r}{r} = \frac{s - 2\sqrt{sr} + r}{r} = \left(\sqrt{\frac{s}{r}} - 1\right)^2.$$

While $s/4 < r < s$, i.e., $1 < \sqrt{s/r} < 2$. Therefore we get $0 < n < 1$ and n increases and approaches to 1 when E approaches to M.

If $M < E < B$ (see Figure 29), then $\sigma = \tau = 1$. Hence

$$n = \frac{|AK| + r}{r} = \frac{|AB| - |BK| + r}{r} = \frac{s - 2\sqrt{sr} + r}{r} = \left(\sqrt{\frac{s}{r}} - 1\right)^2.$$

While $r < s/4$, i.e., $2 < \sqrt{s/r}$. Therefore $1 < n$, and n increase without limit when E approaches to B, since r approaches to 0. If $E = B$, $r = 0$ and $\mathcal{H}(E)$ is denoted by $\mathcal{H}(\overline{0})$ (see Figure 30). While the denominator of the right side of (7) equals 0, where recall the definition (5). Therefore the right side of (7) equals 0, which ensures consistency of our definition $\overline{0} \neq 0$ as symbols but $\overline{0} = 0$ as numbers. Also recall the remark after (5), i.e., $(s - 2\sqrt{sr} + r)/r \neq (\sqrt{s/r} - 1)^2$ in this case.

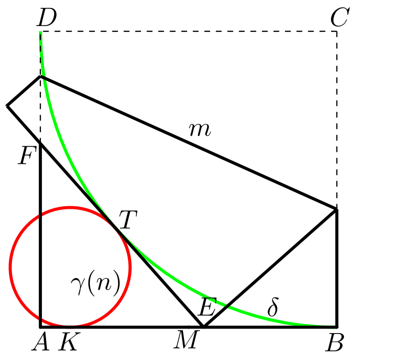

Figure 29: $1 < n$, $M < E < B$.

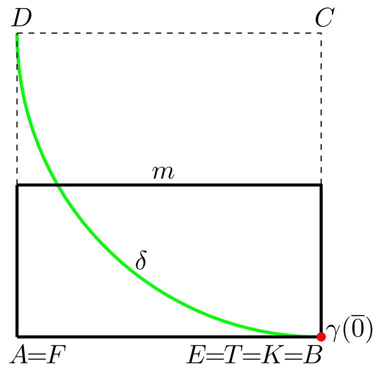

Figure 30: $\mathcal{H}\left(\overline{0}\right)$, $E = B$.

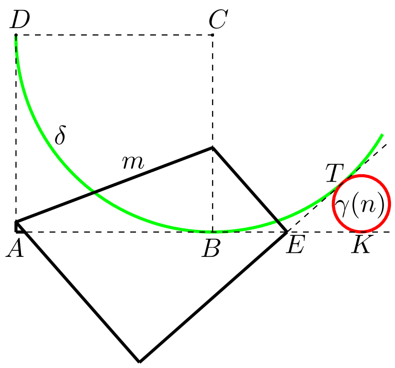

Figure 31: $-1 < n < 0$, $B < E$.

Assume $B < E$ (see Figure 31). Then $\sigma = -1$ and $\tau = 1$. Hence

$$n = \frac{-(|AK| + r)}{r} = \frac{-|AB| - |BK| - r}{r} = \frac{-s - 2\sqrt{sr} - r}{r} = -\left(\sqrt{\frac{s}{r}} + 1\right)^2.$$

While $0 < r$. Therefore n decreases without limit when E approaches to B, since r approaches to 0. Contrarily n increases and approaches to -1 when E moves away from B, since r increases without limit. Therefore $n < -1$ in this case.

We summarize the results in Table 1. The positively sloped arrows mean that n is a monotonically increasing function of E when E moves on the line AB with moving direction same as to $\overrightarrow{AB}$.

case	$E<A$	$E=A$	$A<E<B$	$E=B$	$E<B$
n	$-1<n<0$	0	$0<n$	0	$n<-1$
	$\nearrow$		$\nearrow$		$\nearrow$

Table 1.

Table 1 shows that $n \neq -1$ for $\mathcal{H}(n)$, while the remaining tangent of the circle δ parallel to DA is not a member of the set of circles touching the line DA and δ externally. Therefore the fact suggests us to describe the tangent by $\gamma(-1)$.

7.3. **The case m passing through inside of $ABCD$.** We consider the case in which the line m passes through inside of $ABCD$. In this case we can really fold the square $ABCD$ with the real crease line m (see Figures from 25 to 31). Firstly we consider the case $E < A$. Let I be the point on the line DA such that $I < A$ and $|AI| = \sqrt{2}s$. Then m passes through A if and only if $E = I$ (see Figure 32). In this case $|AK| = |IK|+|AI| = |BK|-|AB|$ holds. Hence we get $\sqrt{rs}+\sqrt{2}s = 2\sqrt{rs}-s$, which implies $\sqrt{s/r} = 3 - 2\sqrt{2}$, i.e., $n = -4(3 - 2\sqrt{2}) = -0.6862\cdots$ by (8). Therefore m does not pass through inside of $ABCD$ if $E \leq I$, and passes through inside of $ABCD$ if $I < E < A$, which is equivalent to $-4(3 - 2\sqrt{2}) < n < 0$.

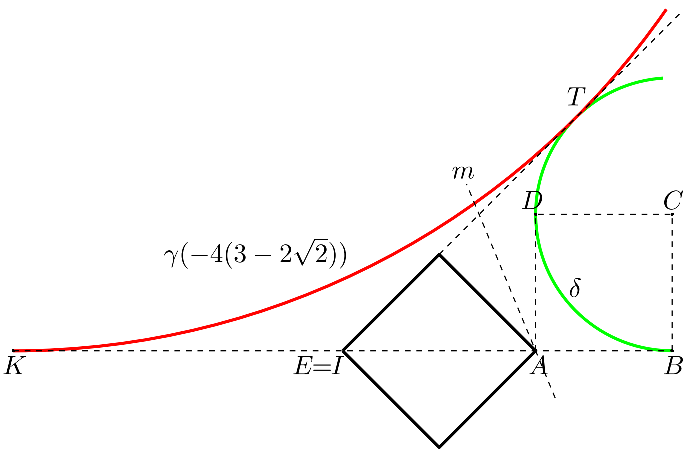

Figure 32: $\mathcal{H}(-4(3 - 2\sqrt{2}))$.

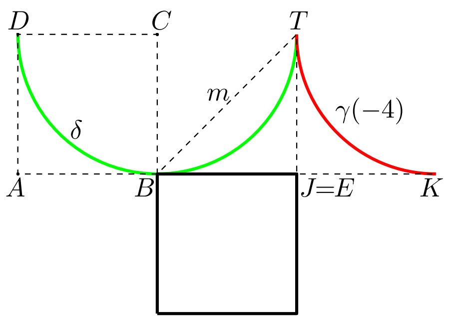

Figure 33: $\mathcal{H}(-4)$.

If $A \leq E \leq B$, m passes through inside of $ABCD$ (see Figures from 26 to 30). Therefore m passes through inside of $ABCD$ if $0 \leq n$. We consider the case $B < E$. Let J be the reflection of A in BC (see Figure 33). It is obvious that m passes through B if $E = J$ and $n = -4$ in this case. Therefore m passes through inside of $ABCD$ if and only if $n < -4$. Hence we get the next theorem.

Theorem 7.1. *For the figure $\mathcal{H}(E)$, the line m passes through A (resp. B) if and only if $E = I$ (resp. $E = J$), which is equivalent to $n = -4(3 - 2\sqrt{2})$ (resp. $n = -4$). Also m passes through inside of $ABCD$ if and only if $I < E < J$, which is equivalent to $-4(3 - 2\sqrt{2}) < n$ or $n < -4$.*

8. INVERSE OF GENERALIZED HAGA'S FOLD

Let $\mathcal{H}(E_i) = \mathcal{H}(n_i)$ $(i = 1, 2)$ for a point E_i on the line DA and a real number n_i. Then $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ are said to be inverses to each other if and only if $n_1 = 1/n_2$, which is equivalent to $n_1 n_2 = 1$ or $\{n_1, n_2\} = \{0, \overline{0}\}$ by (5). In this section we consider two figures $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ which are inverses to each other.

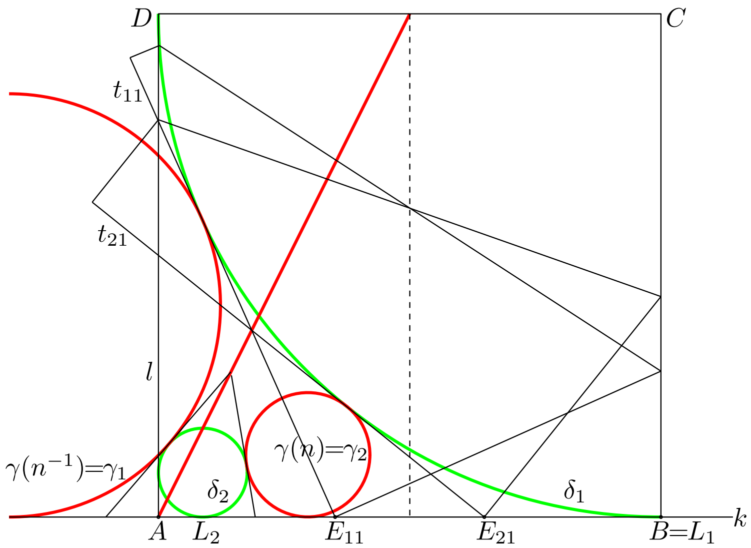

Figure 34: $\mathcal{H}(E_{11}) = \mathcal{H}(n^{-1})$ and $\mathcal{H}(E_{21}) = \mathcal{H}(n)$ for $1 < n$.

We show that any pair of figures made by generalized Haga's fold inverses to each other are derived from the figure $\mathcal{U}(n)$ considered in section 4, where recall that $n = 0$ or $1 \leq n$ for $\mathcal{U}(n)$. Let us define the square $ABCD$ for $\mathcal{U}(n)$ so that $B = L_1$, C is the center of the circle δ_1, D is the point of tangency of δ_1 and the line l. Then $\mathcal{H}(E_{11}) = \mathcal{H}(n^{-1})$ and $\mathcal{H}(E_{21}) = \mathcal{H}(n)$ if $n \neq 0$ (see Figure 34). If $n = 0$, we get $\mathcal{H}(E_{11}) = \mathcal{H}(0)$ and $\mathcal{H}(E_{21}) = \mathcal{H}(\overline{0})$ (see Figures 20, 26 and 30). Assume $n \neq 0$. If we consider the square $ABCD$ for $\mathcal{U}(n)$ such that $B = L_2$, C is the center of δ_2, D is the point of tangency of δ_2 and l, then $\mathcal{H}(E_{12}) = \mathcal{H}(-n^{-1})$ and $\mathcal{H}(E_{22}) = \mathcal{H}(-n)$ (see Figure 35).

Since t_{11} is the radical axis of γ_1 and δ_1, it passes through the radical center of γ_1, γ_2 and δ_1. Similarly t_{21} passes through the radical center of γ_1, γ_2 and δ_1. Therefore the point of intersection of t_{11} and t_{21} passes through the radical center of the three circles, i.e., it lies on the line passing through A and the midpoint of CD by Theorem 4.4. Similarly the point of intersection of t_{12} and t_{22} meet in a point on the same line.

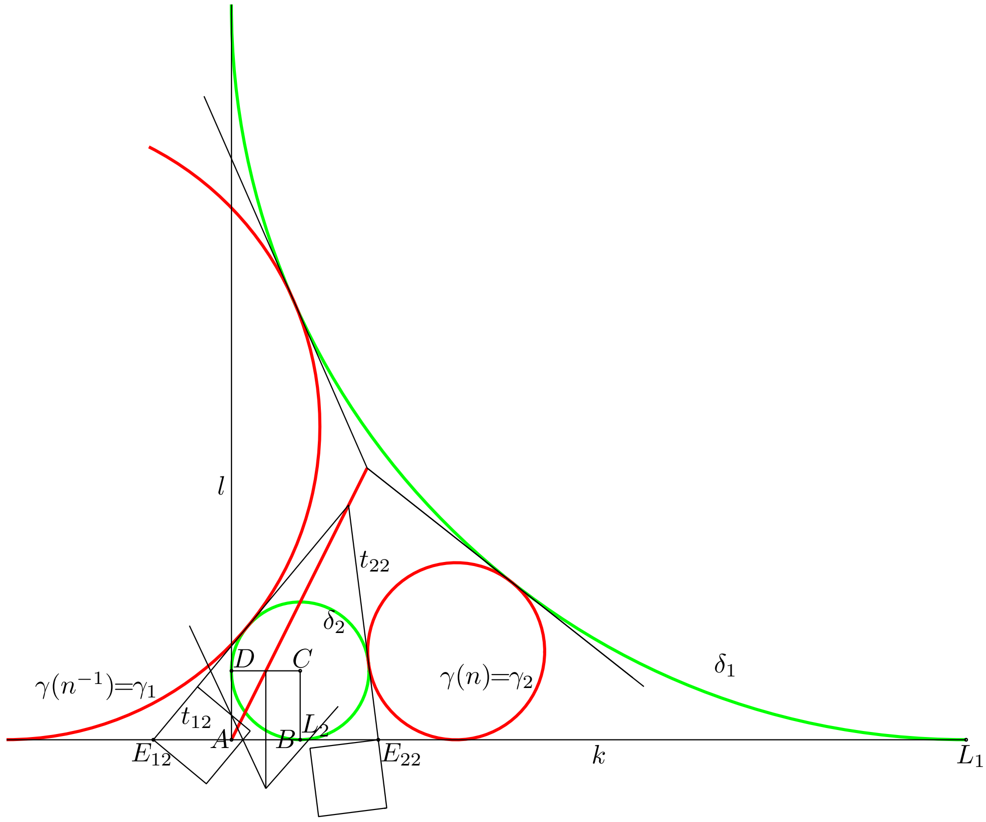

Figure 35: $\mathcal{H}(E_{12}) = \mathcal{H}(-n^{-1})$ and $\mathcal{H}(E_{22}) = \mathcal{H}(-n)$ for $1 < n$.

Theorem 8.1. *The following statements are equivalent for $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$.*
(i) *The figures $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ are inverses to each other.*
(ii) *The points E_1 and E_2 are symmetric about the perpendicular bisector of AB.*
(iii) *E_1 and E_2 coincide with the midpoint of AB, or $E_1 \neq E_2$ and the crease lines of $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ meet in a point on the perpendicular bisector of AB.*

Proof. Theorem 4.3(iv) shows that the points E_{1i} and E_{2i} are symmetric about the perpendicular bisector of AL_i for the figure $\mathcal{U}$. Hence (i) implies (ii). Assume (ii) holds. If $\mathcal{H}(E')$ is the inverse of $\mathcal{H}(E_1)$, then E_1 and E' are symmetric about the perpendicular bisector of AB as just proved. Hence $E_2 = E'$, i.e., $\mathcal{H}(E_2) = \mathcal{H}(E')$. Hence (i) holds. Therefore (i) and (ii) are equivalent. If E_1 and E_2 coincide with the midpoint of AB, then (ii) and (iii) are obviously equivalent. Let us assume $E_1 \neq E_2$. We use a rectangular coordinate system such that the points A and B have coordinates $(-s/2, 0)$ and $(s/2, 0)$, respectively. Let $(e_i, 0)$ be the coordinates of E_i. Then the line m_i has an equation $(-2e_i + s)x + 2sy + (e_i^2 - 5s^2/4) = 0$. Therefore the two lines meet in the point of coordinates

$$\left(\frac{e_1 + e_2}{2}, \frac{-2(e_1 + e_2) + 4e_1e_2/s + 5s}{8}\right).$$

Hence (ii) and (iii) are equivalent. □

9. Haga's theorems

In this section we consider Haga's theorems in origamics [6]. Firstly we consider special cases for the figures $\mathcal{H}(n)$ in the case $A < E < B$, which were often considered in Wasan geometry and are closely related to Haga's theorems. Recall that F is the point of intersection of the line DA and the reflection of the line CD in m if they meet. If E is the midpoint of AB, F divides DA in the ratio $1 : 2$ internally [9, Theorem 3.1] (see Figure 28). The fact is called Haga's first theorem [6]. While Theorem 3.2 shows that this happens if $n = 1$. Therefore the figure of Haga's first theorem is obtained from $\mathcal{H}(1)$. We get $s = 4r$ by Theorem 3.1(i) in this case. A problem considering this relation for $\mathcal{H}(1)$ can be found in [5].

If F is the midpoint of DA, E divides AB in the ratio $2 : 1$ internally [9, Theorem 3.1] (see Figure 36). The fact is called Haga's third theorem [6]. While Theorem 3.2 shows that this happens if $n = 4$. Hence the figure of Haga's third theorem can be obtained from $\mathcal{H}(4)$. Therefore the circle touching $\gamma = \gamma(4)$, AB and DA from inside of $ABCD$ is congruent to γ. Let δ_2 be this circle and let K be the point of tangency of γ and AB. Since E is the midpoint of the segment BK by Proposition 2.1, E and K are the points of trisection of the side AB. The remaining circle touching the line AB and δ and δ_2 externally is $\gamma(1/4)$. The relation (2) shows that K coincides with the point of intersection of AB and the internal common tangent of $\gamma(1/4)$ and δ. It seems that the case $n = 4$ is most frequently considered for $\mathcal{H}(n)$ in Wasan geometry as we have shown in section 2.

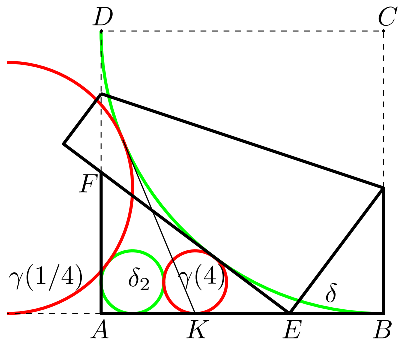

Figure 36: $\mathcal{H}(4)$ with $\gamma(1/4)$.

We have generalized Haga's theorems in [9], which we restate here in terms of $\mathcal{H}(n)$. Notice that the theorem holds for $\mathcal{H}(0)$ and $\mathcal{H}(\overline{0})$ by (5).

Theorem 9.1. *The following relations hold for $\mathcal{H}(n)$.*

$$\frac{|AF|}{|DF|} = 2\frac{|BE|}{|AE|} = \frac{2}{\sqrt{|n|}}.$$

Proof. The first half of the equations is Theorem 3.1 of [9]. The last half of the equations follows from Theorem 3.2. □

10. Conclusion

We argued the merit of considering circles in the geometry of origami in [11, 12]. In these two-part papers we have shown several examples to verify the validity of our assertion. The circles we have considered are tangent circles except the circumcircle of a triangle considered in the first part of the papers. In this sense we may say that many parts of the geometry of origami belong to the geometry of tangent circles. In particular, the incircle and the excircles of a right triangle or circles touching two perpendicular lines play important roles in the geometry of origami using a square piece of paper as shown in the both parts of the papers.

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