Free fall on the spinning planet

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Abstract

The external Schwarzschild metric was extended to the case of a spinning planet, limited to distances from the center correspondingly larger than the planet's radius. Three-dimensional equations of free particle motion are presented. The acceleration of a free falling particle onto a spinning planet in the case of a weakly curved spacetime for low speeds in vector form was recorded, enabling the physical interpretation of individual terms. The resultant torque acting on the test particle causing the precession of its orbital angular momentum was determined. As an example, the expression for the first cosmic velocity in the case of a circular orbit in the equatorial plane was given.

Keywords: general relativity, Schwarzschild metric, Kerr metric, equations of motion, Coriolis acceleration, anomalous rotation of Mercury's perihelion, precession of the orbital angular momentum, de Sitter precession, Lense-Thirring precession, Schiff precession, Gravity Probe B satellite

01. Introduction

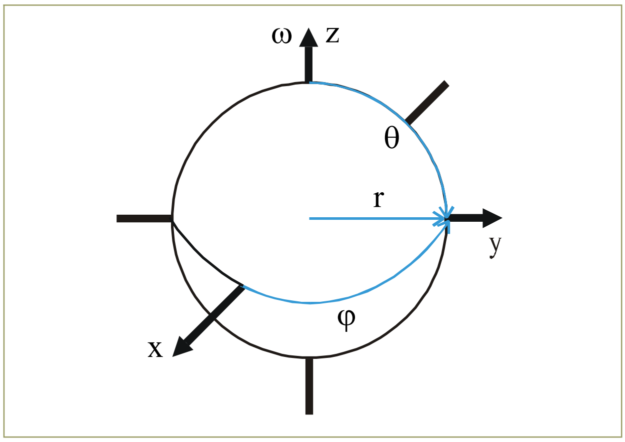
We will extend the external Schwarzschild metric to the case of a spinning planet, limiting ourselves to distances from the center correspondingly larger than the planet's radius. We will present three-dimensional equations of free particle motion.

We will write the acceleration of a free falling particle onto a rotating planet in the case of a weakly curved spacetime for low speeds in vector form enabling the physical interpretation of individual terms.

We will determine the resultant torque $m(\mathbf{r} \times \mathbf{a})$ acting on the test particle with mass m causing the precession of its orbital angular momentum.

As examples we will give, among others:
1. Expression for the first cosmic velocity in the case of a circular orbit in the equatorial plane.
2. A simplified form of the equations of free motion of the test particle in the gravitational field of a rotating planet in variables (t, r, θ, φ).
3. A simplified form of the equations of free motion of the test particle in the gravitational field of a non-rotating planet in variables (t, r, θ, φ).
4. The speed of light propagation according to the Schwarzschild metric for a rotating planet.

02. External Schwarzschild metric for a spinning planet [9]



The non-primed system rotates with the planet relative to the primed system at a constant angular velocity $\boldsymbol{\omega}$ along the axis Z.
So we have $\varphi = \varphi' - \omega t'$. The other coordinates in both systems are identical.

$$(ds')^2 = \left(1 - \frac{r_S}{r'}\right)^{-1}(dr')^2 + r'^2(d\theta')^2 + r'^2\sin^2\theta'(d\varphi')^2 + \left(1 - \frac{r_S}{r'}\right)(dx'^4)^2, \quad r_S = \frac{2GM}{c^2}$$

$$r = r' \qquad dr' = dr$$
$$\theta = \theta' \qquad d\theta' = d\theta$$
$$\varphi = \varphi' - i^{-1}c^{-1}\omega x'^4 \qquad d\varphi' = d\varphi + i^{-1}c^{-1}\omega dx'^4$$
$$x^4 = x'^4 \qquad dx'^4 = dx^4$$

$$ds^2 = \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 + 2i^{-1}c^{-1}\omega r^2\sin^2\theta\, d\varphi\, dx^4 + \left(1 - \frac{r_S}{r} - c^{-2}\omega^2 r^2\sin^2\theta\right)(dx^4)^2$$

Based on the metric form, we can determine the components of the covariant metric tensor and its determinant, and the components of the contravariant metric tensor.

$$g_{rr} = g_{11} = \left(1 - \frac{r_S}{r}\right)^{-1}, \quad g_{\theta\theta} = g_{22} = r^2, \quad g_{\varphi\varphi} = g_{33} = r^2\sin^2\theta,$$

$$g_{\varphi x^4} = g_{34} = i^{-1}c^{-1}\omega r^2\sin^2\theta, \quad g_{x^4\varphi} = g_{43} = i^{-1}c^{-1}\omega r^2\sin^2\theta,$$

$$g_{x^4x^4} = g_{44} = 1 - \frac{r_S}{r} - c^{-2}\omega^2 r^2\sin^2\theta,$$

$$g = g_{11}g_{22}\left(g_{33}g_{44} - g_{34}g_{34}\right) = r^4\sin^2\theta \qquad g_{33}g_{44} - g_{34}g_{34} = r^2\sin^2\theta\left(1 - r_S r^{-1}\right),$$

$$g^{11} = g_{11}^{-1} = 1 - \frac{r_S}{r}, \quad g^{22} = g_{22}^{-1} = r^{-2}, \quad g^{33} = \frac{g_{44}}{g_{33}g_{44} - g_{34}g_{34}},$$

$$g^{34} = g^{43} = \frac{-g_{34}}{g_{33}g_{44} - g_{34}g_{34}} = -i^{-1}c^{-1}\omega\left(1 - \frac{r_S}{r}\right)^{-1}, \quad g^{44} = \frac{g_{33}}{g_{33}g_{44} - g_{34}g_{34}} = \left(1 - \frac{r_S}{r}\right)^{-1}.$$

$$ds^2 = -c^2dt^2 g_{44}\left\{1 - \frac{1}{c^2}\left[\frac{g_{11}}{g_{44}}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{g_{44}}\left(\frac{d\theta}{dt}\right)^2 + \frac{r^2\sin^2\theta}{g_{44}}\left(\frac{d\varphi}{dt}\right)^2\right] - \frac{2\omega r^2\sin^2\theta}{c^2 g_{44}}\frac{d\varphi}{dt}\right\}$$

03. Christoffel symbols of the first kind [9]:

$$\left[{}^{1\,1}_{\;1}\right]=\frac{1}{2}\frac{\partial g_{11}}{\partial r}=-\frac{1}{2}\left(1-\frac{r_S}{r}\right)^{-2}\cdot\frac{r_S}{r^2}$$

$$\left[{}^{2\,2}_{\;1}\right]=-\frac{1}{2}\frac{\partial g_{22}}{\partial r}=-r$$

$$\left[{}^{3\,3}_{\;1}\right]=-\frac{1}{2}\frac{\partial g_{33}}{\partial r}=-r\sin^2\theta$$

$$\left[{}^{3\,3}_{\;2}\right]=-\frac{1}{2}\frac{\partial g_{33}}{\partial \theta}=-r^2\sin\theta\cos\theta$$

$$\left[{}^{4\,4}_{\;1}\right]=-\frac{1}{2}\frac{\partial g_{44}}{\partial r}=-\frac{1}{2}\frac{r_S}{r^2}+\frac{1}{c^2}\cdot\omega^2 r\sin^2\theta$$

$$\left[{}^{4\,4}_{\;2}\right]=-\frac{1}{2}\frac{\partial g_{44}}{\partial \theta}=\frac{1}{c^2}\cdot\omega^2 r^2\sin\theta\cos\theta$$

$$\left[{}^{1\,2}_{\;2}\right]=\left[{}^{2\,1}_{\;2}\right]=\frac{1}{2}\frac{\partial g_{22}}{\partial r}=r$$

$$\left[{}^{1\,3}_{\;3}\right]=\left[{}^{3\,1}_{\;3}\right]=\frac{1}{2}\frac{\partial g_{33}}{\partial r}=r\sin^2\theta$$

$$\left[{}^{1\,3}_{\;4}\right]=\left[{}^{3\,1}_{\;4}\right]=\frac{1}{2}\frac{\partial g_{34}}{\partial r}=\frac{1}{ic}\cdot\omega r\sin^2\theta$$

$$\left[{}^{1\,4}_{\;3}\right]=\left[{}^{4\,1}_{\;3}\right]=\frac{1}{2}\frac{\partial g_{43}}{\partial r}=\frac{1}{ic}\cdot\omega r\sin^2\theta$$

$$\left[{}^{1\,4}_{\;4}\right]=\left[{}^{4\,1}_{\;4}\right]=\frac{1}{2}\frac{\partial g_{44}}{\partial r}=\frac{1}{2}\frac{r_S}{r^2}-\frac{1}{c^2}\cdot\omega^2 r\sin^2\theta$$

$$\left[{}^{2\,3}_{\;3}\right]=\left[{}^{3\,2}_{\;3}\right]=\frac{1}{2}\frac{\partial g_{33}}{\partial \theta}=r^2\sin\theta\cos\theta$$

$$\left[{}^{2\,3}_{\;4}\right]=\left[{}^{3\,2}_{\;4}\right]=\frac{1}{2}\frac{\partial g_{34}}{\partial \theta}=\frac{1}{ic}\cdot\omega r^2\sin\theta\cos\theta$$

$$\left[{}^{2\,4}_{\;3}\right]=\left[{}^{4\,2}_{\;3}\right]=\frac{1}{2}\frac{\partial g_{34}}{\partial \theta}=\frac{1}{ic}\cdot\omega r^2\sin\theta\cos\theta$$

$$\left[{}^{2\,4}_{\;4}\right]=\left[{}^{4\,2}_{\;4}\right]=\frac{1}{2}\frac{\partial g_{44}}{\partial \theta}=-\frac{1}{c^2}\cdot\omega^2 r^2\sin\theta\cos\theta$$

$$\left[{}^{3\,4}_{\;1}\right]=\left[{}^{4\,3}_{\;1}\right]=-\frac{1}{2}\frac{\partial g_{34}}{\partial r}=-\frac{1}{ic}\cdot\omega r\sin^2\theta$$

$$\left[{}^{3\,4}_{\;2}\right]=\left[{}^{4\,3}_{\;2}\right]=-\frac{1}{2}\frac{\partial g_{34}}{\partial \theta}=-\frac{1}{ic}\cdot\omega r^2\sin\theta\cos\theta$$

04. Christoffel symbols of the second kind [9] (approximate values are given for $r >> r_S$):

$$\Gamma^1_{11}=g^{11}\left[{}^{1\,1}_{\;1}\right]=-\frac{1}{2}\left(1-\frac{r_S}{r}\right)^{-1}\cdot\frac{r_S}{r^2}=-\left(1-\frac{r_S}{r}\right)^{-1}\cdot\frac{1}{c^2}\cdot\frac{GM}{r^2}\approx-\frac{1}{c^2}\cdot\frac{GM}{r^2}$$

$$\Gamma^1_{22}=g^{11}\left[{}^{2\,2}_{\;1}\right]=-\left(1-\frac{r_S}{r}\right)\cdot r\approx -r$$

$$\Gamma^1_{33}=g^{11}\left[{}^{3\,3}_{\;1}\right]=-\left(1-\frac{r_S}{r}\right)\cdot r\sin^2\theta\approx -r\sin^2\theta$$

$$\Gamma^1_{34}=\Gamma^1_{43}=g^{11}\left[{}^{3\,4}_{\;1}\right]=-\left(1-\frac{r_S}{r}\right)\cdot\frac{1}{ic}\cdot\omega r\sin^2\theta\approx-\frac{1}{ic}\cdot\omega r\sin^2\theta$$

$$\Gamma^1_{44}=g^{11}\left[{}^{4\,4}_{\;1}\right]=\left(1-\frac{r_S}{r}\right)\left(-\frac{1}{2}\cdot\frac{r_S}{r^2}+\frac{1}{c^2}\cdot\omega^2 r\sin^2\theta\right)\approx-\frac{1}{c^2}\cdot\frac{GM}{r^2}+\frac{1}{c^2}\cdot\omega^2 r\sin^2\theta$$

$$\Gamma^2_{12}=\Gamma^2_{21}=g^{22}\left[{}^{1\,2}_{\;2}\right]=\frac{1}{r}$$

$$\Gamma^2_{33}=g^{22}\left[{}^{3\,3}_{\;2}\right]=-\sin\theta\cos\theta$$

$$\Gamma^2_{34}=\Gamma^2_{43}=g^{22}\left[{}^{3\,4}_{\;2}\right]=-\frac{1}{ic}\cdot\omega\sin\theta\cos\theta$$

$$\Gamma^2_{44}=g^{22}\left[{}^{4\,4}_{\;2}\right]=\frac{1}{c^2}\cdot\omega^2\sin\theta\cos\theta$$

$$g^{33}+g^{34}\cdot\frac{\omega}{ic}=\frac{g_{44}-g_{34}\cdot\frac{\omega}{ic}}{g_{33}g_{44}-g_{34}g_{34}}=\frac{1}{r^2\sin^2\theta},$$

$$g^{43}+g^{44}\cdot\frac{\omega}{ic}=\frac{-g_{34}+g_{33}\cdot\frac{\omega}{ic}}{g_{33}g_{44}-g_{34}g_{34}}=0,$$

$$\Gamma^3_{13}=\Gamma^3_{31}=g^{33}\left[{}^{1\,3}_{\ 3}\right]+g^{34}\left[{}^{1\,3}_{\ 4}\right]=r\sin^2\theta\cdot\left(g^{33}+g^{34}\cdot\frac{\omega}{ic}\right)=\frac{1}{r},$$

$$\Gamma^3_{14}=\Gamma^3_{41}=g^{33}\left[{}^{1\,4}_{\ 3}\right]+g^{34}\left[{}^{1\,4}_{\ 4}\right]=\frac{\omega}{icr}\cdot\left[1-\frac{1}{2}\cdot\frac{r_S}{r}\cdot\left(1-\frac{r_S}{r}\right)^{-1}\right]\underset{c^2>>\omega^2r^2}{\overset{r>>r_S}{\approx}}\frac{\omega}{icr},$$

$$\Gamma^3_{23}=\Gamma^3_{32}=g^{33}\left[{}^{2\,3}_{\ 3}\right]+g^{34}\left[{}^{2\,3}_{\ 4}\right]=r^2\sin\theta\cos\theta\cdot\left(g^{33}+g^{34}\cdot\frac{\omega}{ic}\right)=\mathrm{ctg}\,\theta,$$

$$\Gamma^3_{24}=\Gamma^3_{42}=g^{33}\left[{}^{2\,4}_{\ 3}\right]+g^{34}\left[{}^{2\,4}_{\ 4}\right]=\frac{1}{ic}\cdot\omega r^2\sin\theta\cos\theta\cdot\left(g^{33}+g^{34}\cdot\frac{\omega}{ic}\right)=\frac{1}{ic}\cdot\omega\,\mathrm{ctg}\,\theta,$$

$$\Gamma^4_{13}=\Gamma^4_{31}=g^{43}\left[{}^{1\,3}_{\ 3}\right]+g^{44}\left[{}^{1\,3}_{\ 4}\right]=0,$$

$$\Gamma^4_{14}=\Gamma^4_{41}=g^{43}\left[{}^{1\,4}_{\ 3}\right]+g^{44}\left[{}^{1\,4}_{\ 4}\right]=\frac{1}{c^2}\cdot\frac{GM}{r^2}\cdot\left(1-\frac{r_S}{r}\right)^{-1}\overset{r>>r_S}{\approx}\frac{1}{c^2}\cdot\frac{GM}{r^2},$$

$$\Gamma^4_{23}=\Gamma^4_{32}=g^{43}\left[{}^{2\,3}_{\ 3}\right]+g^{44}\left[{}^{2\,3}_{\ 4}\right]=0,$$

$$\Gamma^4_{24}=\Gamma^4_{42}=g^{43}\left[{}^{2\,4}_{\ 3}\right]+g^{44}\left[{}^{2\,4}_{\ 4}\right]=0.$$

05. Three-dimensional equations of free particle motion [9]:

$$\frac{d^2r}{ds^2}=-\Gamma^1_{11}\frac{dr}{ds}\frac{dr}{ds}-\Gamma^1_{22}\frac{d\theta}{ds}\frac{d\theta}{ds}-\Gamma^1_{33}\frac{d\varphi}{ds}\frac{d\varphi}{ds}-2\Gamma^1_{34}\frac{d\varphi}{ds}\frac{dx^4}{ds}-\Gamma^1_{44}\frac{dx^4}{ds}\frac{dx^4}{ds},$$

$$\frac{d^2\theta}{ds^2}=-2\Gamma^2_{12}\frac{dr}{ds}\frac{d\theta}{ds}-\Gamma^2_{33}\frac{d\varphi}{ds}\frac{d\varphi}{ds}-2\Gamma^2_{34}\frac{d\varphi}{ds}\frac{dx^4}{ds}-\Gamma^2_{44}\frac{dx^4}{ds}\frac{dx^4}{ds},$$

$$\frac{d^2\varphi}{ds^2}=-2\Gamma^3_{13}\frac{dr}{ds}\frac{d\varphi}{ds}-2\Gamma^3_{14}\frac{dr}{ds}\frac{dx^4}{ds}-2\Gamma^3_{23}\frac{d\theta}{ds}\frac{d\varphi}{ds}-2\Gamma^3_{24}\frac{d\theta}{ds}\frac{dx^4}{ds},$$

$$\frac{d}{ds}\approx\frac{1}{ic}\frac{d}{dt},\quad \frac{d^2}{ds^2}\approx-\frac{1}{c^2}\frac{d^2}{dt^2},\quad \text{gdy}\quad r>>r_S,\quad \omega^2r^2<<c^2,\quad v^2<<c^2,$$

$$\mathbf{a}=\frac{d^2r}{dt^2}\sqrt{g_{rr}}\frac{\mathbf{e}_r}{\sqrt{g_{rr}}}+\frac{d^2\theta}{dt^2}\sqrt{g_{\theta\theta}}\frac{\mathbf{e}_\theta}{\sqrt{g_{\theta\theta}}}+\frac{d^2\varphi}{dt^2}\sqrt{g_{\varphi\varphi}}\frac{\mathbf{e}_\varphi}{\sqrt{g_{\varphi\varphi}}},$$

$$\sqrt{g_{rr}}=1,\quad \sqrt{g_{\theta\theta}}=r,\quad \sqrt{g_{\varphi\varphi}}=r\sin\theta,$$

$$\begin{aligned}
\frac{d^2r}{dt^2}\cdot\sqrt{g_{rr}}&=-\frac{GM}{r^2}+\omega^2r\sin^2\theta+2\omega r\sin^2\theta\frac{d\varphi}{dt}+r\frac{d\theta}{dt}\frac{d\theta}{dt}+r\sin^2\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\\
\frac{d^2\theta}{dt^2}\cdot\sqrt{g_{\theta\theta}}&=\omega^2r\sin\theta\cos\theta+2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt}-2\frac{dr}{dt}\frac{d\theta}{dt}+r\sin\theta\cos\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\\
\frac{d^2\varphi}{dt^2}\cdot\sqrt{g_{\varphi\varphi}}&=-2\omega\sin\theta\frac{dr}{dt}-2\omega r\cos\theta\frac{d\theta}{dt}-2\sin\theta\frac{dr}{dt}\frac{d\varphi}{dt}-2r\cos\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}
\end{aligned}\ .$$

The acceleration of a free falling particle on a rotating planet in the case of a weakly curved spacetime for low speeds can be written in a vector form that allows the physical interpretation of individual terms.

$$\boxed{\mathbf{a} = \mathbf{a}_G + \mathbf{a}_O + \mathbf{a}_C + \mathbf{a}_B} \qquad \hat{\mathbf{e}}_r = \frac{\mathbf{e}_r}{\sqrt{g_{rr}}}, \quad \hat{\mathbf{e}}_\theta = \frac{\mathbf{e}_\theta}{\sqrt{g_{\theta\theta}}}, \quad \hat{\mathbf{e}}_\varphi = \frac{\mathbf{e}_\varphi}{\sqrt{g_{\varphi\varphi}}},$$

$\mathbf{a}_G$ – gravitational acceleration,

$\mathbf{a}_O$ – centrifugal acceleration,

$\mathbf{a}_C$ – Coriolis acceleration,

$\mathbf{a}_B$ – ballistic acceleration,

$$\mathbf{a}_G = -\frac{GM}{r^2} \cdot \hat{\mathbf{e}}_r,$$

$$\mathbf{a}_O = \omega^2 r \sin^2\theta \cdot \hat{\mathbf{e}}_r + \omega^2 r \sin\theta\cos\theta \cdot \hat{\mathbf{e}}_\theta,$$

$$\mathbf{a}_C = 2\mathbf{v} \times \boldsymbol{\omega} = a_C^r \cdot \hat{\mathbf{e}}_r + a_C^\theta \cdot \hat{\mathbf{e}}_\theta + a_C^\varphi \cdot \hat{\mathbf{e}}_\varphi,$$

$$\boldsymbol{\omega} = \omega^r \cdot \hat{\mathbf{e}}_r + \omega^\theta \cdot \hat{\mathbf{e}}_\theta + \omega^\varphi \cdot \hat{\mathbf{e}}_\varphi,$$

$$\mathbf{v} = v^r \cdot \hat{\mathbf{e}}_r + v^\theta \cdot \hat{\mathbf{e}}_\theta + v^\varphi \cdot \hat{\mathbf{e}}_\varphi,$$

$$\omega^r = \omega\cos\theta, \quad \omega^\theta = -\omega\sin\theta, \quad \omega^\varphi = 0,$$

$$v^r = \frac{dr}{dt}, \quad v^\theta = r\frac{d\theta}{dt}, \quad v^\varphi = r\sin\theta\frac{d\varphi}{dt},$$

$$a_C^r = 2\left(v^\theta\omega^\varphi - v^\varphi\omega^\theta\right) = 2\omega r\sin^2\theta\frac{d\varphi}{dt},$$

$$a_C^\theta = 2\left(v^\varphi\omega^r - v^r\omega^\varphi\right) = 2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt},$$

$$a_C^\varphi = 2\left(v^r\omega^\theta - v^\theta\omega^r\right) = -2\omega\sin\theta\frac{dr}{dt} - 2\omega r\cos\theta\frac{d\theta}{dt},$$

$$\mathbf{a}_C = 2\omega r\sin^2\theta\frac{d\varphi}{dt} \cdot \hat{\mathbf{e}}_r + 2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt} \cdot \hat{\mathbf{e}}_\theta - \left(2\omega\sin\theta\frac{dr}{dt} + 2\omega r\cos\theta\frac{d\theta}{dt}\right) \cdot \hat{\mathbf{e}}_\varphi,$$

$$\mathbf{a}_C = 2\omega\sin\theta\, v^\varphi \cdot \hat{\mathbf{e}}_r + 2\omega\cos\theta\, v^\varphi \cdot \hat{\mathbf{e}}_\theta - 2\omega\left(\sin\theta\, v^r + \cos\theta\, v^\theta\right) \cdot \hat{\mathbf{e}}_\varphi,$$

$$\mathbf{a}_B = \left(r\frac{d\theta}{dt}\frac{d\theta}{dt} + r\sin^2\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\right) \cdot \hat{\mathbf{e}}_r + \left(-2\frac{dr}{dt}\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\right) \cdot \hat{\mathbf{e}}_\theta + \\ + \left(-2\sin\theta\frac{dr}{dt}\frac{d\varphi}{dt} - 2r\cos\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}\right) \cdot \hat{\mathbf{e}}_\varphi$$

$$\mathbf{a}_B = \frac{1}{r}\left(v^\theta v^\theta + v^\varphi v^\varphi\right) \cdot \hat{\mathbf{e}}_r + \frac{1}{r}\left(-2v^r v^\theta + v^\varphi v^\varphi \mathrm{ctg}\theta\right) \cdot \hat{\mathbf{e}}_\theta + \frac{1}{r}\left(-2v^r v^\varphi - v^\theta v^\varphi \mathrm{ctg}\theta\right) \cdot \hat{\mathbf{e}}_\varphi.$$

NOTE

Due to the apparent singularities at the poles of the spherical coordinate system, it should be assumed that $\theta \neq 0$ and $\theta \neq \pi$.

The acceleration of a free falling particle on a rotating planet in the case of a slightly curved spacetime for low speeds will be saved again in vector form.

$$\boxed{\mathbf{a} = \mathbf{a}_r + \mathbf{a}_\theta + \mathbf{a}_\varphi} \qquad \hat{\mathbf{e}}_r = \frac{\mathbf{e}_r}{\sqrt{g_{rr}}}, \quad \hat{\mathbf{e}}_\theta = \frac{\mathbf{e}_\theta}{\sqrt{g_{\theta\theta}}}, \quad \hat{\mathbf{e}}_\varphi = \frac{\mathbf{e}_\varphi}{\sqrt{g_{\varphi\varphi}}},$$

$$\mathbf{a}_r = \left(-\frac{GM}{r^2} + \omega^2 r\sin^2\theta + 2\omega r\sin^2\theta\frac{d\varphi}{dt} + r\frac{d\theta}{dt}\frac{d\theta}{dt} + r\sin^2\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_r,$$

$$\mathbf{a}_\theta = \left(\omega^2 r\sin\theta\cos\theta + 2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt} - 2\frac{dr}{dt}\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_\theta,$$

$$\mathbf{a}_\varphi = -\left(2\omega\sin\theta\frac{dr}{dt} + 2\omega r\cos\theta\frac{d\theta}{dt} + 2\sin\theta\frac{dr}{dt}\frac{d\varphi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_\varphi.$$

Below we determine the resultant torque $m(\mathbf{r}\times\mathbf{a})$ acting on a test particle with mass m, causing the precession of its orbital angular momentum.

$$m(\mathbf{r}\times\mathbf{a}) = m\mathbf{r}\times(\mathbf{a}_r + \mathbf{a}_\theta + \mathbf{a}_\varphi),$$

$$m(\mathbf{r}\times\mathbf{a}) = m\mathbf{r}\times\mathbf{a}_r + m\mathbf{r}\times\mathbf{a}_\theta + m\mathbf{r}\times\mathbf{a}_\varphi,$$

$$\mathbf{r} = r\hat{\mathbf{e}}_r,$$

$$\mathbf{a}_r = a_r\hat{\mathbf{e}}_r, \quad a_r = -\frac{GM}{r^2} + \omega^2 r\sin^2\theta + 2\omega r\sin^2\theta\frac{d\varphi}{dt} + r\frac{d\theta}{dt}\frac{d\theta}{dt} + r\sin^2\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt},$$

$$\mathbf{a}_\theta = a_\theta\hat{\mathbf{e}}_\theta, \quad a_\theta = \omega^2 r\sin\theta\cos\theta + 2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt} - 2\frac{dr}{dt}\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt},$$

$$\mathbf{a}_\varphi = a_\varphi\hat{\mathbf{e}}_\varphi, \quad a_\varphi = -\left(2\omega\sin\theta\frac{dr}{dt} + 2\omega r\cos\theta\frac{d\theta}{dt} + 2\sin\theta\frac{dr}{dt}\frac{d\varphi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}\right),$$

$$m(\mathbf{r}\times\mathbf{a}) = mr\hat{\mathbf{e}}_r\times a_r\hat{\mathbf{e}}_r + mr\hat{\mathbf{e}}_r\times a_\theta\hat{\mathbf{e}}_\theta + mr\hat{\mathbf{e}}_r\times a_\varphi\hat{\mathbf{e}}_\varphi,$$

$$m(\mathbf{r}\times\mathbf{a}) = mra_r(\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_r) + mra_\theta(\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\theta) + mra_\varphi(\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\varphi),$$

$$\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_r = 0,$$

$$\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\theta = +\hat{\mathbf{e}}_\varphi,$$

$$\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_\theta,$$

$$m(\mathbf{r}\times\mathbf{a}) = +mra_\theta\hat{\mathbf{e}}_\varphi - mra_\varphi\hat{\mathbf{e}}_\theta.$$

The non-zero resultant torque can be represented as the sum of the torques corresponding to the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively

$$\mathbf{a}_1 = \left(-2\frac{dr}{dt}\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{d\varphi}{dt}\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_\theta - \left(2\sin\theta\frac{dr}{dt}\frac{d\varphi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_\varphi,$$

$$\mathbf{a}_2 = \left(\omega^2 r\sin\theta\cos\theta + 2\omega r\sin\theta\cos\theta\frac{d\varphi}{dt}\right)\cdot\hat{\mathbf{e}}_\theta - \left(2\omega\sin\theta\frac{dr}{dt} + 2\omega r\cos\theta\frac{d\theta}{dt}\right)\cdot\hat{\mathbf{e}}_\varphi.$$

Only the second of these torques depends on the value of the angular velocity ω of rotation of the planet.

06. Example: The first cosmic velocity in the case of a circular orbit in the equatorial plane [10, 11]

By assumption

$$r = \text{const}, \quad v^r = \frac{dr}{dt} = 0, \quad \theta = \tfrac{1}{2}\pi, \quad v^\theta = r\frac{d\theta}{dt} = 0, \quad v^\varphi = r\sin\theta\frac{d\varphi}{dt} = r\frac{d\varphi}{dt}.$$

So we have

$$\mathbf{a}_r = 0 \quad \text{or} \quad \mathbf{a}_G + \mathbf{a}_O + \mathbf{a}_C + \mathbf{a}_B = 0,$$

$$-\frac{GM}{r^2} + \omega^2 r + 2\omega\left|v^\varphi\right| + \frac{\left|v^\varphi\right|\left|v^\varphi\right|}{r} = 0 \quad \text{and} \quad -\frac{GM}{r^2} + \omega^2 r - 2\omega\left|v^\varphi\right| + \frac{\left|v^\varphi\right|\left|v^\varphi\right|}{r} = 0.$$

Among the four solutions to the above equations, that are quadratic polynomials relative to $|v^\varphi|$, only two solutions are physical.

$$\left|v_1^\varphi\right| = +v_1^\varphi = -\omega r + \sqrt{\frac{GM}{r}} \quad \text{and} \quad \left|v_2^\varphi\right| = -v_2^\varphi = +\omega r + \sqrt{\frac{GM}{r}}$$

$$\omega = 2\pi\frac{\text{rad}}{\text{day}} \approx 7.3\cdot10^{-5}\frac{\text{rad}}{\text{s}}$$
$$r = 6.4\cdot10^6\,\text{m}$$
$$M = 6\cdot10^{24}\,\text{kg}$$
$$G = 6.67\cdot10^{-11}\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$$

$$\omega r = 467.2\frac{\text{m}}{\text{s}} \approx 0.5\cdot10^3\frac{\text{m}}{\text{s}}, \quad \sqrt{\frac{GM}{r}} = 7.9\cdot10^3\frac{\text{m}}{\text{s}},$$
$$v_1^\varphi = +7.4\cdot10^3\frac{\text{m}}{\text{s}}, \quad \mathbf{v}_1^\varphi = v_1^\varphi\,\mathbf{e}_\varphi,$$
$$v_2^\varphi = -8.4\cdot10^3\frac{\text{m}}{\text{s}}, \quad \mathbf{v}_2^\varphi = v_2^\varphi\,\mathbf{e}_\varphi.$$

The velocity $\mathbf{v}_1^\varphi$ should be given to the satellite in an easterly direction, Coriolis acceleration $\mathbf{a}_{C1} = 2m\left(\mathbf{v}_1^\varphi \times \boldsymbol{\omega}\right)$ is then directed radially from the center of the field source. The velocity $\mathbf{v}_2^\varphi$ should be given to the satellite in a westerly direction, the Coriolis acceleration $\mathbf{a}_{C2} = 2m\left(\mathbf{v}_2^\varphi \times \boldsymbol{\omega}\right)$ is then directed radially towards center of the field source. The orbital period to the east is greater than to the west.

07. Example: A simplified form of the equations of free motion of the test particle in the gravitational field of a rotating planet in variables (t, r, θ, φ) [9]

For $r >> r_S$ and $v_r^2, v_\theta^2, v_\varphi^2, \omega^2 r^2 << c^2$, the equations of free motion of the test particle are reduced to

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \omega^2 r\sin^2\theta + 2\omega\, r\sin^2\theta\frac{d\varphi}{dt} + r\left(\frac{d\theta}{dt}\right)^2 + r\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2$$

$$\frac{d^2\theta}{dt^2} = \omega^2\sin\theta\cos\theta + 2\omega\sin\theta\cos\theta\frac{d\varphi}{dt} - \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} + \sin\theta\cos\theta\left(\frac{d\varphi}{dt}\right)^2$$

$$\frac{d^2\varphi}{dt^2} = -\frac{2\omega}{r}\frac{dr}{dt} - 2\omega\,\text{ctg}\theta\frac{d\theta}{dt} - \frac{2}{r}\frac{dr}{dt}\frac{d\varphi}{dt} - 2\text{ctg}\theta\frac{d\theta}{dt}\frac{d\varphi}{dt}$$

These equations are a simplified description of the effects caused by accelerations – gravitational, centrifugal, Coriolis and ballistic.

08. Example: A simplified form of the equations of free motion of the test particle in the gravitational field of a non-rotating planet in variables (t, r, θ, φ) [9]

When the planet is not spinning, the equations of motion become even simpler.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + r\left(\frac{d\theta}{dt}\right)^2 + r\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 ,$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} + \sin\theta\cos\theta\left(\frac{d\varphi}{dt}\right)^2 ,$$

$$\frac{d^2\varphi}{dt^2} = -\frac{2}{r}\frac{dr}{dt}\frac{d\varphi}{dt} - 2\mathrm{ctg}\theta\frac{d\theta}{dt}\frac{d\varphi}{dt} .$$

From these equations it follows that the free test particle moves in an elliptical orbit (in particular a circular one), the orbital angular momentum (perpendicular to the orbit plane) is constant, and the orbital period does not depend on the direction of motion of the particle.

09. Example: The speed of light propagation according to the Schwarzschild metric for a rotating planet

The external Schwarzschild metric in the case of a rotating planet, for

$$\theta = \mathrm{const} = \frac{\pi}{2}, \quad d\theta = 0, \quad \varphi = \mathrm{const}, \quad d\varphi = 0,$$

is reduced to the form

$$(ds)^2 = \left(1 - \frac{r_S}{r}\right)^{-1}(dr)^2 - \left(1 - \frac{r_S}{r} - \frac{\omega^2 r^2}{c^2}\right)c^2(dt)^2 .$$

The speed (v_{light}) of light propagation be calculated from the condition of

$$(ds)^2 = 0$$

or equivalent

$$0 < v_{light}^2 = \left(\frac{dr}{dt}\right)^2 = c^2\left[\left(1 - \frac{r_S}{r}\right)^2 - \left(1 - \frac{r_S}{r}\right)\frac{\omega^2 r^2}{c^2}\right] .$$

The condition

$$\left[\left(1 - \frac{r_S}{r}\right)^2 - \left(1 - \frac{r_S}{r}\right)\frac{\omega^2 r^2}{c^2}\right] \le 1$$

is met when

$$r \gg r_S, \quad \omega^2 r^2 \ll c^2 .$$

10. Kerr metric

In 1963, Kerr [7] proposed a stationary axially symmetric solution of vacuum equations of gravitational field, whose source is the rotating mass.

$$ds^2 = \left(r^2 + a^2 \cdot \cos^2\theta\right)\left(d\theta^2 + \sin^2\theta \cdot d\varphi^2\right) + \left(2du + a \cdot \sin^2\theta \cdot d\varphi\right)\left(dr + a \cdot \sin^2\theta \cdot d\varphi\right) - \\ - \left(1 - \frac{r \cdot r_S}{r^2 + a^2 \cdot \cos^2\theta}\right)\left(du + a \cdot \sin^2\theta \cdot d\varphi\right)^2.$$

After expanding this metric has the form

$$ds^2 = \left(r^2 + a^2 \cdot \cos^2\theta\right)d\theta^2 + \left(r^2 + a^2 + \frac{a^2 \cdot r \cdot r_S \cdot \sin^2\theta}{r^2 + a^2 \cdot \cos^2\theta}\right)\sin^2\theta \cdot d\varphi^2 + \\ + 2a \cdot \sin^2\theta \cdot dr \cdot d\varphi + 2dr \cdot du + \frac{2a \cdot r \cdot r_S \cdot \sin^2\theta}{r^2 + a^2 \cdot \cos^2\theta} d\varphi \cdot du - \left(1 - \frac{r \cdot r_S}{r^2 + a^2 \cdot \cos^2\theta}\right)du^2.$$

where

$$r_S = \frac{2GM}{c^2},$$

M – mass source,

$$a = \frac{\text{angular momentum of the source relative to the Z axis}}{\text{total mass of the source} \cdot c}, \quad [a] = m,$$

u is a function of r and t.

The Kerr metric is usually written using coordinates and notations proposed by Boyer and Lindquist [1].

$$ds^2 = \frac{r^2 + a^2\cos^2\theta}{r^2 - r\,r_S + a^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + \left(r^2 + a^2 + \frac{a^2 r\,r_S \sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\, d\varphi^2 + \\ - \frac{2a\,r\,r_S\sin^2\theta}{r^2 + a^2\cos^2\theta}d\varphi \cdot d(ct) - \left(1 - \frac{r\,r_S}{r^2 + a^2\cos^2\theta}\right)d(ct)^2,$$

where

$$r = x^1, \quad \theta = x^2, \quad \varphi = x^3, \quad ct = x^4,$$

The Kerr metric is also presented as

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{a^2 r\,r_S \sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\varphi^2 + \\ - \frac{2a\,r\,r_S\sin^2\theta}{\rho^2}d\varphi \cdot d(ct) - \left(1 - \frac{r\,r_S}{\rho^2}\right)d(ct)^2,$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta , \quad \Delta = r^2 - r\, r_S + a^2 , \quad r_S = \frac{2GM}{c^2} .$$

For large r, ($r^2 >> a^2$), the Kerr metric takes the boundary form

$$ds^2 = \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 - 2a\frac{r_S}{r}\sin^2\theta\, d\varphi\, d(ct) - \left(1 - \frac{r_S}{r}\right) d(ct)^2 .$$

For $a = 0$, the Kerr metric is reduced to the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 - \left(1 - \frac{r_S}{r}\right) d(ct)^2 .$$

11. The speed of light propagation according to the Kerr metric

The external Kerr metric, for

$$\theta = \text{const} = \frac{\pi}{2}, \quad d\theta = 0 , \quad \varphi = \text{const} , \quad d\varphi = 0 ,$$

is reduced to the form

$$ds^2 = \frac{r^2}{r^2 - r\, r_S + a^2} dr^2 - c^2\left(1 - \frac{r_S}{r}\right) dt^2 .$$

The speed (v_{light}) of light propagation be calculated from the condition of

$$(ds)^2 = 0$$

or equivalent

$$0 < v_{light}^2 = \left(\frac{dr}{dt}\right)^2 = c^2\left[\left(1 - \frac{r_S}{r}\right)^2 + \left(1 - \frac{r_S}{r}\right)\frac{a^2}{r^2}\right].$$

The conditions

$$\left[\left(1 - \frac{r_S}{r}\right)^2 + \left(1 - \frac{r_S}{r}\right)\frac{a^2}{r^2}\right] \le 1 , \quad \lim_{r\to\infty} v_{light} = c$$

are met when

$$r \ne r_S , \quad \frac{r_S}{a} << 1 , \quad \left(r_S > r > \frac{1}{4} r_S \quad \text{or} \quad r > \left(\frac{1}{4} r_S + \frac{1}{2}\frac{a^2}{r_S}\right)\right).$$

12. A very brief historical sketch

Gaspard Gustave de Coriolis (1792-1843) discovered in 1835 a new force of inertia, which allowed him to formulate equations of motion in a rotating reference system [2].

On the 18 November 1915, Albert Einstein (1879-1955) presented an approximate solution to the equations of a static spherically symmetric gravitational field in an empty space, the source of which is the point mass. He explained qualitatively and quantitatively the anomalous rotation of Mercury's perihelion (and other planets) under the general relativity [3].

Anomalous rotation of the perihelion of Mercury is the phenomenon that the perihelion shifts by 574 arc seconds per century as a result of a movement of the planet along an elliptical rosette, of which 43 arc seconds cannot be explained by Newton's theory.

This phenomenon was observed in 1859 by Urbain Jean Joseph Le Verrier (1811-1877), who supposed that the 43 second surplus could be caused by the hypothetical planet Vulcan or asteroids orbiting closer to the Sun than Mercury [18].

The orbit equation of a particle in a stationary centrally symmetric gravitational field received within general relativity differs from the analogous equation in Newton's theory by the element responsible for orbit rotation. The size of this correction for the rotation of the perihelion of Mercury's orbit confirms a 43-second surplus.

Albert Einstein (1879-1955) finally formulated, within the framework of general relativity, gravitational field equations on 25 November 1915 [4].

Carl Schwarzschild (1873-1916) found on the 13 January 1916 the first exact solution to the Einstein equations in the case of static (and thus stationary) spherically symmetric gravitational field in empty space, whose source is the point mass [16].

Willem de Sitter (1872-1934) described in 1916 the geodetic effect, i.e. the contribution to the precession of **orbital** angular momentum (perpendicular to the orbit plane) of a freely orbiting particle **independent** of the angular velocity of the rotating source body. This precession is called geodetic precession, de Sitter precession and the de Sitter effect [17].

The geodetic effect was then further studied by Jan Arnoldus Schouten (1883-1971) in 1919 [14] and 1922 [15] and Adriaan Daniël Fokker (1887-1972) in 1921 [6].

In 1918, Joseph Lense (1890-1985) and Hans Thirring (1888-1976) analyzed the contribution to the precession of **orbital** angular momentum (perpendicular to the orbit plane) of a freely orbiting particle **depending** on the angular velocity of the rotating source body [8]. This precession is known as Lense-Thirring precession, Lense-Thirring effect, rotational effect, gyroscopic precession, coupling of spin and orbital angular momentum, orbital dragging of inertial systems.

Leonard Isaac Schiff (1915-1971) presented in 1960 [12, 13] the submission of de Sitter's precession and Lense-Thirring's precession of the axis of rotation of a freely orbiting gyroscope in the gravitational field of a rotating source body. This phenomenon is called Schiff precession (Schiff effect).

For a polar orbit 642 km high during the year, de Sitter's precession deflects the gyro axis in the orbit plane by 6.6061 arc seconds, and Lense-Thirring precession deflects the gyro axis perpendicular to the orbit plane by 39.2 arc milliseconds [5].

On 20 March 2004, the Gravity Probe B satellite was placed in polar orbit 642 km high. On board the satellite were four cryogenic gyroscopes, although only one of them was necessary. The gyro rotors were 38 mm spheres made of quartz and covered with a layer of niobium, they were suspended electrostatically.

The results obtained in 2011 [5] confirmed the existence of Schiff precession with great accuracy. The mission was conducted by NASA and Stanford University under the direction of C. W. Francis Everit.

In 1963, Roy Patrick Kerr proposed [7] a stationary axially symmetric solution to vacuum equations of the gravitational field, whose source is the rotating mass.

The Kerr metric is usually written using coordinates and notations that were proposed in 1967 by Robert Hamilton Boyer (1932-1966) and Richard Wallace Lindquist [1].

13. Final remarks

In this paper, we adopted the following notations and constraints:

$$r \geq R,$$

$$\frac{r_S}{r} << 1,$$

$$\frac{\omega^2 r^2}{c^2} << 1,$$

$$v^r = \frac{dr}{dt}, \quad v^\theta = r\frac{d\theta}{dt}, \quad v^\varphi = r\sin\theta\frac{d\varphi}{dt}, \quad \mathbf{v} = v^r \cdot \hat{\mathbf{e}}_r + v^\theta \cdot \hat{\mathbf{e}}_\theta + v^\varphi \cdot \hat{\mathbf{e}}_\varphi,$$

$$v_r^2, v_\theta^2, v_\varphi^2, v^2 << c^2,$$

where

R – the radius of the planet,
ω – the angular speed of rotation of the planet
c – maximum value of signal propagation speed,

$$r_S = \frac{2GM}{c^2},$$

r_S – Schwarzschild radius,

$$G = 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2},$$

M – the mass of the planet.

In the case of the Earth, we have:

$$M = 6 \cdot 10^{24} kg,$$

$$r_S \approx 9 \cdot 10^{-3} m,$$

$$R = 6.4 \cdot 10^6 m,$$

$$\frac{r_S}{R} \approx 1.4 \cdot 10^{-9},$$

$$\omega = 2\pi \frac{rad}{day} \approx 7.3 \cdot 10^{-5} \frac{rad}{s},$$

$$\omega R = 467.2\frac{m}{s} \approx 0.5 \cdot 10^3 \frac{m}{s},$$

$$\frac{\omega^2 r^2}{c^2} = \frac{k^2 \omega^2 R^2}{c^2} << 1, \quad k^2 << \frac{c^2}{\omega^2 R^2}, \quad k << \frac{c}{\omega R} \approx 6 \cdot 10^5,$$

where

$$r = kR, \quad 1 \le k << 6 \cdot 10^5 .$$

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