ON UPPER LIMITS FOR THE HEIGHT OF INFLATED TOWERS.

Ruslan Sharipov

Abstract. The height of buildings is limited by the weight and the strength of constructive materials. Inflated structures are considered as a prospective technology for constructing super-tall buildings, including those reaching the boundary of the cosmic space. The goal of the present paper is to derive and compute theoretical estimates for the upper limit of the height of an inflated cylindrical structure subdivided into segments designed in such a way that they practically do not load each other transmitting all of their weight to the basement through the gas inside them.

1. Introduction.

The idea of a super-tall tower reaching heavens is known from the Bible as the narrative of the tower of Babel. Later on this idea was expressed by Konstantin Tsiolkovsky, one of the pioneers of astronautics. In [1] he wrote “The gravity of the Earth will fade out at the top of a tower whose height is $5\frac{1}{2}$ of its radius (which is 37000 km over its surface, the Moon being 10 times further)”. Jerome Pearson in [2] suggested to use such a tower for launching spacecrafts. Due to the famous science fiction writer Arthur C. Clarke (see [3]) the idea of space tower became known to the broad audience of his readers.

A 37000 km tall tower is too big in order to begin with, 100 km is enough for reaching the space. The altitude of 100 km is known as the Karman line. It is taken as the boundary of the Earth’s atmosphere by convention.

Being 370 times as lower than a 37000 km tall tower, a 100 km tall tower is still too big in order to be built using regular technologies. Indeed, the 100 km long steel rod standing vertically produces the pressure of 7800 MPa[1], while the compressive yield strength of steel is about 150 MPa (see [4]). Inflatable structures are considered as a prospective technology for super-tall towers by several authors (see [5–7]). There are two patents [8] and [9] on inflated towers. The second one is based upon the results of [7].

Like in [7], we consider a cylindrical structure standing vertically and divided into segments by horizontal diaphragms (see Fig. 2.1 below). However, unlike [7], the diaphragms in our case do not serve for keeping a constant gas pressure along the height of the whole structure. Each diaphragm in our case is designed in such a way that it holds the weight of the upper segment (along with its payload and the gas inside it) and transmits this load to the gas within the lower segment. Ultimately, the lowermost diaphragm transmits the weight of the whole structure to

2010 *Mathematics Subject Classification.* 70C20, 74B05, 65Q30.

Key words and phrases. Ultra-tall building, inflated core structure.

[1] The density of steel is taken as 8.05 g/cm^3.

the basement of the tower. Thus the walls of the structure in our case do experience only a minor stress in the axial direction basically due to their own weight within one segment. Our goal is to derive and compute the upper limits for the height of the whole structure under these constructive assumptions.

2. The construction of a single segment.

A single segment of the structure is shown in Fig. 2.1. It is a cylindrical pipe enclosed between two diaphragms. Note that the diaphragms extend outside the pipeline and form two rectangular platforms. These platforms are used in order to connect one pipeline with another thus forming a multicolumn and multistory core structure. The height of the segment is equal to the floor-to-floor distance between platforms. It is denoted through L in Fig. 2.1. The internal radius of the cylinders in Fig. 2.1 is denoted through R. Relying upon data for regular gas pipelines (see [10]) and for the sake of simplicity we choose

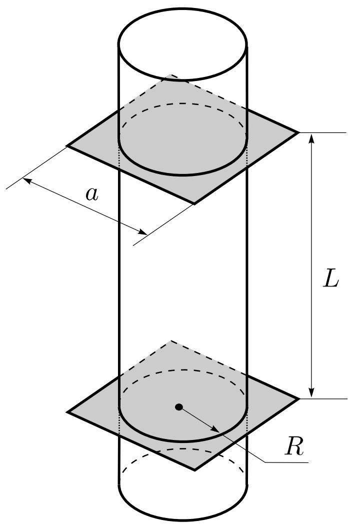

Fig. 2.1

$$L = 3 \text{ m}, \quad R = 0.5 \text{ m} \tag{2.1}$$

in our computations below.

Cylindrical segments of the structure are assumed to be made of rubber enforced with kevlar cord. Like car tires they consist of a soft inner gas-holding cylindrical tube and an outer shell, which could be called a tire. The thickness of the walls of track tire tubes varies from 2.5 to 5.0 millimeters. We choose

$$d_0 = 5 \text{ mm} = 0.005 \text{ m} \tag{2.2}$$

for this value in our computations below.

The walls of the outer shell are formed by a cylindrical kevlar carcass surrounded with a rubber envelope as shown in Fig. 2.2. The kevlar carcass is made of circular kevlar threads. In axial section they are shown in yellow (see Fig. 2.2). The envelope surrounding them is shown in blue. It serves for holding kevlar threads together and for gluing them to each other with inlaid rubber. The thickness of the kevlar winding is denoted through d_1, while the thickness of the outer shell as a whole is denoted through d_2. Then

$$R_2 = R + d_0 + d_2 \tag{2.3}$$

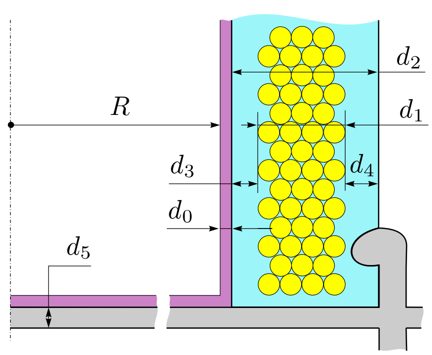

Fig. 2.2

is the outer radius of the cylindrical structure shown in Fig. 2.1. Unlike d_0, the thickness of the outer shell d_2 as well as the thickness of its kevlar carcass d_1 are not constant. However the thickness of the internal rubber layer d_3 of the shell and the thickness of its external rubber layer d_4 are assumed to be constant. We choose

$$d_3 = 7 \text{ mm} = 0.007 \text{ m}, \qquad\qquad d_4 = 12 \text{ mm} = 0.012 \text{ m}. \tag{2.4}$$

Looking at Fig. 2.2 we easily derive the relationship

$$d_2 = d_3 + d_1 + d_4. \tag{2.5}$$

The inner gas-holding tube is shown in purple in Fig. 2.2. It is also made of rubber. Its walls thickness is denoted through d_0 in (2.2) and in (2.3).

Along with other elements we see a hook-shaped outgrouth in Fig. 2.2. It presents the rim of the circular diaphragm, which is similar to wheel rims in cars and bicycles and which serves the same purposes. The diaphragm itself is a part of the square platform (see Fig. 2.1). For its thickness we choose the value

$$d_5 = 10 \text{ mm} = 0.01 \text{ m}. \tag{2.6}$$

The platforms are used for connecting several cylindrical columns together thus forming a multicolumn tower or a multicolumn core structure of a building. In order to keep each segment of each column serviceable one needs to access them for surveying, for repairing, and for replacing if necessary. Therefore one needs to preserve some space between them. We do it by setting the inequality $a \geqslant 3\,R_2$, where a is the length of the edge of each square platform (see Fig. 2.1). Since d_2 in (2.3) is not constant, the radius R_2 is not constant too (lower segments are thicker than upper ones). Hence we can satisfy the inequality $a \geqslant 3\,R_2$ only if we choose

$$a = 3\,\max(R_2). \tag{2.7}$$

Thus we have defined all geometric parameters of our structure and therefore we can proceed to further steps.

3. Loads on a single segment of the structure.

Let's consider one column in our multicolumn structure and enumerate its segments from up to down, the topmost being the first and that which is at the bottom of the structure being the last. Then let's consider i-th segment as shown in Fig. 2.1. The weight of the segment is given by the formula

$$w(i) = w_0 + w_1 + w_2 + w_3, \tag{3.1}$$

where w_0 is the weight of the inner gas-holding tube, w_1 is the weight of the lower platform (the upper platform is considered as a part of the previous $(i-1)$-th segment), w_2 is the weight of the outer shell, and w_3 is the weight of some payload attached to i-th segment. The weight w_0 in (3.1) is calculated as follows:

$$w_0 = 2\,\pi\,R^2\,d_0\,\rho_0\,g + \pi\,[(R + d_0)^2 - R^2]\,L\,\rho_0\,g, \tag{3.2}$$

where ρ_0 is the rubber density and $g = 9.81$ m/s^2 is the standard gravitational acceleration for the Earth.

The weight w_2 in (3.1) is calculated by mens of the following formula:

$$\begin{aligned} w_2 = \pi\,[(R+d_0+d_2)^2-(R+d_0)^2]\,L\,\rho_0\,g+\\ +\,\pi\,[(R+d_0+d_3+d_1)^2-(R+d_0+d_3)^2]\,L\,(\rho_1-\rho_0)\,g. \end{aligned} \tag{3.3}$$

Here ρ_1 is the density of kevlar. The weight of the platform is given by the formula

$$w_1 = a^2\,d_5\;\rho_2\,g, \tag{3.4}$$

where ρ_2 is the density of steel. And finally, we should take a decision on the payload weight w_3 in (3.1). For the sake of simplicity we assume that the payload weight is the half of the total weight $w(i)$. Then formula (3.1) reduces to

$$w(i) = 2\,(w_0+w_1+w_2). \tag{3.5}$$

Static equilibrium of a body in mechanics is a state of balance of opposite forces in all directions. The vertical forces acting upon i-th segment are shown in Fig. 3.1. The $(i-1)$-th segment acts downward upon the i-th segment with the force $N(i)$. Similarly, the i-th segment acts downward upon the $(i+1)$-th segment with the force $N(i+1)$. However, according to Newton's third law, each action produces the equal but opposite reaction. The upward force $N(i+1)$ in Fig. 3.1 is the force of reaction from the lower $(i+1)$-th segment acting upon the i-th segment of the structure.

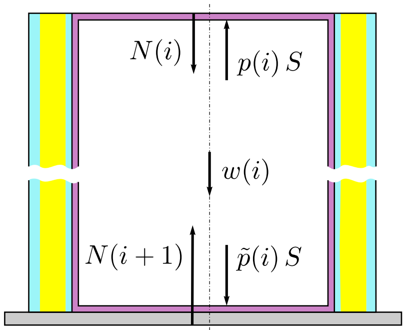

Fig. 3.1

The force $w(i)$ in Fig. 3.1 is a gravity force. It is due to the weight of the solid elements of i-th segment (see (3.1) and (3.5)). Note that we treat the gas inside i-th segment as a separate body. Its weight is not included into the sum (3.1). The gas acts upon solid elements of the i-th segment by means of the pressure forces $p(i)\,S$ and $\tilde p(i)\,S$. Their difference is equal to the weight of the gas w_4:

$$w_4 = \tilde p(i)\,S - p(i)\,S. \tag{3.6}$$

The area S in (3.6) is determined by the internal radius of the structure:

$$S = \pi\,R^2. \tag{3.7}$$

Looking at Fig. 3.1, we can write the vertical equilibrium condition for the i-th segment of the structure. It is given by the equality

$$N(i+1) + p(i)\,S = N(i) + \tilde p(i)\,S + w(i). \tag{3.8}$$

The horizontal equilibrium of the i-th segment of our structure is set up by the gas pressure forces and the tensile stress forces in the vertical walls of the structure. We consider it below in section 5.

4. Recurrence formula for the gas pressure.

Looking at Fig. 3.1 we see that downward force $N(i)$ is applied to the upper boundary of the i-th segment of the structure. The backward reaction to this force is produced partly by the vertical walls of the segment and partly by the gas pressure $p(i)$ within it. The compressive hardness of the rubber walls (even though they arec reinforced with kevlar) is very small. Therefore their contribution to the reaction force is negligible and we can write

$$p(i)\,S = N(i) \tag{4.1}$$

as a good approximation. The same reasons applied to the $(i+1)$-th segment yield

$$p(i+1)\,S = N(i+1) \tag{4.2}$$

Substituting (4.1) and (4.2) into (3.8), we derive

$$p(i+1) = \tilde{p}(i) + \frac{w(i)}{S}. \tag{4.3}$$

The pressure in a vertical gas column at constant temperature varies exponentially in the vertical direction (see section 5 in [7]). Therefore

$$\tilde{p}(i) = k\,p(i), \quad \text{where} \quad k = \exp\Bigl(\frac{\mu\,g\,L}{\mathcal{R}\,T}\Bigr). \tag{4.4}$$

Here $g = 9.81$ m/s^2 is the standard gravitational acceleration for the Earth, μ is the molecular mass of the gas, T is the absolute temperature of the gas, and R is the universal gas constant. Its value in SI units is

$$\mathcal{R} = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}. \tag{4.5}$$

For the temperature T in our calculations below we choose the value

$$T = 293\ \mathrm{K} = 20^\circ\mathrm{C}, \tag{4.6}$$

which is a comfortable room temperature. In the case of helium, we have

$$\mu = 4\ \mathrm{g/mol} = 0.004\ \mathrm{kg/mol}. \tag{4.7}$$

Substituting (4.5), (4.6), and (4.7) into (4.4), we derive

$$k = k_{\,\mathrm{He}} = 1.000048. \tag{4.8}$$

In the case of nitrogen N_2 we have

$$\mu = 28\ \mathrm{g/mol} = 0.028\ \mathrm{kg/mol}. \tag{4.9}$$

Substituting (4.5), (4.6), and (4.9) into (4.4), we derive

$$k = k_{\,\mathrm{N_2}} = 1.00034. \tag{4.10}$$

In both cases k is a definite known constant. Substituting (4.4) into (4.3), we derive

$$p(i+1)=k\,p(i)+\frac{w(i)}{S}. \tag{4.11}$$

The equality (4.11) is a recurrence equation for the gas pressure within segments of our structure. This equation is somewhat similar to ordinary differential equations. Once the initial value $p(1)$ is known, the values of $p(i)$ for all i are calculated or at least computed numerically.

5. Stress in a thick walled inflated cylinder.

As we see in Fig. 2.2, the side walls of our structure are composed by several rubber layers and one kevlar layer. Rubber is a soft material. Practically it cannot resist a tensile stress, just transmitting the load of inner gas pressure to the kevlar layer. The kevlar layer is a winding of kevlar threads. This winding is assumed to be maximally dense so that in our calculations we can approximate it by a solid thick walled kevlar cylinder. According to Fig. 2.2 the inner and outer radii of this cylinder are given by the formulas

$$R_3=R+d_0+d_3, \qquad\qquad R_4=R+d_0+d_3+d_1. \tag{5.1}$$

We use R_3 and R_4 for these radii in (5.1) since R_2 is already used in (2.3).

Note that $k_{\,\mathrm{He}}$ in (4.8) and $k_{\,\mathrm{N}_2}$ in (4.10) both are very close to the unity. Hence

$$\tilde{p}(i)\approx p(i) \tag{5.2}$$

in (4.4). The relationship (5.2) means that it would be a good approximation if we consider the thick walled kevlar cylinder with the radii (5.1) loaded by the uniform internal gas pressure $p_3=p(i)$ and by some uniform external gas pressure $p_4<p_3$.

The problem of calculating the stress distribution in a thick walled inflated cylinder is not new (see [11]). Nevertheless, we reproduce its solution here using the covariant derivatives technique that comes from differential geometry (see [12]).

In solving problems with cylindrical symmetry cylindrical coordinates used. We denote them $y^1=\rho$, $y^2=\varphi$, $y^3=h$. The upper indices in y^1, y^2, y^3 are chosen according to Einstein's tensorial notation (see § 20 in Chapter I of [13]). The regular Cartesian coordinates x^1, x^2, x^3 are expressed through the cylindrical coordinates y^1, y^2, y^3 by means of the formulas

$$\mathbf{r}=\begin{Vmatrix}x^1\\ x^2\\ x^3\end{Vmatrix}=\begin{Vmatrix}x\\ y\\ z\end{Vmatrix}=\begin{Vmatrix}\rho\,\cos\varphi\\ \rho\,\sin\varphi\\ h\end{Vmatrix}. \tag{5.3}$$

Each curvilinear coordinate system has its associated moving frame (see § 2 in Chapter III of [12]). It is a triple of vectors obtained by differentiation of the radius-vector (5.3) with respect to the curvilinear coordinates y^1, y^2, y^3:

$$\mathbf{E}_1=\frac{\partial\,\mathbf{r}}{\partial y^1}, \qquad\qquad \mathbf{E}_2=\frac{\partial\,\mathbf{r}}{\partial y^2}, \qquad\qquad \mathbf{E}_3=\frac{\partial\,\mathbf{r}}{\partial y^3}. \tag{5.4}$$

In the case of cylindrical coordinates from (5.4) we derive

$$\mathbf{E}_1=\begin{Vmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{Vmatrix},\qquad \mathbf{E}_2=\begin{Vmatrix}-\rho\,\sin\varphi\\ \rho\,\cos\varphi\\ 0\end{Vmatrix},\qquad \mathbf{E}_3=\begin{Vmatrix}0\\ 0\\ 1\end{Vmatrix}.\tag{5.5}$$

The frame (5.5) is not orthonormal, though it is orthogonal. In physical literature the following orthonormal frame is used:

$$\mathbf{e}_1=\begin{Vmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{Vmatrix},\qquad \mathbf{e}_2=\begin{Vmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{Vmatrix},\qquad \mathbf{e}_3=\begin{Vmatrix}0\\ 0\\ 1\end{Vmatrix}.\tag{5.6}$$

Due to the difference of frames (5.5) and (5.6) some formulas below are different from those in [11] and in [14].

Using (5.5), one easily derives the formulas for the components of the direct and inverse metric tensors (see formula (4.8) in § 4 of Chapter III in [12]):

$$g_{ij}=\begin{Vmatrix}1&0&0\\ 0&\rho^2&0\\ 0&0&1\end{Vmatrix},\qquad g^{ij}=\begin{Vmatrix}1&0&0\\ 0&\rho^{-2}&0\\ 0&0&1\end{Vmatrix}.\tag{5.7}$$

Statics of solid media is described through the following three concepts:

$$\boxed{\text{Displacement}}\rightleftarrows\boxed{\text{Strain}}\rightleftarrows\boxed{\text{Stress}}\,.\tag{5.8}$$

In linear theory the displacement is treated as a vector field. Its components are defined using its expansion in the moving frame (5.5):

$$\mathbf{u}=\sum_{i=1}^{3}u^i\,\mathbf{E}_i.\tag{5.9}$$

The relation of displacement and strain is given by the following formula:

$$u_{ij}=\frac{\nabla_i u_j+\nabla_j u_i}{2}=\frac{1}{2}\biggl(\frac{\partial u_j}{\partial y^i}+\frac{\partial u_i}{\partial y^j}\biggr)-\sum_{k=1}^{3}\Gamma^k_{ij}\,u_k.\tag{5.10}$$

The Christoffel symbols Γ^k_{ij} in (5.10) are calculated using the components of the direct and inverse metric tensors given in (5.7):

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{r=1}^{3}g^{kr}\left(\frac{\partial g_{rj}}{\partial u^i}+\frac{\partial g_{ir}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^r}\right).\tag{5.11}$$

The values of Γ^k_{ij} for cylindrical coordinates are given in § 9 of Chapter III in [12]):

$$\Gamma^1_{11}=0,\qquad \Gamma^1_{12}=0,\qquad \Gamma^1_{21}=0,$$

$$\begin{aligned}
&\Gamma^1_{13} = 0, && \Gamma^1_{31} = 0, && \Gamma^1_{22} = -\rho,\\
&\Gamma^1_{23} = 0, && \Gamma^1_{32} = 0, && \Gamma^1_{33} = 0,
\end{aligned}\tag{5.12}$$

$$\begin{aligned}
&\Gamma^2_{11} = 0, && \Gamma^2_{12} = \rho^{-1}, && \Gamma^2_{21} = \rho^{-1},\\
&\Gamma^2_{13} = 0, && \Gamma^2_{31} = 0, && \Gamma^2_{22} = 0,\\
&\Gamma^2_{23} = 0, && \Gamma^2_{32} = 0, && \Gamma^2_{33} = 0,
\end{aligned}\tag{5.13}$$

$$\begin{aligned}
&\Gamma^3_{11} = 0, && \Gamma^3_{12} = 0, && \Gamma^3_{21} = 0,\\
&\Gamma^3_{13} = 0, && \Gamma^3_{31} = 0, && \Gamma^3_{22} = 0,\\
&\Gamma^3_{23} = 0, && \Gamma^3_{32} = 0, && \Gamma^3_{33} = 0.
\end{aligned}\tag{5.14}$$

The formula (5.11) is known as the formula for the components of the metric Levi-Civita connection (see [15]). The reader can verify the formulas (5.12), (5.13), and (5.14) by applying (5.11) to (5.7).

Note that the covariant and contravariant components of the displacement vector $\mathbf{u}$ in the formulas (5.9) and (5.10) are different. They are related to each other through the components of the metric tensors (5.7):

$$u_i = \sum_{j=1}^{3} g_{ij}\, u^j, \qquad\qquad u^i = \sum_{j=1}^{3} g^{ij}\, u_j. \tag{5.15}$$

In linear theory the stain to stress relation in (5.8) is written as

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{q=1}^{3} C^{kq}_{ij}\, u_{kq}, \tag{5.16}$$

where C^{kq}_{ij} are the components of the stiffness tensor (see [16]). In the case of an isotropic solid medium the relationship (5.16) reduces to the following one:

$$\sigma_{ij} = \frac{3\,K - 2\,G}{3}\left(\,\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\, u_{kq}\right) g_{ij} + 2\,G\, u_{ij}. \tag{5.17}$$

Two constants K and G in (5.17) are known as the bulk modulus and the shear modulus. The inverse relationship for (5.17) looks like:

$$u_{ij} = \left(\frac{1}{9\,K} - \frac{1}{6\,G}\right)\left(\,\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\sigma_{kq}\right) g_{ij} + \frac{1}{2\,G}\,\sigma_{ij}.$$

In the absence of body forces, the static equilibrium of any solid medium is described by the following differential equation:

$$\sum_{k=1}^{3}\sum_{j=1}^{3} g^{kj}\,\nabla_k\sigma_{ij} = 0. \tag{5.18}$$

Body forces in our case do exist. They are due to the Earth gravity. However, they act in vertical direction. Therefore they do not affect the hoop stress produced in the side walls by the gas pressure inside them. Replacing covariant derivatives in (5.18) by partial derivatives we transform the equation (5.18) to

$$\sum_{k=1}^{3}\sum_{j=1}^{3} g^{kj}\left(\frac{\sigma_{ij}}{\partial y^k}-\sum_{s=1}^{3}\Gamma^s_{ki}\,\sigma_{sj}-\sum_{s=1}^{3}\Gamma^s_{kj}\,\sigma_{si}\right)=0. \tag{5.19}$$

The equation (5.19) is the main differential equation for our further calculations.

Due to the cylindrical symmetry of our structure and since we approximate the inner and outer gas pressures by two constant values p_3 and p_4. The displacement vector $\mathbf{u}$ is directed radially, i. e. it has only one component in (5.9):

$$\mathbf{u} = u^1\,\mathbf{E}_1. \tag{5.20}$$

For the same reasons the unique nonzero component of the vector $\mathbf{u}$ in (5.20) is a function of the radial variable $y^1 = \rho$ only, i. e. we have

$$u^1 = u(\rho), \qquad\qquad u^2 = 0, \qquad\qquad u^3 = 0. \tag{5.21}$$

Further steps are the following:

1) applying the first of the two formulas (5.15), we derive the covariant components u_1, u_2, u_3 of the vector $\mathbf{u}$;
2) applying (5.10) to these covariant components u_1, u_2, u_3, we derive the formulas for the components of the strain tensor u_{ij};
3) using (5.17), we calculate the components of the stress tensor σ_{ij};
4) and finally, substituting σ_{ij} into (5.19), we derive an ordinary differential equation for the function $u(\rho)$ in (5.21):

$$\frac{3\,K+4\,G}{3}\left(u''_{\rho\rho}+\frac{1}{\rho}\,u'_{\rho}-\frac{1}{\rho^2}u\right)=0. \tag{5.22}$$

The common factor

$$M=\frac{3\,K+4\,G}{3} \tag{5.23}$$

in (5.22) is known as the P-wave modulus, as the longitudinal modulus, and as the constrained modulus of a solid medium (see table with elastic moduli conversion formulas in [17]). This modulus M in (5.23) is nonzero. Therefore it can be omitted in (5.22) and the equation (5.22) reduces to

$$u''_{\rho\rho}+\frac{1}{\rho}\,u'_{\rho}-\frac{1}{\rho^2}u=0. \tag{5.24}$$

The equation (5.24) is explicitly solvable. Its solution is written as

$$u(\rho)=C_1\,\rho+\frac{C_2}{\rho}. \tag{5.25}$$

Here C_1 and C_2 are two arbitrary constants which are called integration constants. Substituting (5.25) back into the formulas for the strain tensor, we find that the strain tensor is given by a diagonal matrix with two nonzero diagonal elements:

$$u_{11} = C_1 - \frac{C_2}{\rho^2}, \qquad\qquad u_{22} = C_1\,\rho^2 + C_2.$$

The stress tensor is also given by a diagonal matrix. However in this case we have not two, but three nonzero diagonal elements:

$$\begin{aligned}
\sigma_{11} &= \frac{6\,K + 2\,G}{3}\,C_1 - \frac{2\,G\,C_2}{\rho^2},\\
\sigma_{22} &= \frac{6\,K + 2\,G}{3}\,C_1\,\rho^2 + 2\,G\,C_2,\\
\sigma_{33} &= \frac{6\,K - 4\,G}{3}\,C_1.
\end{aligned}\tag{5.26}$$

Looking at (5.26), we can introduce the other two arbitrary constants

$$\tilde{C}_1 = \frac{6\,K + 2\,G}{3}\,C_1, \qquad\qquad \tilde{C}_2 = 2\,G\,C_2.\tag{5.27}$$

In terms of these two constants (5.27) the formulas (5.26) are written as

$$\begin{aligned}
\sigma_{11} &= \tilde{C}_1 - \frac{\tilde{C}_2}{\rho^2},\\
\sigma_{22} &= \tilde{C}_1\,\rho^2 + \tilde{C}_2,\\
\sigma_{33} &= \frac{3\,K - 2\,G}{3\,K + G}\,\tilde{C}_1 = 2\,\nu\,\tilde{C}_1.
\end{aligned}\tag{5.28}$$

The constant ν in (5.28) is known as the Poisson's ratio (see table with elastic moduli conversion formulas in [17]). This means that the stress in the axial direction $\sigma_{33} \neq 0$ arises in our calculations purely due to the Poisson effect (see [18]). As we already noted above we do not consider the vertical stress from any sources. Therefore the third formula (5.28) is inessential for us.

Boundary conditions for the components of the stress tensor σ_{ij} on any solid-to-gas interface are derived from the following formula:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,n^i\,n^j = p.\tag{5.29}$$

Here p is the gas pressure and n^1, n^2, n^3 are the components of the unit vector perpendicular to the interface. Applying (5.29) to our problem of thick walled kevlar cylinder with the radii (5.1), we derive

$$\sigma_{11}\biggr|_{\rho=R_3} = p_3, \qquad\qquad \sigma_{11}\biggr|_{\rho=R_4} = p_4.\tag{5.30}$$

Substituting the first formula (5.28) into (5.30), we derive two linear equations for the integration constants $\tilde{C}_1$ and $\tilde{C}_2$. Their solution is

$$\tilde{C}_1 = \frac{p_4\,R_4^2 - p_3\,R_3^2}{R_4^2 - R_3^2}, \qquad\qquad \tilde{C}_2 = -\frac{R_3^2\,R_4^2\,(p_3 - p_4)}{R_4^2 - R_3^2}. \tag{5.31}$$

Applying these formulas (5.31) to (5.28), we get

$$\sigma_{11} = \frac{\bigl(p_4\,R_4^2 - p_3\,R_3^2\bigr)\rho^{\,2} + R_3^2\,R_4^2\,(p_3 - p_4)}{\bigl(R_4^2 - R_3^2\bigr)\rho^{\,2}}, \tag{5.32}$$

$$\sigma_{22} = \frac{\bigl(p_4\,R_4^2 - p_3\,R_3^2\bigr)\rho^{\,2} - R_3^2\,R_4^2\,(p_3 - p_4)}{R_4^2 - R_3^2}. \tag{5.33}$$

As we shall see below, these two formulas (5.32) and (5.33) are in agreement with the results of [11].

6. Strength limits for the gas pressure and the side walls thickness.

Since the moving frame of a curvilinear coordinate system is not always orthonormal (see (5.5) as an example), the values of physical stress are not always given by the components of the stress tensor. The physical stress in the direction given by a unit vector $\mathbf{n}$ is expressed by the formula similar to (5.29):

$$\sigma = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,n^i\,n^j. \tag{6.1}$$

Choosing $\mathbf{n} = \mathbf{E}_1$ and $\mathbf{n} = \mathbf{E}_2/\rho$ and substituting these two unit vectors into (6.1), we derive the formulas for physical values of the radial and circumferential stress:

$$\sigma_r = \sigma_{11} = \frac{\bigl(p_4\,R_4^2 - p_3\,R_3^2\bigr)\rho^{\,2} + R_3^2\,R_4^2\,(p_3 - p_4)}{\bigl(R_4^2 - R_3^2\bigr)\rho^{\,2}}, \tag{6.2}$$

$$\sigma_\varphi = \frac{\sigma_{22}}{\rho^{\,2}} = \frac{\bigl(p_4\,R_4^2 - p_3\,R_3^2\bigr)\rho^{\,2} - R_3^2\,R_4^2\,(p_3 - p_4)}{\bigl(R_4^2 - R_3^2\bigr)\rho^{\,2}}. \tag{6.3}$$

The formulas (6.2) and (6.3) do coincide with the corresponding formulas in [11].

Looking at (5.28), we see that $\sigma_r = \sigma_{11}$ is a monotonic function of ρ. Since $p_3 > p_4$ and due to (5.30) it decreases from its maximal value

$$\sigma_{r\,\max} = p_3 = p(i) \tag{6.4}$$

at the inner radius $\rho = R_3$ to its minimal value p_4 at the outer radius $\rho = R_4$. The formula (6.3) for the circumferential stress is equivalent to

$$\sigma_\varphi = \tilde{C}_1 + \frac{\tilde{C}_2}{\rho^{\,2}}. \tag{6.5}$$

Looking at (6.5), we see that σ_φ is also a monotonic function. Since $R_4 > R_3$ and $p_3 > p_4$ in (5.31), the second constant $\tilde{C}_2$ in (6.5) is negative. Though our cylinder is thick walled, its outer radius R_4 does not substantially exceed its inner radius R_3. The pressures p_3 and p_4 are different. The outer pressure p_4 is less or equal to the normal atmospheric pressure, while the inner pressure p_3 can exceed many times the normal atmospheric pressure. Therefore a typical value of the first constant $\tilde{C}_1$ in our case is also negative. As a result we get that $\sigma_\varphi < 0$ and σ_φ is an increasing function. The negative value of σ_φ means that the circumferential stress is a tensile stress, while the radial stress is compressive. Typically the absolute value of the tensile stress $|\sigma_\varphi|$ decreases from its maximal value

$$|\sigma_\varphi|_{\max} = \frac{p_3\left(R_4^2 + R_3^2\right) - 2\,p_4\,R_4^2}{R_4^2 - R_3^2} \tag{6.6}$$

at the inner radius $\rho = R_3$ to its minimal value

$$|\sigma_\varphi|_{\min} = \frac{2\,p_3\,R_3^2 - p_4\left(R_4^2 + R_3^2\right)}{R_4^2 - R_3^2} \tag{6.7}$$

at the outer radius $\rho = R_4$.

The external air pressure p_4 contributes to our efforts in restraining the internal gas pressure p_3. However, as we noted above, its contribution is small since $p_4 \ll p_3$. Therefore it would be a good approximation to set

$$p_4 = 0 \tag{6.8}$$

in (6.6) and (6.7). As a result of (6.8) we reduce (6.6) and (6.7) to

$$|\sigma_\varphi|_{\max} = \frac{p_3\left(R_4^2 + R_3^2\right)}{R_4^2 - R_3^2}, \qquad |\sigma_\varphi|_{\min} = \frac{2\,p_3\,R_3^2}{R_4^2 - R_3^2}. \tag{6.9}$$

There are two strength limitations for the internal gas pressure p_3:

$$\sigma_{r\,\max} \leqslant \sigma_1 = \text{compressive yield strength of rubber}, \tag{6.10}$$

$$|\sigma_\varphi|_{\max} \leqslant \sigma_2 = \text{tensile yield strength of kevlar}. \tag{6.11}$$

The expressions for $\sigma_{r\,\max}$ and $|\sigma_\varphi|_{\max}$ in (6.10) and (6.11) are taken from (6.4) and (6.9). In [7] we find the following value of σ_2:

$$\sigma_2 = 3.6 \text{ GPa}. \tag{6.12}$$

The value (6.12) corresponds to Kevlar 49 which is commercially available. Wikipedia gives approximately the same value for kevlar tensile yield strength (see [19]).

The value (6.12) cannot be used in practical design. In order to convert it to the practical design value we should choose some safety factor (SF). For pressure vessels the recommended value of the safety factor is 4 (see [20]). Taking into account this safety factor, we derive the maximum working tensile stress value for kevlar:

$$\sigma_4 = \frac{\sigma_2}{4} = 0.9 \text{ GPa}. \tag{6.13}$$

We choose the value (6.13) for $|\sigma_\varphi|_{\max}$, i.e. we set $\sigma_4 = |\sigma_\varphi|_{\max}$ thus obeying the inequality (6.11) and the recommended safety requirements. Then we summarize kevlar data in the following table:

Manufacturer/Supplier	Material	σ_4	σ_2	SF
DuPont, International	Kevlar 49	0.9 GPa	3.6 GPa	4

The case of rubber is more complicated. Most data for rubber available in the Internet are tensile data. Therefore we shall gather the required data from the product properties. Most suitable product for us is high pressure hoses.

In describing practical pressure vessels two parameters are crucial: maximal working pressure (Max WP) and minimal burst pressure (Min BP). Their ratio is known as the safety factor (SF). We gather these product data in a table:

Manufacturer/Supplier	Material /product	Max WP	Min BP	SF
Hebei Qianli Rubber Products Co., Ltd., China	rubber SHP200-04	45.2 MPa	180.6 MPa	4
Polyfluor Plastics BV, Netherlands	PTFE, PFA SHP Hose	47.5 MPa	190 MPa	4
Boltorq, India	Nylon BH94 Ser.	112 MPa	280 MPa	2.5
SITEC Sieber Engineering AG, Switzerland	POM HP Hose	175 MPa	700 MPa	4

Relying on this table, we choose the following value for σ_1 in (6.10):

$$\sigma_1 = 180.6 \text{ MPa}.$$

Then, choosing the safety factor 4, we define

$$\sigma_3 = \frac{\sigma_1}{4} \approx 45.2 \text{ MPa} \tag{6.14}$$

and set $\sigma_{r\,\max} = \sigma_3$. As a result we satisfy the inequality (6.10) and the recommended safety requirements from [20].

The value $\sigma_4 = |\sigma_\varphi|_{\max}$ in (6.13) determines the kevlar winding thickness for each particular segment of the structure through the formulas (6.4) and (6.9) thus determining the overall side walls thickness. The value $\sigma_3 = \sigma_{r\,\max}$ in (6.14) delimits the number of segments and the overall height of the structure through (6.4) and the recurrence formula (4.11).

7. Computation of the height and other parameters of the structure.

Let N be the number of segments in the structure. The height of the structure is limited by the maximal gas pressure in its lowermost segment $p_3 = p(N)$ (see (6.4)). For this segment from $\sigma_3 = \sigma_{r\,\max}$ and (6.14) we derive

$$p(N) = p_3 \approx \sigma_3. \tag{7.1}$$

Applying (7.1) to the first formula (6.9) and taking into account $|\sigma_\varphi|_{\max} = \sigma_4$ we derive the equation for R_4 in the case of the lowermost segment:

$$\frac{\sigma_3\left(R_4^2 + R_3^2\right)}{R_4^2 - R_3^2} \approx \sigma_4. \tag{7.2}$$

The value of R_3 in (7.2) does not depend on a particular segment of the structure. It is a constant given by the first formula (5.1). Substituting the values of R, d_0 and d_3 from (2.1), (2.2), and (2.4) into (5.1) we compute

$$R_3 = 0.512 \text{ m}. \tag{7.3}$$

Then, solving the equation (7.2) with respect to R_4, we get

$$R_4(N) \approx \sqrt{\frac{\sigma_4 + \sigma_3}{\sigma_4 - \sigma_3}}\, R_3 \approx 0.539 \text{ m}. \tag{7.4}$$

Comparing (7.4) and (7.3), we see that the kevlar carcass is not thick. Its maximum thickness is 2.7 centimeters.

The outer radius of the structure is given by the formula (2.3). Using (2.5) and (5.1) and looking at Fig. 2.2, we can transform it as follows:

$$R_2 = R + d_0 + d_2 = R + d_0 + d_3 + d_1 + d_4 = R_4 + d_4. \tag{7.5}$$

Applying (2.4) and (7.4) to (7.5), we find the maximal value of the outer radius:

$$\max(R_2) = R_2(N) = R_4(N) + d_4 \approx 0.551 \text{ m}. \tag{7.6}$$

Now let's recall the formula (2.7). It determines the size of square platforms separating segments of the structure from each other (see Fig. 2.1). These platforms are also used for connecting vertical cylindrical structures with each other. Substituting (7.6) into the formula (2.7), we compute

$$a \approx 1.653 \text{ m}. \tag{7.7}$$

The weight of each platform is given by the formula (3.4). As we already noted above, for the density of steel ρ_2 in this formula we choose the value

$$\rho_2 = 8.05 \text{ g/cm}^3 = 8050 \text{ kg/m}^3, \tag{7.8}$$

which is in agreement with the data from Wikipedia [21]. Substituting (2.6), (7.7), and (7.8) into the formula (3.4), we compute the value of w_1:

$$w_1 \approx 2157.8 \text{ N}. \tag{7.9}$$

Note that we use the force units (newtons) for expressing weight in (7.9).

The weight of the inner gas-holding tube is given by the formula (3.2), where ρ_0 is the rubber density. In the case of butyl rubber, which is commonly used in tires and hoses, its density varies from 1.15 g/cm^3 to 1.35 g/cm^3 (see [22]). We choose

$$\rho_0 = 1.35 \text{ g/cm}^3 = 1350 \text{ kg/m}^3 \tag{7.10}$$

for our computations. Substituting (2.1) and (7.10) into (3.2), we get

$$w_1 \approx 731.22 \text{ N}. \tag{7.11}$$

The formula (3.3) is more complicated than (3.2) and (3.4). We can simplify it using the radii introduced in (5.1) and taking into account (2.5):

$$\begin{aligned} w_2 = \pi\,[(R_4+d_4)^2-(R+d_0)^2]\,L\,\rho_0\,g+\\ +\pi\,[R_4^2-R_3^2]\,L\,(\rho_1-\rho_0)\,g. \end{aligned} \tag{7.12}$$

Here ρ_1 is the density of kevlar. Its value can be found in [7] and in [19]:

$$\rho_1 = 1.44 \text{ g/cm}^3 = 1440 \text{ kg/m}^3. \tag{7.13}$$

Note that R_3, d_0, d_4, ρ_0, and ρ_1 in (7.12) are constants. Their values are given in (7.3), (2.2), (2.4), (7.10), and (7.13) respectively. However R_4 in (7.12) is not constant. The outer radius of the kevlar carcass depends on the pressure it should withstand. In (7.4) we have its maximal value corresponding to the lowermost segment of the structure.

Since w_2 is not constants, we continue our computations programmatically using the Maple package. The formula (7.12) is transformed to a Maple procedure:

```
w2:=proc(R_4) local w:
 global R,R_3,d_0,d_4,rho_0,rho_1,L:
 w:=Pi*((R_4+d_4)^2-(R+d_0)^2)*L*rho_0*g:
 w:=w+Pi*(R_4^2-R_3^2)*L*(rho_1-rho_0)*g:
 return evalf(w):
end proc:
```

The radius R_4 is given by a formula similar to (7.4). Due to (6.4) and (7.1) this formula is produced from (7.4) by replacing σ_3 with p_3:

$$R_4 = \sqrt{\frac{\sigma_4+p_3}{\sigma_4-p_3}}\,R_3. \tag{7.14}$$

The formula (7.14) is also transformed to a Maple procedure:

```
R4:=proc(p_3) local r:
 global R_3,sigma_4:
 r:=sqrt((sigma_4+p_3)/(sigma_4-p_3))*R_3:
 return evalf(r):
end proc:
```

Now we can proceed to the recurrence equation (4.11). This equation is integrated numerically by means of the following Maple code:

```
for i from 1 by 1 while p[i]<=sigma_3 do
 w[i]:=2*(w_0+w_1+w2(R4(p[i]))):
 N:=N+1:  t_m:=t_m+w[i]/g:  t_p:=t_p+w[i]/2/g:
 p[i+1]:=k*p[i]+w[i]/S:
end do:
```

The area of each diaphragm S in (4.11) in the above code (see Fig. 2.1) is given by the formula (3.7). Its value is computed using (2.1):

$$S \approx 0.79\ \mathrm{m}^2.$$

The total weight of i-th segment is given by the formula (3.5). This formula is built into the above code. Two auxiliary variables `t_m` and `t_p` in the code stand for total total mass per column and total payload mass per column in the structure. For the coefficient `k` in the code we have two options. If the gas used is helium, the value of k is given in (4.8). If the nitrogen gas is used, its value is given in (4.10).

Like the equation (4.11), the above code is recurrent. In order to run it we need to set the initial value of the internal pressure, i. e. the value of the variable `p[1]`. Assume that the structure is designed to hold on its top a payload of the mass M per each column. Then the initial pressure is given by the formula

$$p(1) = \frac{M\,g + w_1}{S}. \tag{7.15}$$

Here w_1 is the weight of the topmost platform of the structure. Its value is taken from (7.11). For M in (7.15) we choose

$$M = 20 \text{ tonnes} = 20\,000 \text{ kg}. \tag{7.16}$$

Programmatically the variables are initialized as follows:

```
M:=20000:
p[1]:=(M*g+w_1)/S;
N:=0:  t_m:=M+w_1/g:  t_p:=M:
```

Then the recurrence equation (4.11) with the initial data in (7.15) and (7.16) is solved by means of the code given in the previous page. The results of running this code are presented in the following table.

Gas	He	N_2
Payload mass on the top per one column (tonnes)	20	20
Number of segments (storeys) in the structure	2391	1878
Maximal height of the structure (km)	7.173	5.634
Total payload mass per one column (tonnes)	1726	1336
Total mass[1] per one column (tonnes)	3433	2653
Pressure (stress) upon the basement (MPa)	45.2	45.2

8. Summary and conclusions.

The data of the above table in the previous section is the main result of the present paper. Comparing maximal heights in the last two columns of the table,

[1] The mass of the gas is not included into the total mass of the structure. However it is taken into account when computing the pressure upon the basement

we see that they do not differ many times, though their difference is substantial. As far as prices are concerned, nitrogen is many times as cheaper, than helium.

Tower structures considered in [5–7]) are light single column mast-like towers. Unlike them, multicolumn tower structures suggested in the present paper can hold substantial amount of payload (more than one thousand tonnes per one column). Though they are lower than those in [5–7]) and cannot reach the cosmic space, they can form core structures for buildings which are several times taller than presently available skyscribers.

The kevlar strength limit set by the inequality (6.11) and by the safety coefficient $\mathrm{SF} = 4$ in (6.13) is reached in each segment of our structure. Therefore the major limiting factor for the height of our structure is the rubber compressive strength in (6.10). Looking at the second table on page 13 above, we see that there are some polymers stronger than rubber. These are polytetrafluoroethylene (PTFE), perfluoroalkoxy alkanes (PFA), nylon, and polyoxymethylene (POM). However, all of them are not elastomers. Using them in place of rubber would produce difficulties in transporting, mounting and replacing parts of the structure when they become damaged or ramshackle. Moreover, being inelastic, the inner gas-holding tube would badly fit the outer shell thus producing risks of cracking and tearing. Further prospects for increasing the height of ultra-tall buildings in our approach are expected from some future elastomers that would have the flexibility of rubber and the compressive strength of harder polymers.

References

1. Tsiolkovsky K. E., *Dreams about the Earth and the Sky*, GONTI, Moscow, Leningrad, 1938.
2. Pearson J., *The orbital tower: a spacecraft launcher using Earth's rotational energy*, Acta astronautica **2** (1975), 785-799.
3. Clarke A. C., *Fountains of Paradise*, Harcourt Brace Jovanovich, New York, 1978.
4. *ASTM A36 steel*, MatWeb, material property data, online resource.
5. Bolonkin A. A., *Optimal inflatable space towers with 3-100 km height*, Journal of the British Interplanetary Society **56** (2003), 87–97.
6. Bolonkin A. A., *Optimal solid space tower*, e-print arXiv:physics/0701093.
7. Seth R. K., Quine B. M., Zhu Z. H., *Feasibility of 20 km free-standing inflatable space tower*, Journal of the British Interplanetary Society **62** (2009), 342–353.
8. Bird W. W., *Air inflated tower or the like*, Patent US 2849011 (1958).
9. Quine B. M., *Space elevator*, Patent US 9085987 B2 (2015), Assignee: Thoth Technology Inc., Pembroke ON, Canada.
10. *Nord Stream 2. Pipes (Russian)*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
11. Labossiere P. E., Flores D., Vant C., *Thick walled cylinders*, Mechanics of materials laboratory, University of Washington, 2007; see http://courses.washington.edu/me354a/Thick Walled Cylinders.pdf.
12. Sharipov R. A., *Course of differential geometry*, Bashkir State University, Ufa, 1996; see also arXiv:math/0412421 [math.HO].
13. Sharipov R. A., *Course of analytical geometry*, Bashkir State University, Ufa, 2011; see also arXiv:1111.6521 [math.HO].
14. Landau L. D., Lifshits E. M., *Theory of elasticity. Theoretical physics, Vol. VII*, Nauka publishers, Moscow, 1987.
15. *Levi-Civita connection*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
16. *Hooke's law*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
17. *Elastic modulus*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
18. *Poisson's ratio*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
19. *Kevlar, structure and properties*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.

20. *Factor of safety*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
21. *Steel, material properties*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
22. *Butyl rubber*, MatWeb, material property data, online resource.

Bashkir State University, 32 Zaki Validi street, 450074 Ufa, Russia
E-mail address: r-sharipov@mail.ru
URL: http://ruslan-sharipov.ucoz.com
http://freetextbooks.narod.ru