There exist infinitely many couples of primes (p,p+2n) ,with 2n ⩾2 is a fixed distance between p and p+2n.

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Abstract.

For any real number x > 0, let $\lfloor x \rfloor$ be the largest integer not exceeding x and $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}}$ p is the product of all primes not exceeding $\lfloor\sqrt{x}\rfloor$ with $\mathcal{P}$ is the set of primes .
let 2n≥2 denotes the distance between two primes .
let $\Pi_{2n}(x)$=card{(p,p+2n)/ p+2n $\leq x$, (p,p+2n) $\epsilon\mathcal{P}^2$ } denotes the number of couple of primes (p,p+2n) not exceeding x .
in this paper . we will prove that for any n≥1 .there is a constant A(n) such that $\Pi_{2n}(x) \geq \frac{(x-2n-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)$- $A(n)$
this result will help us to prove that , there is infinite couples of primes (p,p+2n), with 2n is a fixed distance between p and p+2n .

We will also prove the next results :

1. there exist infinite twin primes .
2. there exist infinite cousin primes .
3. The cousin primes are equivalent to twin primes in infinity.

Introduction.

let n > 0 a positive integer , and 2n denotes the distance between the couple of primes (p,p+2n) .
(just to be obvious we don't talk in this paper, about gabs between primes)
let $\Pi_{2n}(x)$=card{(p,p+2n)/ p+2n $\leq x$, (p,p+2n) $\epsilon\mathcal{P}^2$ } denotes the number of couple of primes (p,p+2n) not exceeding x . The aim of this paper is to prove that for any n≥1 , $\Pi_{2n}(x)\to\infty$,when $x\to\infty$, which means that there exists infinitly many couple of primes (p,p+2n) .
Furthermore we would have to special cases .
Case 1 , 2n=2 .
$\Pi_2(x)$=card{(p,p+2)/ p+2 $\leq x$, (p,p+2) $\epsilon\mathcal{P}^2$ } will denotes the number of couples

of twin primes not exceeding x , in fact will prove that Π_2(x)$\sim 2\frac{x}{\log(x)^2}$ which means that the conjecture of twin primes is true .
Case 2 , 2n=4 .
Π_4(x)=card{(p,p+4)/ p+4 $\leq x$, (p,p+4) $\epsilon \mathcal{P}^2$ } will denotes the number of couples of cousin primes not exceeding x , will prove also that Π_4(x)$\sim 2\frac{x}{\log(x)^2}$

In Theorem B , we will prove an extraordinarily powerful discovery is that | Π_4(x)-Π_2(x)| $\leq$ ln(x) for succiently large x.

Respectively.

Theorem A.
let n $\geq$ 1 and fix 2n as the distance between two primes, then

there exist infinitely many couples (p, p+2n), where p and p+2n are both prims.

Corollary 1.
1. Π_2(x)$\sim \frac{2x}{\log(x)^2}$ for sufficietly large x .

2. Π_4(x)$\sim \frac{2x}{\log(x)^2}$ for sufficietly large x .

Theorem B.

| Π_4(x)-Π_2(x)| $\leq$ ln(x) for sufficietly large x .

Lemma 1. For any real number x > 0, let $\lfloor x \rfloor$ be the largest integer

not exceeding x and $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}}$ p is the

product of all primes not exceeding $\lfloor\sqrt{x}\rfloor$, with $\mathcal{P}$ is the set of primes

$\mathcal{P} = \{2, 3, 5, 7 \ldots\ldots\}$ and let gcd(a,b) denotes the greatest

common divisor of the elements (a,b)

then $\lfloor\sqrt{x}\rfloor + 1 \leq n \leq x$ and gcd(n, $N_{\lfloor\sqrt{x}\rfloor}$)=1 $\Rightarrow$ n is a prime

Proof of Lemma 1. let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}}$ p
we suppose that gcd(n, $N_{\lfloor\sqrt{x}\rfloor}$)=1.

let d be a prime divisor of n $\Longrightarrow$ $1<d\leq\lfloor\sqrt{x}\rfloor$
$\Longrightarrow$ $d/N_{\lfloor\sqrt{x}\rfloor}$
$\Longrightarrow$ gcd(n, $N_{\lfloor\sqrt{x}\rfloor}$)$\neq$1 Absurd

then n is a prime

Lemma 2 . (see [01]) let μ denotes the Mobius function then .

$$\sum_{d'/\gcd(n,d)} \mu(d') = \begin{cases} 1 & \text{if } \gcd(n,d)=1 \\ 0 & \text{if not} \end{cases}$$

Lemma 3 . (see [01])

let f be a multiplicative function then $\sum_{d/n}$ f(d) is also multiplicative .

Lemma 4 .(see[04])

$\prod_{p\leqslant x, p\neq 2}\left(1-\frac{2}{p}\right) \sim \frac{1}{\log(x)^2}$, for all sufficiently large x

Lemma 5 .(see [05])

Let a,b and c , any given integers and let ax+by=c

be a diophantine equation, then ax+by=c

has a solution iff gcd(a,b)/c .

And if (x_0,y_0) is a particular solution of ax+by=c

then there exist an integer k such that ($x_0+\frac{kb}{\gcd(a,b)}$, $y_0-\frac{ka}{\gcd(a,b)}$)

is the set of solutions .

Lemma 6.

let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}}$ p and $d_1/N_{\lfloor\sqrt{x}\rfloor}$.

then $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1\wedge d_2=1 \quad\Leftrightarrow\quad d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$,

Proof of Lemma 6.

let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}}$ p and $d_1/N_{\lfloor\sqrt{x}\rfloor}$.

1- we suppose that $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$.

we have $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1} \Rightarrow d_2d_1/N_{\lfloor\sqrt{x}\rfloor} \Rightarrow d_2/N_{\lfloor\sqrt{x}\rfloor}$

and since $N_{\lfloor\sqrt{x}\rfloor}$ is squarefree and $d_1 d_2/N_{\lfloor\sqrt{x}\rfloor}$ then $d_1 \wedge d_2 = 1$

this means that $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1} \Rightarrow d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1 \wedge d_2 = 1$

2- we suppose that $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1 \wedge d_2 = 1$.

we have $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1/N_{\lfloor\sqrt{x}\rfloor}, d_1 \wedge d_2 = 1 \Rightarrow d_2 d_1/\mathrm{N}_{\lfloor\sqrt{x}\rfloor}$

$$\Rightarrow d_2/\frac{\mathrm{N}_{\lfloor\sqrt{x}\rfloor}}{\mathrm{d}_1}$$

then from 1 and 2 we obtain the equivalence .

$$d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1 \wedge d_2 = 1 \quad \Leftrightarrow \quad d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$$

Proof of theorem A.

let x >9 and fix n≥1 .

let Π_{2n}(x)=card{(p,p+2n)/ p+2n $\leq x$, (p,p+2n) $\epsilon\mathcal{P}^2$ } denotes the

number of couple of primes (p,p+2n) not exceeding x .

and Π'_{2n}(x)=card{(p,p+2n)/$\lfloor\sqrt{x}\rfloor$<p$\leq x-2n$,(p,p+2n) $\epsilon\mathcal{P}^2$ }

denotes the number of couple of primes (p,p+2n) that are between

$\lfloor\sqrt{\mathrm{x}}\rfloor$ and x .

it is evident that Π_{2n}(x) ≥ Π'_{2n}(x) (Π_{2n}(x)= $\Pi_{2n}(\sqrt{x}$)+ Π'_{2n}(x)).

Then if we can prove that Π'_{2n}(x)→+∞ when x→+∞

this will be sufficient to prove Theorem A .

in fact this will be our aim for the next sections .

Remark 1. let z=$\lfloor\sqrt{x}\rfloor$
by Lemma 1, if $\lfloor\sqrt{x}\rfloor$<p $\leq x-2n$, gcd(p,N_z)=1 and gcd(p+2n,N_z)=1
then (p, p+2n) is a couple of primes , with distance 2n .

we will exploit Remark 1 to calculate Π'_{2n}(x).

Π'_{2n}(x)=card{(p,p+2n)/$\lfloor\sqrt{x}\rfloor$<p$\leq x-2n$,gcd$(p, N_z) = 1$, gcd$(p + 2n, N_z) = 1$ }

$$=\sum_{\gcd(p,N_z)=1,\lfloor\sqrt{x}\rfloor<p\le x-2n}\sum_{\gcd(p+2n,N_z)=1,\lfloor\sqrt{x}\rfloor+2n<p+2n\le x}1$$

$$=\sum_{\gcd(p,N_z)=1,\gcd(p+2n,N_z)=1,\lfloor\sqrt{x}\rfloor<p\le x-2n}1.$$

If we apply Lemma 2 ,we obtain .

$$\Pi_{2n}^{'}(\mathrm{x})=\sum_{d_1/p\wedge N_z,d_2/(p+2n\wedge N_z),\lfloor\sqrt{x}\rfloor\le p\le x-2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)$$

$$=\sum_{d_1/N_z,d_1/p,d_2/N_z,d_2/p+2n,\lfloor\sqrt{x}\rfloor\le p\le x-2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)$$

$$=\sum_{d_1/N_z,d_2/N_z}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{d_1/p,d_2/p+2n,\lfloor\sqrt{x}\rfloor\le p\le x-2n}1$$

But we have the equivalence

$$d_1/p,d_2/p+2n \quad\Leftrightarrow \exists j,k\epsilon\mathcal{N}^{\star 2}\text{such that p=j}d_1\text{ et p+2n=k}d_2$$

Then .

$$\Pi_{2n}^{'}(\mathrm{x})=\sum_{d_1/N_z,d_2/N_z}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{p=\mathrm{j}d_1,p+2n=\mathrm{k}d_2,\lfloor\sqrt{x}\rfloor\le p\le x-2n}1$$

$$=\sum_{d_1/N_z,d_2/N_z}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{\mathrm{j}d_1+2n=\mathrm{k}d_2,\lfloor\sqrt{x}\rfloor\le p=\mathrm{j}d_1\le x-2n}1$$

Remark 2 .

we remark that the sum $\sum_{\mathrm{j}d_1+2n=\mathrm{k}d_2,\lfloor\sqrt{x}\rfloor\le p=\mathrm{j}d_1\le x-2n}1$

depends only on the diophantine equation $\mathrm{j}d_1+2n=\mathrm{k}d_2$

with j and k are the variables ,

Problem 1 .

if we want to give a explicit formula to $\Pi_{2n}^{'}(\mathrm{x})$ we would have to

calculate the sum $\sum_{\mathrm{j}d_1+2n=\mathrm{k}d_2,\lfloor\sqrt{x}\rfloor\le p=\mathrm{j}d_1\le x-2n}1.$

In fact we will find that if $\gcd(d_1,d_2)/2n$ then .

$$\sum_{\mathrm{j}d_1+2n=\mathrm{k}d_2,\lfloor\sqrt{x}\rfloor\le p=\mathrm{j}d_1\le x-2n}1=\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1}\times\frac{\gcd(d_1,d_2)}{d_2}+1+\mathrm{O}(1)$$

Proof of Problem 1 .

we set $\delta(j,k)=\{$ 1 if the equation $\mathrm{j}d_1+2n=\mathrm{k}d_2$ has a solution

$0 \quad$ if not

based on Lemma 5 we have $\delta(j,k)=\{\begin{matrix} 1 & \text{if} \quad \gcd(d_1,d_2)/2\text{n} \\ 0 & \text{if} \quad \text{not} \end{matrix}$

$$\text{then } L \quad = \sum_{\text{jd}_1+2n=\text{kd}_2, \lfloor\sqrt{x}\rfloor \le p=\text{jd}_1 \le x-2n} 1$$

$$= \sum_{\lfloor\sqrt{x}\rfloor \le p=\text{jd}_1 \le x-2n} \delta(j,k)$$

$$= \quad \sum_{\frac{\lfloor\sqrt{x}\rfloor}{d_1} \le j \le \frac{x-2n}{d_1}, j\epsilon\mathcal{N}^*} \delta(j,k)$$

$$= \sum_{\frac{\lfloor\sqrt{x}\rfloor}{d_1} \le j \le \frac{x-2n}{d_1}, \gcd(d_1,d_2)/2n, j\epsilon\mathcal{N}^*} 1$$

If we have $\gcd(d_1,d_2)/2n$, by Lemma 5 , we will have also

$\text{j}=\text{j}_0+\frac{\text{td}_2}{\gcd(d_1,d_2)}$ et k $=\text{k}_0+\frac{\text{td}_1}{\gcd(d_1,d_2)}$ with t is an integer

and (j_0,k_0) is a particular solution of $\text{jd}_1+2n=\text{kd}_2$

$$\text{then} \quad \text{L} = \sum_{\frac{\lfloor\sqrt{x}\rfloor}{d_1} \le j=j_0+\frac{\text{td}_2}{\gcd(d_1,d_2)} \le \frac{x-2n}{d_1}, j\epsilon\mathcal{N}^*} 1$$

$$= \sum_{\frac{\lfloor\sqrt{x}\rfloor}{d_1} \le j_0+\frac{\text{td}_2}{\gcd(d_1,d_2)} \le \frac{x-2n}{d_1}, t\epsilon\mathcal{N}^*} 1$$

$$= \sum_{\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0 \le \frac{\text{td}_2}{\gcd(d_1,d_2)} \le \frac{x-2n}{d_1} - j_0, t\epsilon\mathcal{N}^*} 1$$

$$= \sum_{\left(\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \le t \le \left(\frac{x-2n}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2}, t\epsilon\mathcal{N}^*} 1$$

$$= \left\lfloor \left(\frac{x-2n}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor - \left\lfloor \left(\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor + 1$$

Result 1 .

if $\gcd(d_1,d_2)/2n$ then the sum $\sum_{\text{jd}_1+2n=\text{kd}_2, \lfloor\sqrt{x}\rfloor \le p=\text{jd}_1 \le x-2n} 1$ is equal to $\text{L}=\left\lfloor \left(\frac{x-2n}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor - \left\lfloor \left(\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor + 1$

by **Result 1** we have

$$\text{L}=\left\lfloor \left(\frac{x-2n}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor - \left\lfloor \left(\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0\right) \times \frac{\gcd(d_1,d_2)}{d_2} \right\rfloor + 1$$

$$\text{then L}=\left(\frac{x-2n}{d_1} - j_0\right)\frac{\gcd(d_1,d_2)}{d_2} - \left(\frac{\lfloor\sqrt{x}\rfloor}{d_1} - j_0\right)\frac{\gcd(d_1,d_2)}{d_2} + 1 + \text{O}(1)$$

$$= \frac{x-2n}{d_1} \times \frac{\gcd(d_1,d_2)}{d_2} - \frac{\lfloor\sqrt{x}\rfloor}{d_1} \times \frac{\gcd(d_1,d_2)}{d_2} + 1 + \text{O}(1)$$

$=(\frac{x-2n}{d_1}$ - $\frac{\lfloor\sqrt{x}\rfloor}{d_1})\frac{\gcd(d_1,d_2)}{d_2}+1+O(1)$

$=\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1}\times\frac{\gcd(d_1,d_2)}{d_2}+1+O(1)$

$=\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1 d_2}\gcd(d_1,d_2)+1+O(1)$

then if $\gcd(d_1,d_2)/2n$ we will have .

$$\sum_{jd_1+2n=kd_2,\lfloor\sqrt{x}\rfloor\le p=jd_1\le x-2n}1=\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1 d_2}\gcd(d_1,d_2)+1+O(1)$$

Let us now return to calculate $\Pi_{2n}'(x)$.

we have .

$$\Pi_{2n}'(x)=\sum_{d_1/N_z,d_2/N_z}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{jd_1+2n=kd_2,\lfloor\sqrt{x}\rfloor\le p=jd_1\le x-2n}1$$

By Problem 1 ,we will obtain .

$$\Pi_{2n}'(x)=\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)(\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1 d_2}\gcd(d_1,d_2))+$$
$$\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)(\ 1+O(1))$$

Problem 2 . let $\tau(n)=\sum_{d/n}1$ denotes the number of divisors of n.

then the error term $\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)(\ 1+O(1))$

is equal to $O(2\tau(\mathrm{rad}(2n)))$

Proof of Problem 2 .

let K $=\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)(\ 1+O(1))$

$=\sum_{d_1/N_z,}\boldsymbol{\mu}(d_1)\sum_{d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_2)(\ 1+O(1))$

we set F $=\{d=d_1\wedge d_2/,\ d_1/N_z,d_2/N_z,d/2n\}$

then we will obtain .

K $=\sum_{d_1/N_z,}\boldsymbol{\mu}(d_1)\sum_{d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_2)(\ 1+O(1))$

$=\sum_{d\epsilon F}\sum_{d_1/N_z,}\boldsymbol{\mu}(d_1)\sum_{d_2/N_z,\gcd(d_1,d_2)=d}\boldsymbol{\mu}(d_2)(\ 1+\mathrm{O}(1))$

$=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\boldsymbol{\mu}(d_2)(\ 1+\mathrm{O}(1))$

By Lemma 6 we have .

$=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)(\ 1+\mathrm{O}(1))$

$=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)+\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

now we have tow sums to calculate .let us calculate them.

let $\mathrm{T}=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)$

and $\mathrm{R}=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d},}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

by Lemma 2,we have $\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)=\{\begin{array}{ll}1 & \text{if } \frac{\frac{N_z}{d}}{\frac{d_1}{d}}=1\\ 0 & \text{if not}\end{array}$

then $\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)=\{\begin{array}{ll}1 & \text{if } \frac{N_z}{d}=\frac{d_1}{d}\\ 0 & \text{if not}\end{array}$

then we obtain $\mathrm{T}=\sum_{d\epsilon F}\boldsymbol{\mu}\left(\frac{N_z}{d}\right)$

$=\sum_{d\epsilon F}(\text{-}1\ )^{\omega\left(\frac{N_z}{d}\right)}$

we have also $\mathrm{T}\leq\sum_{d\epsilon F}|(\text{-}1\ )^{\omega\left(\frac{N_z}{d}\right)}|$

$\leq\sum_{d\epsilon F}1$

$\leq\mathrm{rad(F)}$

Remark 3.

since N_z is squarefree then $\mathrm{F}=\{\mathrm{d}=d_1\wedge d_2/,\ d_1/N_z,d_2/N_z,d/\mathrm{rad}(2n)\}$

$=\{d/\mathrm{rad}(2n)\ /\ \mathrm{d}\leqslant\lfloor\sqrt{x}\rfloor\}$

We have $\tau(\text{rad}(2n))$=card{ d/rad(2n)}

=card{{ d/rad(2n)/ d$\leqslant \lfloor\sqrt{x}\rfloor$}∪{d/rad(2n) / d$\geqslant \lfloor\sqrt{x}\rfloor$}}

=card{F∪{d/rad(2n) / d$\geqslant \lfloor\sqrt{x}\rfloor$}}

Then F⊂ { d/rad(2n)} .

Which that rad(F) $\leq \tau(\text{rad}(2n))$.

And if we exploit Remark 3. we will have T$\leq$ rad(F) $\leq \tau(\text{rad}(2n))$

then T = O($\tau(\text{rad}(2n))$)

we remain to calculate the sum R=$\sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\text{O}(1)$.

$$R = \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\text{O}(1)$$

$$= \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\text{O}(1)$$

we have
$$\sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2) \leq \left|\sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d \epsilon F} \left|\sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \left|\boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} |\boldsymbol{\mu}(d_1)| \left|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \left|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$$

we have
$$\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2) = \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}\left(\frac{d_2}{d} d\right)$$

since gcd($\frac{d_2}{d}, d$)=1 ,then $\boldsymbol{\mu}\left(\frac{d_2}{d} d\right) = \boldsymbol{\mu}\left(\frac{d_2}{d}\right)\boldsymbol{\mu}(d)$

then
$$\sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \boldsymbol{\mu}(d_1) \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2) \leq \sum_{d \epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z,}{d}} \left|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}\left(\frac{d_2}{d}\right)\boldsymbol{\mu}(d)\right|$$

$$\leq\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z,}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}\left(\frac{d_2}{d}\right)|$$

and if we apply again the Lemma 2 . we obtain

$$\sum_{\frac{d_1}{d}/\frac{N_z,}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}\left(\frac{d_2}{d}\right)|=\left|\boldsymbol{\mu}\left(\frac{N_z}{d}\right)\right|=1 \quad , \quad \text{then}$$

$$\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z,}{d}}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)\leq\sum_{d\epsilon F}1$$

we already know that $\sum_{d\epsilon F}1\leq\tau(\text{rad}(2n))$ from the above calculations

then $\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z,}{d}}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)\leq\tau(\text{rad}(2n))$

then we obtain R=O(τ(rad($2n$)))

Result 2 of Problem 2.
we have T = O(τ(rad($2n$))) and R=O(τ(rad($2n$)))

then K=T+R =O(τ(rad($2n$))) +O(τ(rad($2n$)))

=O(2τ(rad($2n$)))

now we have obtained the must interesting result in this article

$$\sum_{d_1/N_z,}\boldsymbol{\mu}(d_1)\sum_{d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_2)(\ 1+O(1))=O(2\tau(\text{rad}(2n)))$$

because O(2τ(rad($2n$))) will be the error term of Π'_{2n}(x)

in the next sections you will see that this error term is much smaller than

the main term of Π'_{2n}(x).

by Problem 2. we obtain

$$\Pi'_{2n}(x)=\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\left(\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1d_2}\gcd(d_1,d_2)\right)+$$
$$+O(2\tau(\text{rad}(2n)))$$

$$=\sum_{d\epsilon F}\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)=d}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\left(\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1d_2}\text{d }\right)+$$
$$+O(2\tau(\text{rad}(2n)))$$

$$=\sum_{d\epsilon F}\sum_{d_1/N_z}\mu(d_1)\sum_{d_2/N_z,\gcd(d_1,d_2)=d}\mu(d_2)(\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1d_2}d)+O(2\tau(\mathrm{rad}(2n)))$$

$$=\sum_{d\epsilon F}\sum_{d_1/N_z}\mu(d_1)\sum_{d_2/N_z,\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\mu(d_2)(\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1d_2}d)+O(2\tau(\mathrm{rad}(2n)))$$

$$=\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\mu\left(\frac{d_1}{d}d\right)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\mu\left(\frac{d_2}{d}d\right)\frac{x-2n-\lfloor\sqrt{x}\rfloor}{d_1d_2}\mathrm{d}+O(2\tau(\mathrm{rad}(2n$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor\quad)\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\mu\left(\frac{d_1}{d}d\right)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}d\right)}{d_1d_2}\mathrm{d}+$$
$$O(2\tau(\mathrm{rad}(2n)))$$

since gcd (d,$\frac{d_1}{d}$)=1 and gcd (d,$\frac{d_2}{d}$)=1 . then $\mu\left(\frac{d_1}{d}d\right)=\mu\left(\frac{d_1}{d}\right)\mu(d)$ and $\mu\left(\frac{d_1}{d}d\right)=\mu\left(\frac{d_1}{d}\right)\mu(d)$, then we obtain.

$$\Pi'_{2n}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\mu\left(\frac{d_1}{d}\right)\mu(d)^2\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}\right)}{d_1d_2}\mathrm{d}+$$
$$+O(2\tau(\mathrm{rad}(2n)))$$

but if $d\epsilon F$, then d is a squarefree then $\mu(d)^2=1$

we obtain .

$$\Pi'_{2n}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\mu\left(\frac{d_1}{d}\right)d\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}\right)}{d_1d_2}+$$
$$O(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{d_1}d\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}\right)}{d_2}+$$
$$O(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{d_1}d\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}\times\frac{1}{d}+$$
$$+O(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}+$$
$$O(2\tau(\mathrm{rad}(2n)))$$

If we apply Lemma 6 we obtain.

$$\Pi'_{2n}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}+O(2\tau(\mathrm{rad}(2n)))$$

since $\frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}$ is multiplicative , then by Lemma 3 ,$\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}$ is also multiplicative .

then $\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \frac{\mu\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}} = \prod_{p/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} (1\text{-}\frac{1}{p})$

$$= \frac{\prod_{p/\frac{N_z}{d}}\left(1-\frac{1}{p}\right)}{\prod_{p/\frac{d_1}{d}}\left(1-\frac{1}{p}\right)}$$

we will obtain .

$$\Pi_{2n}^{'}(\mathrm{x}) =(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\frac{\prod_{\mathrm{p}/\frac{\mathrm{N_z}}{\mathrm{d}}}\left(1\text{-}\frac{1}{\mathrm{p}}\right)}{\prod_{\mathrm{p}/\frac{\mathrm{d_1}}{\mathrm{d}}}\left(1\text{-}\frac{1}{\mathrm{p}}\right)}+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{\mathrm{p}/\frac{\mathrm{N_z}}{\mathrm{d}}}\left(1\text{-}\frac{1}{\mathrm{p}}\right)\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}\left(1-\frac{1}{p}\right)}+ \mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

we apply again Lemma 3 on $\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}\left(1-\frac{1}{p}\right)}$ we obtain.

$$\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\mu\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}\left(1-\frac{1}{p}\right)} =\prod_{p/\frac{N_z}{d}}(1\text{-}\frac{1}{p\left(1-\frac{1}{p}\right)})$$

then .

$$\Pi_{2n}^{'}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{\mathrm{p}/\frac{\mathrm{N_z}}{\mathrm{d}}}\left(1\text{-}\frac{1}{\mathrm{p}}\right)\prod_{p/\frac{N_z}{d}}(1\text{-}\frac{1}{p\left(1-\frac{1}{p}\right)})+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{\mathrm{p}/\frac{\mathrm{N_z}}{\mathrm{d}}}\left(1\text{-}\frac{1}{\mathrm{p}}\right)(1\text{-}\frac{1}{p\left(1-\frac{1}{p}\right)})+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(\frac{p-1}{p}\right)\left(1-\frac{1}{p-1}\right)+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(\frac{p-1}{p}\right)\left(\frac{p-2}{p-1}\right)+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(\frac{p-2}{p}\right)+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

$$=(x-2n-\lfloor\sqrt{x}\rfloor)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right)+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

We already have F $=\{\mathrm{d}=d_1\wedge d_2/,\ d_1/N_z, d_2/N_z, d/2n\}$

from the definition of F , we can deduce that .

F=$\{1, 2, \ldots\ldots d_r\}$

then Π'_{2n}(x)=$(x-2n-\lfloor\sqrt{x}\rfloor)(\prod_{p/N_z}\left(1-\frac{2}{p}\right)+\frac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)+$

$$\sum_{d\epsilon F, d\neq 1, d\neq 2}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right))\ +O(2\tau(\text{rad}(2n)))$$

sice 2 is prime and $2/N_z$ then $(\prod_{p/N_z}\left(1-\frac{2}{p}\right)=0$.

then Π'_{2n}(x)=$(x-2n-\lfloor\sqrt{x}\rfloor)(\frac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)+$

$$\sum_{d\epsilon F, d\neq 1, d\neq 2}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right))\ +O(2\tau(\text{rad}(2n)))$$

Result 3.

$$\Pi'_{2n}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)(\frac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)+$$

$$\sum_{d\epsilon F, d\neq 1, d\neq 2}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right))+O(2\tau(\text{rad}(2n)))$$

This result is very important we will need them to prove Corollary 1 and Theorem B.

from Result 3 we deduce that .

$$\Pi'_{2n}(\mathrm{x}) \geq \frac{(x-2n-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)+O(2\tau(\text{rad}(2n)))$$

$$\geq \frac{(x-2n-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\neq 2}\left(1-\frac{2}{p}\right)\text{- } 2\tau(\text{rad}(2n))$$

$$\geq \frac{(x-2n-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x}, p\neq 2}\left(1-\frac{2}{p}\right)\text{- } 2\tau(\text{rad}(2n))$$

by Lemma 4 , we have $\prod_{p\leq\sqrt{x}, p\neq 2}\left(1-\frac{2}{p}\right)\sim\frac{1}{\log(\sqrt{x})^2}$

$$\sim\frac{4}{\log(x)^2} \quad \text{for sufficiely large x .}$$

We don't have to forget that 2n is fixed then $\tau(\text{rad}(2n))$ is also will be fix .

then $\frac{(x-2n-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x}, p\neq 2}\left(1-\frac{2}{p}\right)\sim\frac{2(x-2n-\lfloor\sqrt{x}\rfloor)}{\log(x)^2}$

$$\sim\frac{2x}{\log(x)^2} \text{ for sufficiently large x}$$

this means that Π'_{2n}(x)$\rightarrow+\infty$, when x $\rightarrow+\infty$ (because $\frac{2x}{\log(x)^2}\rightarrow+\infty$)

we already have Π_{2n}(x)$\geq$ Π'_{2n}(x) then Π_{2n}(x)$\rightarrow+\infty$ when x $\rightarrow+\infty$

this prove Theorem A .

Proof of corollary 1.

By Result 3 we have for any n$\geqslant$1 .

$$\Pi'_{2n}(\mathrm{x})=(x-2n-\lfloor\sqrt{x}\rfloor)(\tfrac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\tfrac{2}{p}\right)+\sum_{d\epsilon F,d\neq1,d\neq2}\tfrac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\tfrac{2}{p}\right))+$$
$$+\mathrm{O}(2\tau(\mathrm{rad}(2n)))$$

Case 1. if 2n=2

$$\Pi'_{2}(\mathrm{x})=(x-2-\lfloor\sqrt{x}\rfloor)(\tfrac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\tfrac{2}{p}\right)+0)+\mathrm{O}(2\tau(\mathrm{rad}(2)))$$
$$=\frac{(x-2-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x},p\neq2}\left(1-\frac{2}{p}\right)+\mathrm{O}(4) \quad (**)$$

It is known that if (p ,p+2) is a couple of twin primes then there is no prime between them. Then, Π'_2(x) denotes the number of twin primes not exceeding x.

We have Π_2(x)=Π'_2(x)+$\Pi_2(\sqrt{x})$.

then, Π_2(x)=Π'_2(x)+$\Pi'_2(\sqrt{x})$+$\Pi_2(\sqrt{\sqrt{x}})$

$$=\Pi'_2(\mathrm{x})+\Pi'_2(\sqrt{x})+\Pi'_2(\sqrt{\sqrt{x}})........$$

let $\sqrt[b]{x}=4 \quad \Rightarrow \ln(\mathrm{x}) = 4\mathrm{b}$

$$\Rightarrow \quad \mathrm{b} = \frac{\ln(x)}{4}$$

then we obtain , $\Pi_2(\mathrm{x}) = \sum_{i=1}^{\frac{\ln(x)}{4}} \Pi'_2(\sqrt[i]{x})$

$$= \Pi'_2(\mathrm{x}) \sum_{i=1}^{\frac{\ln(x)}{4}} \frac{\Pi'_2(\sqrt[i]{x})}{\Pi'_2(\sqrt{x})}$$

but from (**) we have $\frac{\Pi'_2(\sqrt[i]{x})}{\Pi'_2(\sqrt{x})} = \frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)}$

then $\Pi_2(\mathrm{x})=\Pi'_2(\mathrm{x}) \sum_{i=1}^{\frac{\ln(x)}{4}} \frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)}$

$$=\Pi'_2(\mathrm{x})(1+\sum_{i=2}^{\frac{\ln(x)}{4}} \frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq2}\left(1-\frac{2}{p}\right)+O(4)})$$

$$=\Pi_2^{'}(x)(1+\frac{(\sqrt[2]{x}-2-\sqrt[3]{x})\prod_{p\leqslant\sqrt[3]{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}+.........)$$

but by Lemma 4 , $\frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}\sim\frac{\frac{\sqrt[i]{x}}{\ln\left(\sqrt[i+1]{x}\right)^2}}{\frac{x}{\ln(\sqrt{x})^2}}$

for sufficiently large x and $\frac{\ln(x)}{4}\geqslant i\geqslant 2$.

then, $\frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}\sim\frac{\sqrt[i]{x}}{x}\times\frac{\ln(\sqrt{x})^2}{\ln(\sqrt[i+1]{x})^2}$

$$\sim\frac{\sqrt[i]{x}}{x}\times\frac{(i+1)^2}{4}\quad\to 0 \text{ when } x\to\infty$$

we can deduce that $\sum_{i=2}^{\frac{\ln(x)}{4}}\frac{(\sqrt[i]{x}-2-\sqrt[i+1]{x})\prod_{p\leqslant\sqrt[i+1]{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}{(x-2-\sqrt{x})\prod_{p\leqslant\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)}\to 0$ when $x\to\infty$

then , $\Pi_2(x)\sim\Pi_2^{'}(x)$

$$\sim\frac{(x-2-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)$$

by Lemma 4 we have , $\Pi_2(x)\sim\frac{x}{2\ln(\sqrt{x})^2}$ for sufficiently large x.

$$\sim\frac{4x}{2\ln(x)^2}$$

$$\sim\frac{2x}{\ln(x)^2}$$

then , $\Pi_2(x)\sim\frac{2x}{\ln(x)^2}$

Case 2 . if 2n=4 F=$\{1,2\}$

then, $\Pi_4^{'}(x)=(x-4-\lfloor\sqrt{x}\rfloor)(\frac{1}{2}\prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)+0)+O(2\tau(\text{rad}(4)))$

$$=\frac{(x-4-\lfloor\sqrt{x}\rfloor)}{2}\prod_{p\leq\sqrt{x},p\neq 2}\left(1-\frac{2}{p}\right)+O(4)$$

but it is known that if (p ,p+4) is a couple of cousin primes then

there is no prime between them , except (3, 7).then $\Pi_4^{'}(x)$ denotes the number of cousin primes not exceeding x.

By the seem method developped in Case 1 we can prove that $\Pi_4^{'}(x) \sim \frac{2x}{\ln(x)^2}$.

Proof of Theorem B.

by the Result 3 we have .

$$\Pi_{2n}^{'}(x) = (x - 2n - \lfloor\sqrt{x}\rfloor)\left(\frac{1}{2}\prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) + \sum_{d \epsilon F, d \neq 1, d \neq 2} \frac{1}{d}\prod_{p/\frac{N_z}{d}.}\left(1 - \frac{2}{p}\right)\right) + \\ + O(2\tau(\text{rad}(2n)))$$

Case 1 . if 2n=2 we have F={1, 2}

$$\text{then } \Pi_2^{'}(x) = \frac{(x - 2 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(2)))$$
$$= \frac{(x - 2 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p \leq \sqrt{x}, p \neq 2}\left(1 - \frac{2}{p}\right) + O(4)$$

Case 2 . if 2n=4 we have F={1, 2}

$$\text{then } \Pi_4^{'}(x) = \frac{(x - 4 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(4)))$$
$$= \frac{(x - 4 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p \leq \sqrt{x}, p \neq 2}\left(1 - \frac{2}{p}\right) + O(4)$$

From case 1 and case 2 we obtain .

$$\Pi_2^{'}(x) - \Pi_4^{'}(x) = \frac{(x - 2 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) + O(4) -$$
$$- \frac{(x - 4 - \lfloor\sqrt{x}\rfloor)}{2}\prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) - O(4)$$
$$= \prod_{p/\frac{N_z}{2}.}\left(1 - \frac{2}{p}\right) + O(4)$$

by Lemma 4 , we have $\prod_{p \leq \sqrt{x}, p \neq 2}\left(1 - \frac{2}{p}\right) \sim \frac{1}{\log(\sqrt{x})^2}$,for sufficiently large x .

$\sim \frac{4}{\log(x)^2}$,for sufficiently large x .

then $\Pi_2^{'}(x)$- $\Pi_4^{'}(x)$=O(4) , for sufficiently large x .

we have $\Pi_4(x) - \Pi_2(x) = \Pi_4^{'}(x) - \Pi_2^{'}(x) + \Pi_4(\sqrt{x}) - \Pi_2(\sqrt{x})$

$$=\Pi_4'(x)-\Pi_2'(x)+\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x})+\Pi_4(\sqrt{\sqrt{x}})-\Pi_2(\sqrt{\sqrt{x}})$$

$$=\Pi_4'(x)-\Pi_2'(x)+\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x})+\Pi_4'(\sqrt{\sqrt{x}})-\Pi'_2(\sqrt{\sqrt{x}})..$$

$$= O(4)+O(4)\ +O(4)\ldots$$

let $\sqrt[b]{x} = 4 \quad \Rightarrow \ln(x) = 4b$

$$\Rightarrow \quad b = \frac{\ln(x)}{4}$$

then $\Pi_4(x)-\Pi_2(x) \quad = \sum_{i=1}^{\frac{\ln(x)}{4}} (\Pi_4'(\sqrt[i]{x})-\Pi_2'(\sqrt[i]{x}))$

$$=\sum_{i=1}^{\frac{\ln(x)}{4}} O(4)$$

$$=O(\ln(x))$$

then for sufficiently large x we have $|\ \Pi_4(x)-\Pi_2(x)| \leq \quad \ln(x)$

this means that $\left| \frac{\Pi_4(x)}{\Pi_2(x)} - 1\right| \leq \frac{\ln(x)}{\Pi_2(x)}$

but $\Pi_2(x) \sim \frac{2x}{\log(x)^2}$ then $\frac{2\ln(x)}{\Pi_2(x)} \sim \frac{\ln(x)}{\frac{2x}{\log(x)^2}}$

$$\sim \frac{\ln(x)^3}{2x} \rightarrow 0 \text{ when } x \rightarrow \infty$$

then $\Pi_4(x) \sim \Pi_2(x)$ for sufficiently large x.

this means that The cousin primes are equivalent to twin primes in infinity

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