Dirac theory in Euclidean 3D Geometric algebra (*Cl*3)

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Abstract

This article is intended as an addition to the book [10], since, in the first edition, I was double minded whether to introduce the Dirac theory for young students. Now I am quite sure that it should be introduced, and for several reasons. First, the *Cl*3 formulation of the Dirac theory is simple and the derivation of the Dirac's formula is straightforward. Second, it is relatively easy to show that gamma matrices are not the only possibility in linearizing the Klein-Gordon equation (we even do not need it in *Cl*3). Finally, the fact that it is possible to use the same mathematical (3D) formalism for classical mechanics, the special (and general) theory of relativity (without Minkowski space), electromagnetism, and both non-relativistic and relativistic quantum mechanics (without the imaginary unit) is remarkable. Not to mention the geometric clarity and possibilities of unifications, as well as generalizations. Moreover, all this without coordinates, matrices, tensors… In addition, we should appreciate the new concept of oriented numbers and simple fact that *Cl*3 contains complex, hypercomplex, and dual numbers, quaternions, spinors, etc. Geometric algebra of 3D Euclidean vector space (*Cl*3) is truly rich in structure and the question remains as to how physics would have developed had the ideas of Grassmann and Clifford been accepted in the late nineteenth and early twentieth centuries.

In the text, APS (algebra of physical space) and *Cl*3 (*Cl* is due to Clifford) mean the same: geometric algebra of 3D Euclidean vector space. The abbreviation Q/C is for quantum/classical.

Keywords: geometric algebra, spinors, Clifford algebra, standard mathematics, formalism, the cross product, geometric product, the Pauli matrices, Dirac theory, relativistic quantum mechanics, quaternions

Problems with the Klein-Gordon equation

We are using $\hat{\sigma}_i$ for the Pauli matrices, e_i for orthonormal vectors in 3D Euclidean vector space (sometimes we also use σ_i), and $j = e_1 e_2 e_3$ for the commutative unit pseudoscalar in *Cl*3.

Trying to solve the problems of the Klein-Gordon equation (see [8]), Dirac started with the equation

$$i\hbar\left(\partial_t + \alpha^k \partial_k\right)\left|\psi\left(\mathbf{x},t\right)\right\rangle = m\beta\left|\psi\left(\mathbf{x},t\right)\right\rangle,\ k = 1, 2, 3,$$

for $\left|\psi\left(\mathbf{x},t\right)\right\rangle \in \mathbb{C}^N$, α^k and β are supposed to be $N \times N$ matrices over a field (it will be $\mathbb{C}$). In order for this equation to be squared to the Klein-Gordon equation, we use a kind of conjugation and get (see [13])

$$\alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{ij} I_N, \quad (1)$$

$$\alpha^j\beta+\beta\alpha^j=0,\ \beta^2=I_N. \quad (2)$$

where I_N is the identity matrix. The smallest number N for which the above relations are satisfied is $N=4$ and this is how Dirac came to his gamma-matrices.

Why matrices? Well, because we need non-commutative quantities here. However, quaternions are also non-commutative and we can represent the quaternion units by $-i\hat{\sigma}_k$, $k=1,2,3$ (see [7] and the Sect. 1.9.10 in [10]). Moreover, what about vectors? Obviously, the relation (1) could be satisfied by the Pauli matrices or orthonormal vectors from *Cl*3. Unfortunately, people like to say things like "3D vector space is not relativistic enough". In fact, it is intrinsically relativistic; the problem is how to multiply vectors. A multivector in *Cl*3 can be written in the form

$$a^\mu e_\mu+jb^\mu e_\mu,\ \mu=0,1,2,3,\ e_0=1,\ a^\mu,\ b^\mu\in\mathbb{R},$$

where both a^μ and b^μ satisfy the Lorentz transformations (see the Sect. 5.4.7 in [10]). The argument against the Pauli matrices is that the relation (2) cannot be satisfied if α^k and β are 3D matrices, which is true. However, what if α^k are the Pauli matrices and β is – something different?

In addition to the 4×4 gamma-matrices, Dirac introduced spinors

$$|\psi\rangle=\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\end{pmatrix},$$

where z_i are complex numbers. Note that all information in $|\psi\rangle$ can also be organized in a 2×2 complex matrix, like

$$\psi=\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix},$$

which, contrary to $|\psi\rangle$, is possibly invertible (it could be important, see[13]).

Now, what if the quantity β in $\beta\psi$ is not a 2×2 complex matrix, but a **matrix operation** (*operator*). Then, instead of $\beta\psi$, we can write $\beta(\psi)$, which, instead of (2), gives new conditions (for the details, see [13])

$$\alpha^j\beta(\psi)+\beta(\alpha^j\psi)=0,$$
$$(i\beta)^2(\psi)\equiv i\beta(i\beta(\psi))=-\psi.$$

The operation $\beta(\psi)$ has several properties (see [13])

$$\beta(\alpha^j)=-\alpha^j\beta(I),$$
$$\beta(\phi\psi)=\beta(\phi)\beta^{-1}(I)\beta(\psi),$$
$$\beta(i\psi)=-i\beta(\psi).$$
$$\beta\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} d^* & -c^*\\ -b^* & a^*\end{pmatrix}\beta(I),$$
$$\beta(I)=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\equiv\hat{\sigma}_3,$$

and, finally

$$\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & c^* \\ -b^* & -a^* \end{pmatrix}.$$

Note that we can also write

$$\beta(\psi) = \mathrm{adj}\left(\psi^\dagger\right)\hat{\sigma}_3,$$

where $\mathrm{adj}\left(\psi^\dagger\right)$ denotes the classical adjoint (*adjugate*) of $\psi^\dagger$ and † denotes a Hermitian conjugation. As an example, for 2×2 matrices we have

$$\mathrm{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

adjA = A^{-1} det A (with the conditions of existence).

If we write ($|\eta\rangle$ and $|\xi\rangle$ are columns of two complex numbers)

$$\psi = \left(|\xi\rangle \quad -i\hat{\sigma}_2|\eta\rangle^*\right),$$

it fallows that

$$\beta(\psi) = \left(|\eta\rangle \quad i\hat{\sigma}_2|\xi\rangle^*\right)$$

and

$$i\hbar\left(\partial_t + \alpha^k\partial_k\right)|\xi\rangle = m|\eta\rangle,$$
$$i\hbar\left(\partial_t + \alpha^k\partial_k\right)|\eta\rangle = m|\xi\rangle.$$

We have left and right spinors $|\eta\rangle$ and $|\xi\rangle$, combined in the form of a square matrix.

Here is a citation from [13]:

"But, is there any advantage in using 2 × 2 complex matrices instead of $\mathbb{C}^4$-valued column matrices in Dirac theory? We believe that there is, and we classify the advantages as computational, didactical and epistemological. The computational advantages are seen, for instance, when we notice that, in many cases, a square matrix possesses an inverse matrix, whereas a column matrix does not. The existence of an inverse element makes it easier to manipulate some mathematical expressions, and the proof of Fierz identities (see [13], A/N) is a very good example of this computational advantage. The didactical advantages are manifested by the fact that the same mathematical structure that can be used to study mechanics, in particular rigid body kinematics (in terms of the Cayley - Klein parameters), and electromagnetism can be used to study quantum mechanics. In other words, there is no need for an additional mathematical structure in relativistic quantum mechanics besides the one already used in classical mechanics and electromagnetism. In order to grasp the epistemological advantages, we must take into account the fact that the 2 × 2 complex matrix algebra is in fact a representation of an algebra constructed from entities with a clear geometrical meaning. This is the APS. The elements of this algebra are the representatives of geometrical objects that are oriented line segments, oriented plane fragments, and oriented volumes. For this reason, the original denomination given by Clifford for this mathematical structure was geometric algebra."

Dirac's equation in Cl3

Here we use the units $\hbar = 1$, $c = 1$, the nabla operator

$$\nabla = \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k},$$

and the paravectors

$$\partial \equiv \partial_t + \nabla, \quad A = \phi - \mathbf{A},$$

where A is EM potential.

Momentum in *Cl*3 formulation is (see [10])

$$p = \Lambda m \Lambda^{\dagger} = mu,$$

where Λ is an *eigenspinor* (a special Lorentz rotor that transforms a particle from the rest frame to the lab frame), $\Lambda\bar{\Lambda}=1$, and u is the *proper velocity*, $u\bar{u} = 1$. In the rest frame, we have $u_0 = 1$, $p_0 = m$, and in the lab frame

$$p = \Lambda m \Lambda^{\dagger} = mu.$$

It fallows that

$$p\bar{\Lambda}^{\dagger} \equiv p\overleftarrow{\Lambda} = m\Lambda, \quad (3)$$

and we can add

$$p\overleftarrow{\Lambda} = -m\Lambda,$$

for negative energies (see the Literature). The relation (3) is the classical version of the Dirac's equation. The eigenspinors technique is simple and powerful. Current density associated with the rest-frame (real) distribution $\rho(x)$ is

$$J(x) = \Lambda\rho(x)\Lambda^{\dagger} = \psi\psi^{\dagger},$$

which gives Dirac's spinors in the form

$$\psi = \sqrt{\rho}\Lambda,$$

which is also a solution of (3), so we have the equation

$$p\overleftarrow{\psi} = m\psi.$$

Thus, Dirac's spinors are just Lorentz rotors with dilatation (for the comparison, see the Sect. 2.10.6 in [10]).

The proper velocity of the particle can also be written in the form

$$u = p/m = \Lambda e^{j\beta}\Lambda^{\dagger} = e^{j\beta}B^2,$$

where B is a boost, and β is *Yvon-Takabayasi angle* (see [11]). Note that for a complex phase we have $\overline{\exp(j\beta)} = \exp(-j\beta)$ and we still have the condition $u\bar{u} = 1$, because j commutes with all elements of *Cl*3.

The differential form of the standard formulation of Dirac equation is obtained by replacing the conjugate momentum $p^{\mu} + eA^{\mu}$ by $j\partial^{\mu}$. In the paravector formulation, $(p + eA)\overleftarrow{\psi}$ is replaced by $j\partial\overleftarrow{\psi}e_3$ (we move to the quantum side of the Q/C interface), which gives

$$p\overleftarrow{\psi} = j\partial\overleftarrow{\psi}e_3 - eA\overleftarrow{\psi} = m\psi$$

or

$$j\bar{\partial}\psi e_3 - e\bar{A}\psi = m\overleftarrow{\psi}.$$

Under a Lorentz transformation L, we have

$$\bar{\partial} \to \overleftarrow{L}\,\bar{\partial}\bar{L}\,,\ \psi \to L\psi\,,\ \ \bar{A} \to \overleftarrow{L}\,\bar{A}\bar{L}\,,$$

(e_3 is invariant because it always represents the spin in the rest frame of the electron). From this, the Lorentz covariance fallows easily (see [11]).

Here is a useful relation between standard and $Cl3$ spinors (see the Sect. 2.10 in [10]):

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \\ -b^3 + ib^0 \\ -b^1 - ib^2 \end{pmatrix} \leftrightarrow \psi = a^0 + ja^k e_k + j\left(b^0 + jb^k e_k\right).$$

The reader can show (or see in [11]) that the $Cl3$ formulation is equivalent to the standard formulation. *Hint*: use the projectors

$$P_{\pm} = \left(1 \pm e_3\right)/2\,.$$

A simple solution

For $A = 0$, we have $p = const$, therefore, a possible solution for $j\partial\bar{\psi}e_3 = m\psi$ is

$$\psi(x) = \psi(0)\exp\left(-j\left\langle p\bar{x}\right\rangle_S e_3\right),$$

where $\left\langle p\bar{x}\right\rangle_S$ is just $m\tau$ (τ is the *proper time*, see [3]). We see that

$$\psi(x) = \psi(0)\exp\left(-m\tau\, je_3\right) \quad (4)$$

describes a rapid rotation at the frequency $\omega = 2m$, which is called the *Zitter-bewegung frequency*, about the direction e_3 in the particle frame. For the details, see [11], where you can find about positive and negative solutions, standing waves, Zitterbewegung frequency, Klein paradox, the basic symmetry transformations (CPT, see also [13]), the Schrödinger equation, complex APS (i.e. $Cl3$) formulation of the Dirac theory (see also [6]), etc.

The reader can investigate some quantities that are useful in applications. Defining

$$\rho = \sqrt{\sigma^2 + \omega^2}\,,\ \tan\beta = \frac{\omega}{\sigma}\,,$$

we have

$\psi\bar{\psi} = \sigma + j\omega = \rho\exp\left(j\beta\right)$, *complex scalar*;

$J = \psi\psi^{\dagger}$, the current, a *paravector*;

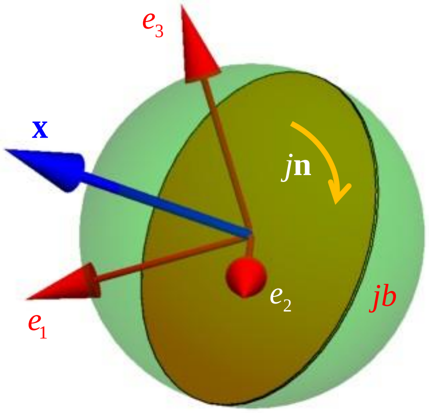
$S = \psi\, je_3\bar{\psi}$, a *complex vector*;

$s = \psi e_3\psi^{\dagger}$, the spin distribution, a *paravector*;

$\psi = \sqrt{\rho}\,\exp\left(j\beta/2\right)L\,,\ L\bar{L} = 1.$

The reader may try to apply these quantities to the solution (4).

The reader may also be interested in STA, a 4D geometric algebra with the signature (1,3). However, note that STA does not fulfill the simple idea about the same formalism for many physical theories. Not to mention that we need just the even part of the STA. Possible advantages of STA for the Dirac theory could be fulfilled by CAPS (a complex version of APS, i.e. *Cl*3, see [6]). However, note that *Cl*3 is the minimal algebra with the highest possibilities in describing the physical theories, and with clear geometric interpretation at every step of calculations. I recommend the reader to master *Cl*3 and the main physical theories in *Cl*3 first. Connections between the theories and similarities in mathematical formulations are breathtaking.



A multivector in 3D:
the **disk** represents an oriented bivector, the transparent **sphere** represents a pseudoscalar, which orientation is given by a color.

This simple geometric object is enough to formulate all main physical theories.

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