On Evaluation of an Improper Real Integral Involving a Logarithmic Function

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Abstract

In this paper we use the methods in complex analysis to evaluate an improper real integral involving the natural logarithmic function. Our presentation is somewhat unique because we use traditional notation in performing the calculations.

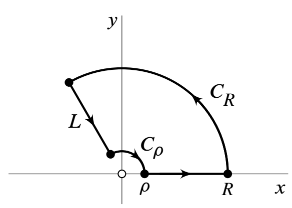
Using the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0\,, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the logarithmic function, integrate the function

$$f(z) = \frac{\log z}{z^3 + 1}$$

around the following closed contour. Let C_ρ and C_R denote arcs of the circles $|z| = \rho$ and $|z| = R$, respectively, where $\rho < 1 < R$. The leg L of this contour is a directed line segment along the ray $\arg z = \frac{2\pi}{3}$. The point $z = e^{i\pi/3}$ is the only singularity of $f(z)$ which is interior to this positively oriented simple closed contour.



Theorem 1 $\lim_{R\to\infty} \operatorname{Re} \int_{C_R} f(z)\, dz = 0.$

Proof.
Since

$$\left|\operatorname{Re} \int_{C_R} f(z)\, dz\right| \leqq \left|\int_{C_R} f(z)\, dz\right|$$

and, whenever $R > 1$ and $z = Re^{i\theta}$ is a point on C_R,

$$\left|\int_{C_R} f(z)\, dz\right| \leqq \frac{\frac{2\pi}{3} R \ln R + \frac{4\pi^2}{9} R}{R^3 - 1},$$

it follows that

$$0 \leqq \left|\operatorname{Re} \int_{C_R} f(z)\, dz\right| \leqq \frac{\frac{2\pi}{3} R \ln R + \frac{4\pi^2}{9} R}{R^3 - 1}.$$

In the last equation, by squeeze theorem,

$$\lim_{R\to\infty} \left|\operatorname{Re} \int_{C_R} f(z)\, dz\right| = 0$$

and hence Theorem 1.

Theorem 2 $\lim_{R\to\infty}\int_1^R \frac{1}{r^3+1}\,dr$ exists.

Proof.
For $r \geqq 1$,

$$0 \leqq \frac{1}{r^3+1} \leqq \frac{1}{r^3}.$$

By comparison test, $\lim_{R\to\infty}\int_1^R \frac{1}{r^3+1}\,dr$ exists.

Theorem 3 $\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr$ exists.

Proof.
From Theorem 2 and by the fact that

$$\int_\rho^R \frac{1}{r^3+1}\,dr = \int_\rho^1 \frac{1}{r^3+1}\,dr + \int_1^R \frac{1}{r^3+1}\,dr,$$

it follows that

$$\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr = \int_\rho^1 \frac{1}{r^3+1}\,dr + \lim_{R\to\infty}\int_1^R \frac{1}{r^3+1}\,dr. \tag{1}$$

Theorem 4 $\lim_{\rho\to 0^+}\operatorname{Re}\int_{C_\rho} f(z)\,dz = 0.$

Proof.
Since

$$\left|\operatorname{Re}\int_{C_\rho} f(z)\,dz\right| \leqq \left|\int_{C_\rho} f(z)\,dz\right|$$

and, whenever $\rho < 1$ and $z = \rho e^{i\theta}$ is a point on C_ρ,

$$\left|\int_{C_\rho} f(z)\,dz\right| \leqq \frac{-\frac{2\pi}{3}\rho\ln\rho + \frac{4\pi^2}{9}\rho}{1-\rho^3},$$

it follows that

$$0 \leqq \left|\operatorname{Re}\int_{C_\rho} f(z)\,dz\right| \leqq \frac{-\frac{2\pi}{3}\rho\ln\rho + \frac{4\pi^2}{9}\rho}{1-\rho^3}.$$

In the last equation, by squeeze theorem,

$$\lim_{\rho\to 0^+}\left|\operatorname{Re}\int_{C_\rho} f(z)\,dz\right| = 0$$

and hence Theorem 4.

Theorem 5 $\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr$ exists.

Proof.
For $0 < r \leqq 1$,

$$0 \leqq \frac{1}{r^3+1} \leqq 1$$

By comparison test, $\lim_{\rho\to 0^+}\int_\rho^1 \frac{1}{r^3+1}\,dr$ exists. From Theorem 2, $\lim_{R\to\infty}\int_1^R \frac{1}{r^3+1}\,dr$ exists. It follows from (1),

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr = \lim_{\rho\to 0^+}\int_\rho^1 \frac{1}{r^3+1}\,dr + \lim_{R\to\infty}\int_1^R \frac{1}{r^3+1}\,dr.$$

Theorem 6 $\lim_{R\to\infty} \operatorname{Im}\int_{C_R} f(z)\,dz = 0.$

Proof.
The proof is the same as the proof of Theorem 1 except that every occurrence of Re should be replaced by Im.

Theorem 7 $\lim_{\rho\to 0^+} \operatorname{Im}\int_{C_\rho} f(z)\,dz = 0.$

Proof.
The proof is the same as the proof of Theorem 4 except that every occurrence of Re should be replaced by Im.

Theorem 8 $\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{\ln r}{r^3+1}\,dr = -\frac{2\pi^2}{27}.$

Proof.

The residue theorem suggests that

$$\int_\rho^R \frac{\ln r}{r^3+1}\,dr + \int_{C_R} f(z)\,dz - e^{i2\pi/3}\int_\rho^R \frac{\ln r + i\frac{2\pi}{3}}{r^3+1}\,dr + \int_{C_\rho} f(z)\,dz = \frac{\pi^2}{9} + i\,\frac{\pi^2}{3\sqrt{3}}. \tag{2}$$

By equating the imaginary parts on each side of equation (2),

$$\operatorname{Im}\int_{C_R} f(z)\,dz - \frac{\sqrt{3}}{2}\int_\rho^R \frac{\ln r}{r^3+1}\,dr + \frac{\pi}{3}\int_\rho^R \frac{1}{r^3+1}\,dr + \operatorname{Im}\int_{C_\rho} f(z)\,dz = \frac{\pi^2}{3\sqrt{3}}$$

and hence

$$\int_\rho^R \frac{\ln r}{r^3+1}\,dr = \frac{2}{\sqrt{3}}\operatorname{Im}\int_{C_R} f(z)\,dz - \frac{2\pi^2}{9} + \frac{2}{\sqrt{3}}\operatorname{Im}\int_{C_\rho} f(z)\,dz + \frac{2\pi}{3\sqrt{3}}\int_\rho^R \frac{1}{r^3+1}\,dr.$$

Moreover,

$$\lim_{R\to\infty}\int_\rho^R \frac{\ln r}{r^3+1}\,dr = -\frac{2\pi^2}{9} + \frac{2}{\sqrt{3}}\operatorname{Im}\int_{C_\rho} f(z)\,dz + \frac{2\pi}{3\sqrt{3}}\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr$$

and hence this improper integral exists. From the last equation,

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{\ln r}{r^3+1}\,dr = -\frac{2\pi^2}{9} + \frac{2\pi}{3\sqrt{3}}\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr$$

and thus

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{\ln r}{r^3+1}\,dr - \frac{2\pi}{3\sqrt{3}}\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_\rho^R \frac{1}{r^3+1}\,dr = -\frac{2\pi^2}{9}. \tag{3}$$

By equating the real parts on each side of equation (2),

$$\int_\rho^R \frac{\ln r}{r^3+1}\,dr + \operatorname{Re}\int_{C_R} f(z)\,dz + \frac{1}{2}\int_\rho^R \frac{\ln r}{r^3+1}\,dr + \frac{\pi}{\sqrt{3}}\int_\rho^R \frac{1}{r^3+1}\,dr + \operatorname{Re}\int_{C_\rho} f(z)\,dz = \frac{\pi^2}{9}.$$

It follows that

$$\frac{3}{2}\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr+\frac{\pi}{\sqrt{3}}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr+\operatorname{Re}\int_{C_\rho}f(z)\,dz=\frac{\pi^2}{9}$$

and hence

$$\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr=\frac{2\pi^2}{27}-\frac{2}{3}\operatorname{Re}\int_{C_\rho}f(z)\,dz-\frac{2\pi}{3\sqrt{3}}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr.$$

From the last equation,

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr=\frac{2\pi^2}{27}-\frac{2\pi}{3\sqrt{3}}\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr$$

and thus

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr+\frac{2\pi}{3\sqrt{3}}\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr=\frac{2\pi^2}{27}. \tag{4}$$

Consider

$$x-\frac{2\pi}{3\sqrt{3}}y=-\frac{2\pi^2}{9} \tag{5}$$

and

$$x+\frac{2\pi}{3\sqrt{3}}y=\frac{2\pi^2}{27}. \tag{6}$$

From (3) and (4), $\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr$ and $\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr$ are solutions of (5) and (6). Moreover, $-\frac{2\pi^2}{27}$ and $\frac{2\pi}{3\sqrt{3}}$ are solutions of (5) and (6) as well. By uniqueness of solutions of (5) and (6),

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{\ln r}{r^3+1}\,dr=-\frac{2\pi^2}{27}$$

and

$$\lim_{\rho\to 0^+}\lim_{R\to\infty}\int_{\rho}^{R}\frac{1}{r^3+1}\,dr=\frac{2\pi}{3\sqrt{3}}.$$

References

[1] R. V. Churchill, and J. W. Brown, *Complex Variables and Applications*, 5th ed., McGraw-Hill, Inc., New York, 1990.

[2] D. V. Widder, *Advanced Calculus*, 2nd ed., Dover Publications, Inc., New York, 1989.