TIME-PERIODIC SOLUTION TO THE COMPRESSIBLE VISCOELASTIC FLOWS IN PERIODIC DOMAIN

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ABSTRACT. In this paper, we are concerned with the time-periodic solutions to the three-dimensional compressible viscoelastic flows with a time-periodic external force in a periodic domain. By using an approach of parabolic regularization and combining with the topology degree theory, we show the existence and uniqueness of the time-periodic solution to the model under some smallness and symmetry assumptions on the external force.

1. INTRODUCTION

In this paper, we consider the existence and uniqueness of time-periodic solution for the compressible viscoelastic flows (cf. [7, 11, 14, 22, 31]):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P - \nabla \cdot \mathbb{S} = \alpha \nabla \cdot (\rho F F^T) + \rho f(x,t), \\ F_t + u \cdot \nabla F = \nabla u F. \end{cases} \tag{1.1}$$

Here $x \in \Omega = (-L, L)^N$ $(N \geq 1)$, $\rho \geq 0$, $u = (u_1, u_2, \cdot, u_N)$, and $F \in M^{N \times N}$(the set of $N \times N$ matrices with positive determiants) are, respectively, the density, the velocity, and the deformation gradient. P is the pressure function, for the case of ideal gas, it satisfies

$$P(\rho) = A\rho^\gamma, \tag{1.2}$$

While $\mathbb{S}$ is the viscous stress tensor, which is given by Newton's viscosity formula:

$$\mathbb{S} = \mu(\nabla u + (\nabla u)^\top) + \lambda \nabla \cdot u \mathbb{I},$$

where $(\nabla u)^\top$ is the transpose matrix of ∇u and $\mathbb{I}$ is the $n \times n$ identity matrix, the constants μ and λ are the viscosity coefficients, which satisfy the physical restrictions

$$\mu > 0, \quad N\lambda + 2\mu > 0.$$

The parameter $\alpha > 0$ denotes the speed of propagation of shear waves which we set to unity. For system (1.1), the corresponding elastic energy is chosen to be the special form of the Hookean linear elasticity

$$W(F) = \frac{\alpha}{2}|F|^2 + \frac{1}{\rho}\int_0^\rho P(s)ds, \alpha > 0.$$

In addition, $f(x,t)$ is a given external force with periods $2L$ and T both in space and time, respectively. We also assume that

$$\operatorname{div}(\rho F^T) = 0, \ F^{lk}(0)\nabla_l F^{ij}(0) = F^{lj}(0)\nabla_l F^{ik}(0). \tag{1.3}$$

The condition (1.3) is preserved by the flow, please refer [13, 29].

In the past few decades, there are a lot of research on the viscoelastic flows. For the incompressible cases, the existence of classical solutions of both the Cauchy problem and the initial-boundary value problem are extensively studied, in [5, 6, 21, 23–25, 35]. The Long-time behavior and weak-strong uniqueness of solutions was proved by Hu-Wu in [15]. The global

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existence of weak solution to the two-dimensional incompressible viscoelastic flows with discontinuous initial data was proved by Hu-Lin in [16]. On the other hand, the global existence of weak solutions with large initial data is still an open problem. For the compressible viscoelastic flows, when the external force $f = 0$, the global existence of classical solutions to the two-dimensional Oldroyd model was proved by Lei-Zhou [22] via the incompressible limit. Hu-Wang [12] proved the local existence of strong solutions. Later, the global existence to the system (1.1) with the initial data close to constant equilibrium in the critical L^2 Besov space was stuied by Hu-Wang [13] and Qian-Zhang [29]. See also [14, 28] for the global existence and optimal time decay for the Cauchy problem to the system (1.1), respectively, for the initial data are close to the constant equilibrium state in H^2 and in L^p critical spaces. As for the initial boundary value problem, a global-in-time solution was proved to exist close to the equilibrium state, please refer to [15, 30] and references. However, the existence and uniqueness of time-periodic solution to the system (1.1) in bounded periodic domain or unbounded domain remains open. It is worth noting that when F is a constant matric, the system (1.1) reduces to compressible Navier-Stokes equation. There has been much nice work on the periodic solution to compressible Navier-Stokes equation and related models; refer to [1–4, 10, 17–20, 26, 27, 33] and references therein. Here we only mention some of them for bounded domain. The existence and uniqueness of time-periodic solution to the compressible Navier-Stokes equations in bounded bounded domain and periodic domain was obtained by Jin in [17] and Jin-Yang in [18], respectively. And for the works on the time-periodic solutions to some models related to the compressible Navier-Stokes equations, see [3, 34], for instance, and references therein.

In this paper, we shall establish the existence and the uniqueness of a time-periodic solution to the system (1.1) for (ρ, u, F) around the constant equilibrium state $[\bar{\rho}, 0, I]$ in a periodic domain, which can be reformulated problem as follows:

$$\begin{cases} \sigma_t + \mathrm{div} u = -\mathrm{div}(\sigma u), \\ u_t - \dfrac{\mu}{\sigma+\bar{\rho}}\Delta u - \dfrac{\mu+\lambda}{\sigma+\bar{\rho}}\nabla \mathrm{div} u + \gamma\bar{\rho}\nabla\sigma - \mathrm{div}E = -(u\cdot\nabla)u - g(\sigma)\nabla\sigma \\ \quad +(E^T\cdot\nabla)E + f(x,t), \\ E_t - \nabla u = -u\cdot\nabla E + \nabla u E. \end{cases} \tag{1.4}$$

Here, $\sigma = \rho - \bar{\rho}$, $E = F - I$, $\gamma = \frac{P'(\bar{\rho})}{\bar{\rho}^2}$, $g(\sigma) = \frac{P'(\sigma+\bar{\rho})}{\sigma+\bar{\rho}}$. The proof is based on the combination of topology degree theory with some *a priori* estimates under the oddness and smallness assumption on the periodic external force. The key of matter of the present paper is the uniform estimates of the dissipation on $\|\nabla^{m+2}\sigma\|_{L^2}$ and $\|\nabla^{m+2}E\|_{L^2}$. For this goal, special attention has to be paid on the coupling between the second and third equations as well as the condition $\mathrm{div}(\rho F^T) = 0$ and structure of the equation for F:

- The presence of the deformation gradient F in the transport equation gives rises to the the unexpected extra linear term $\mathrm{div}E$ in the reformulated system. In spite of can be handled directly by virtue of the linearized equation for E, however, we can't directly get the estimate of dissipation on $\|\nabla^{m+2}\sigma\|_{L^2}$ and $\|\nabla^{m+2}E\|_{L^2}$ by virtue of multiplying by $\nabla^{m+1}\sigma$ and $\nabla^{m+1}E$ respectively. To get over this difficulty, we take the Hodge decomposition of the momentum equation, and then the linear term $\nabla\rho$ and $\mathrm{div}E$ are separated, which enables us to obtain the estimates of dissipation on $\|\nabla^{m+2}\sigma\|_{L^2}$ and $\|\nabla^{m+2}E\|_{L^2}$.
- To get the dissipation of $\|\nabla^{m+2}E\|$, making use of the structure of the equation for F and the condition $\mathrm{div}(\rho F^T) = 0$, namely, $\mathrm{curl}E$ is a high order term, we succeed in establishing estimates of dissipation for $\|\nabla^{m+2}E\|$. Please refer proposition 3.1 for the details.

Before stating the main results, we explain the notations and conventions throughout this paper. We denote by C a generic positive constant. For two quantities A and B, we write $A \sim B$ if $C^{-1}A \leq B \leq CA$. The notation $A \lesssim B$ means that $A \leq CB$ for a universal constant $C > 0$

independent of time t. We denote $Q_T = \Omega \times (0,T)$ and let

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_N})$$

and put $\partial_x^l f = \nabla^l f = \nabla(\nabla^{l-1} f)$.For any integer $m \geq 0$, we use H^m to denote the standard Sobolev space $H^m(\Omega)$. Let $L^2 = H^m$ when $m = 0$. For simplicity, the norm of H^m is denoted by $\|\cdot\|_m$, and in particular, denote $\|\cdot\| =: \|\cdot\|_0$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space $L^2(\Omega)$, i.e.

$$\langle f, g \rangle = \int_\Omega f(x)g(x)dx, \ f = f(x), \ g = g(x) \in L^2(\Omega).$$

We define that

$$\Theta = \big\{(\sigma, u, E); \sigma, E \in L^\infty(0,T; H^{m+2}(\Omega)), u \in L^\infty(0,T; H^{m+2}(\Omega)) \cap L^2(0,T; H^{m+3}); \\ \sigma, u, E \ satisfy \ (a), \ (b) \ and \ (c) \ in \ Theorem 1.1\big\},$$

and set the space

$$\Gamma = \{(\rho, \omega, e) \in L^\infty(0,T; H^{m+1}(\Omega)) \cap L^2(0,T; H^{m+2}(\Omega)), \rho, \omega, e \ satisfies \ (a), (b), (c)\}$$

and

$$\Gamma_R = \{(\rho, \omega, e) \in \Gamma; \sup_{0 \leq t \leq T} \|(\rho, \omega, e)(t)\|^2_{H^{m+1}} + \int_0^T \|(\rho, \omega, e)(t)\|^2_{H^{m+2}} dt < R^2\}.$$

Now it is the place to state our main results on the existence and uniqueness of time-periodic solution to the system (1.4).

Theorem 1.1. *Assume that the integer* $m \geq [\frac{N}{2}] + 1$ *and* $f(x,t) \in L^2(0,T; H^{m+1})$ *with* $f(-x,t) = -f(x,t)$, *in addition* $\int_0^T \|f(x,t)\|^2_{H^{m+1}}$ *is suitably small, Then there exists a time-periodic solution* $(\sigma, u, E) \in \Theta \cap \Gamma_R$ *to the system* (1.4), *where* Θ *and* Γ_R *are defined in section 2. Here, the solution* (σ, u, E) *also satisfies the following property*
(a) (σ, u, E) *is periodic with the space period 2L and time period* T;
(b) $\int_\Omega \sigma(x,t)dx = 0$, $\int_\Omega u(x,t)dx = 0$, $\int_\Omega E(x,t)dx = 0$;
(c) $\sigma(x,t) = \sigma(-x,t)$, $u(x,t) = -u(-x,t)$, $E(x,t) = E(-x,t)$.
Moreover, if $\sup_{t \in [0,T]} \|(\sigma, u, E)(t)\|_{H^{m+2}}$ *is small enough, the uniqueness of time-periodic solution* (σ, u, E) *holds.*

2. PRELIMINARIES

In this section, we collect some facts and inequalities which will be frequently used in the subsequent analysis. In what follows, we shall introduce some Sobolev inequalities for later use (cf. [9, 32]). Let us begin with the following interpolation inequality.

Lemma 2.1. *Let* $0 \leq m, k \leq l$ *and the function* $f \in C_0^\infty(\Omega)$, *then we have*

$$\|\nabla^k f\|_{L^p} \lesssim \|\nabla^m f\|^{1-\theta}_{L^2} \|\nabla^l f\|^\theta_{L^2}, \tag{2.1}$$

where $0 \leq \theta \leq 1$ *and* k *satisfy*

$$\frac{1}{p} - \frac{k}{3} = \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta) + \left(\frac{1}{2} - \frac{l}{3}\right)\theta.$$

The second inequality is the L^p estimate on any two product terms with the sum of the order of their derivatives equal to a given integer.

Lemma 2.2. *Let* $n \geq 1$. *Let* $\alpha^1 = (\alpha_1^1, \cdots, \alpha_n^1)$ *and* $\alpha^2 = (\alpha_1^2, \cdots, \alpha_n^2)$ *be two multi-indices with* $|\alpha^1| = k_1$, $|\alpha^2| = k_2$ *and set* $k = k_1 + k_2$. *Let* $1 \leq p, q, r \leq \infty$ *with* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *Then, for* $u_j : R^n \to R$ $(j = 1, 2)$, *one has*

$$\|\partial^{\alpha^1} u_1 \partial^{\alpha^2} u_2\|_{L^p(\Omega)} \leq C\left(\|u_1\|_{L^q(\Omega)} \|\nabla^k u_2\|_{L^r(\Omega)} + \|u_2\|_{L^q(\Omega)} \|\nabla^k u_1\|_{L^r(\Omega)}\right) \tag{2.2}$$

for some constant $C > 0$ *independent of* u_1 *and* u_2.

As a generalization of Lemma 2.2, we have also

Lemma 2.3. *Let $n \geq 1, l > 2$ be integers. Let $\alpha^j = (\alpha_1^j, \cdots, \alpha_n^j)$, $1 \leq j \leq l$ be multi-indices with $|\alpha^j| = k_j$, $1 \leq j \leq l$ and $k = k_1 + k_2 + \cdots + k_l$. Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, for $u = (u_1, \cdots, u_l) : \mathbb{R}^n \to \mathbb{R}^l$, one has*

$$\left\| \prod_{j=1}^{l} \partial^{\alpha^j} u_j \right\|_{L^p(\Omega)} \leq C \|u\|_{L^\infty(\Omega)}^{l-2} \|u\|_{L^q(\Omega)} \|\nabla^k u\|_{L^r(\Omega)} \tag{2.3}$$

for some constant $C > 0$ independent of u.

To study the existence of time-periodic solutions for (1.4), let us first consider the following regularized problem

$$\begin{cases} \sigma_t - \varepsilon\Delta\sigma + \bar{\rho}\mathrm{div}u = -\mathrm{div}(\sigma u), \\ u_t - \dfrac{\mu}{\sigma+\bar{\rho}}\Delta u - \dfrac{\mu+\lambda}{\sigma+\bar{\rho}}\nabla\mathrm{div}u + \gamma\bar{\rho}\nabla\sigma - \nabla\cdot E = -(u\cdot\nabla)u - g(\sigma)\nabla\sigma \\ \qquad\qquad\qquad\qquad\qquad +(E^T\cdot\nabla)E + f(x,t), \\ E_t - \nabla u - \varepsilon\Delta E = -u\cdot\nabla E + \nabla u E. \end{cases} \tag{2.4}$$

Now, let's use the topology degree theory to establish the existence of solutions $(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)$. Define an operator

$$\mathcal{G} : \Gamma_R \times [0,1] \to \Gamma,$$
$$((\rho,\omega,e),\tau) \to (\sigma,u,E)$$

with R being suitably small, where (σ, u, E) is the solution of the following linear parabolic problem

$$\begin{cases} \sigma_t - \varepsilon\Delta\sigma + \bar{\rho}\mathrm{div}u = G_1(\rho,\omega,e,\tau), \\ u_t - \dfrac{\mu}{\bar{\rho}+\tau\rho}\Delta u - \dfrac{\mu+\lambda}{\bar{\rho}+\tau\rho}\nabla\mathrm{div}u + \gamma\bar{\rho}\nabla\sigma - \nabla\cdot E = G_2(\rho,\omega,e,\tau) + \tau f(x,t), \\ E_t - \nabla u - \varepsilon\Delta E = G_3(\omega,e,\tau), \end{cases} \tag{2.5}$$

where

$$\begin{aligned} G_1(\rho,\omega,e,\tau) &= -\tau\mathrm{div}(\rho\omega), \\ G_2(\rho,\omega,e,\tau) &= \left(\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\bar{\rho}+\tau\rho)}{(\bar{\rho}+\tau\rho)}\right)\nabla\rho - \tau\omega\cdot\nabla\omega + \tau(e^T\cdot\nabla)e, \\ G_3(\omega,e,\tau) &= \tau(-\omega\cdot\nabla e + \nabla\omega e). \end{aligned}$$

First, we shall prove that $\mathcal{G}$ is well defined in the following:

Lemma 2.4. *Assume that R is suitable small and $(\rho,\omega,e) \in \Gamma_R$, then for any $\tau \in [0,1]$, there exists a time-periodic solution $(\sigma, u, E) \in \Gamma$ to the problem (2.5).*

Proof. Firstly, in view of $\|\rho\|_{L^\infty} \leq C \sup\limits_{0\leq t\leq T} \|\rho\|_{H^m} \leq CR$, we get for suitably small R that

$$\frac{\bar{\rho}}{2} \leq \bar{\rho} + \tau h \leq 2\bar{\rho},$$

which implies

$$\frac{1}{2\bar{\rho}} \leq \frac{1}{\bar{\rho}+\tau\rho} \leq \frac{2}{\bar{\rho}}. \tag{2.6}$$

Set the operator

$$\mathbb{B} = \begin{pmatrix} \varepsilon\Delta & -\bar\rho\mathrm{div} & 0 \\ -\gamma\bar\rho\nabla & \frac{\mu}{\bar\rho+\tau\rho}\Delta + \frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla\mathrm{div} & \mathrm{div} \\ 0 & \nabla & \varepsilon\Delta \end{pmatrix} \tag{2.7}$$

and let $U=(\sigma,u,E)$, $W=(\rho,\omega,e)$, $G(W)=(G_1,G_2,G_3)$, $Q=(0,\tau f,0)$. The system (2.5) can be reformulated as follows:

$$U_t = \mathbb{B}U + G(W) + Q.$$

To obtain the solution $U\in\Gamma$, we first consider the following initial value problem of the linear system $U_t=\mathbb{B}U$ in Ω with periodic boundary

$$\begin{cases} \sigma_t - \varepsilon\Delta\sigma + \bar\rho\mathrm{div}u = 0, \\ u_t - \dfrac{\mu}{\bar\rho+\tau\rho}\Delta u - \dfrac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla\mathrm{div}u + \gamma\bar\rho\nabla\sigma - \nabla\cdot E = 0, \\ E_t - \nabla u - \varepsilon\Delta E = 0, \\ (\sigma,u,E)(x,0) = (\sigma_0,u_0,E_0)(x), \end{cases} \tag{2.8}$$

where $\sigma_0(x)$ and $E_0(x)$ are even function with $\int_\Omega \sigma_0(x)dx=0$ and $\int_\Omega E_0(x)dx=0$, $u_0(x)$ is odd functions. Obviously, these properties are remained for the corresponding solution (σ,u,E). Applying ∇^{m+1} to (2.8) and multiplying the resulting equations by $\gamma\nabla^{m+1}\sigma$, $\nabla^{m+1}u$ and $\nabla^{m+1}E$, respectively, then integrating the resulting equations by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+1}\sigma|^2 + |\nabla^{m+1}u|^2 + |\nabla^{m+1}E|^2)dx \\ &\quad + \int_\Omega\left(\varepsilon(\gamma|\nabla^{m+2}\sigma|^2 + |\nabla^{m+2}E|^2) + \frac{\mu}{\bar\rho+\tau\rho}|\nabla^{m+2}u|^2 + \frac{\mu+\lambda}{\bar\rho+\tau\rho}|\nabla^{m+1}\mathrm{div}u|^2\right) \\ &= \int_\Omega\left(\sum_{1\le k\le m+1} C_{m+1}^k\nabla^k\frac{\mu}{\bar\rho+\tau\rho}\nabla^{m+1-k}\Delta u\nabla^{m+1}u - \nabla\frac{\mu}{\bar\rho+\tau\rho}\nabla^{m+2}u\nabla^{m+1}u\right)dx \\ &\quad + \int_\Omega\left(\sum_{1\le k\le m+1} C_{m+1}^k\nabla^k\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla^{m+1-k}\nabla\mathrm{div}u\nabla^{m+1}u - \nabla\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla^{m+1}\mathrm{div}u\nabla^{m+1}u\right)dx \\ &\le C(\|\nabla\rho\|_{L^\infty}\|\nabla^2u\|_{H^m} + \|\nabla^2u\|_{L^\infty}\|\rho\|_{H^{m+1}})\|\nabla^{m+1}u\|_{L^2} + C\|\nabla\rho\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}\|\nabla^{m+1}u\|_{L^2} \\ &\le C(\|\rho\|_{H^{m+1}}\|u\|_{H^{m+2}}^2), \\ &\le CR\|u\|_{H^{m+2}}^2. \end{aligned} \tag{2.9}$$

If R is small enough, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+1}\sigma|^2 + |\nabla^{m+1}u|^2 + |\nabla E|^2)dx + 2\varepsilon\int_\Omega(\gamma|\nabla^{m+2}\sigma|^2 + |\nabla^{m+2}E|^2)dx \\ &\quad + \int_\Omega\left(\frac{\mu}{3\bar\rho}|\nabla^{m+2}u|^2 + \frac{\mu+\lambda}{\bar\rho}|\nabla^{m+1}\mathrm{div}u|^2\right)dx \le 0. \end{aligned} \tag{2.10}$$

By Poincaré inequality, we have

$$\|\nabla^{m+1}(\sigma,u,E)\|_{L^2} \le \|\nabla^{m+1}(\sigma_0,u_0,E_0)\|_{L^2}e^{-C\varepsilon t},$$

which means that

$$\|e^{t\mathbb{B}}U_0\|_{H^{m+1}} \le \|U_0\|_{H^{m+1}}e^{-C\varepsilon t}.$$

By Duhamel's principle, the solution $U=[\sigma,u,E]$ to the system (2.5) can be formally written as

$$U(t) = \int_{-\infty}^t e^{(t-s)\mathbb{B}}(G(W)(s)+Q(s))ds. \tag{2.11}$$

Utilizing the time-periodic property of W and Q, we have

$$\begin{aligned}U(t+T) &= \int_{-\infty}^{t+T} e^{(t+T-s)\mathbb{B}}(G(W)(s)+Q(s))ds \\ &= \int_{-\infty}^{t+T} e^{(t-(s-T))\mathbb{B}}(G(W)(s-T)+Q(s-T))ds \\ &= \int_{-\infty}^{t} e^{(t-s)\mathbb{B}}(G(W)(s)+Q(s))ds = U(t),\end{aligned}$$

which means $U(t)$ is periodic with period T. Combing (2.11) with the property of W and F, we obtain

$$\begin{aligned}\|U(t)\|_{H^{m+1}} &\le \int_{-\infty}^{t} \|e^{(t-\tau)\mathbb{B}}(G(W(\tau))+Q(\tau))\|_{H^{m+1}}d\tau \\ &\le \int_{-\infty}^{t} e^{-C\varepsilon(t-\tau)}\|G(W(\tau))+Q(\tau)\|_{H^{m+1}}d\tau \\ &\le C_\varepsilon\left(\int_0^T \|G(W(t))+Q(t)\|^2_{H^{m+1}}dt\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, utilizing the classical theory of parabolic equations, we have that the problem (2.5) admits a time-periodic solution $(\sigma, u, E)\in\Gamma$ for any $(\rho,\omega,e)\in\Gamma_R, \tau\in[0,1]$. On the other hand, if there exists another solution $\bar{U}=(\bar{\sigma},\bar{u},\bar{E})$ satisfied (2.5), then we have

$$(U-\bar{U})_t = \mathbb{B}(U-\bar{U}).$$

Using (2.10) again, we have $(U-\bar{U})\equiv(0,0,0)$, which means the uniqueness is proved. Noting that if $(\sigma(x,t),u(x,t),E(x,t))$ is the periodic solution of (2.5), then $(\sigma(-x,t),-u(-x,t),E(-x,t))$ is also the solution of (2.5), thus using the uniqueness of solutions, we have $(\sigma(x,t),u(x,t),E(x,t)) = (\sigma(-x,t),-u(-x,t),E(-x,t))$. We complete the proof of Lemma 2.4. □

Next, we shall prove that $\mathcal{G}$ is compact and continuous. We first give the complete proof of compactness of $\mathcal{G}$ in the following lemma.

Lemma 2.5. *If R is small enough, then the operator $\mathcal{G}$ is compact.*

Proof. Let $|\alpha|=m+2$, applying ∂_x^α to (2.5), it follows that

$$\begin{cases}\partial_x^\alpha\sigma_t - \varepsilon\partial_x^\alpha\Delta\sigma + \bar{\rho}\partial_x^\alpha \mathrm{div} u = \partial_x^\alpha G_1(\rho,\omega,e,\tau), \\ \partial_x^\alpha u_t - \partial_x^\alpha(\dfrac{\mu}{\tau\rho+\bar{\rho}}\Delta u) - \partial_x^\alpha(\dfrac{\mu+\lambda}{\tau\rho+\bar{\rho}}\nabla \mathrm{div} u) + \gamma\bar{\rho}\partial_x^\alpha\nabla\sigma - \partial_x^\alpha \mathrm{div} E = \partial_x^\alpha G_2(\rho,\omega,e,\tau) \\ +\tau\partial_x^\alpha f(x,t), \\ \partial_x^\alpha E_t - \partial_x^\alpha\nabla u - \varepsilon\partial_x^\alpha\Delta E = \tau\partial_x^\alpha G_3(\omega,e,\tau).\end{cases} \tag{2.12}$$

Multiplying (2.12)$_1$-(2.12)$_3$ by $\gamma\partial_x^\alpha\sigma$, $\partial_x^\alpha u$, and $\partial_x^\alpha E$, respectively, and integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\sigma|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}E|^2)dx+\varepsilon\int_\Omega(\gamma|\nabla^{m+3}\sigma|^2+|\nabla^{m+3}E|^2)dx\\
&\quad+\int_\Omega\frac{\mu}{\bar\rho+\tau\rho}|\nabla^{m+3}u|^2dx+\int_\Omega\frac{\mu+\lambda}{\bar\rho+\tau\rho}|\nabla^{m+2}\mathrm{div}u|^2dx\\
&=-\int_\Omega\tau\gamma\nabla^{m+2}\mathrm{div}(\rho\omega)\nabla^{m+2}\sigma dx+\int_\Omega\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l\frac{\mu}{\bar\rho+\tau\rho}\nabla^{m+2-l}\Delta u\nabla^{m+2}udx\\
&\quad+\int_\Omega\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla\mathrm{div}u\nabla^{m+2}udx-\int_\Omega\nabla\frac{\mu}{\bar\rho+\tau\rho}\nabla^{m+3}u\nabla^{m+2}udx\\
&\quad-\int_\Omega\nabla\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla^{m+2}\mathrm{div}u\nabla^{m+2}udx+\int_\Omega\nabla^{m+2}(-\tau(\omega\cdot\nabla)\omega)\nabla^{m+2}udx\\
&\quad-\int_\Omega\nabla^{m+2}\left((\frac{p'(\bar\rho)}{\bar\rho}-\frac{P'(\bar\rho+\tau\rho)}{\bar\rho+\tau\rho})\nabla\rho+\tau e^T\cdot\nabla e+\tau f\right)\nabla^{m+2}udx\\
&\quad+\tau\int_\Omega\nabla^{m+2}(-\omega\cdot\nabla e+\nabla\omega e)\nabla^{m+2}E\\
&=I_1+I_2+\cdots+I_{10}.
\end{aligned}$$

By virtue of the periodic boundary condition, we have $\|\nabla^k(\rho,\omega,e)\|_{L^2}\le C\|\nabla^k\nabla(\rho,\omega,e)\|_{L^2}$ for all $k\ge 0$. For any $m\ge[\frac{N}{2}]+1$, similar to [18], using lemmas 2.2-2.3, we have

$$\begin{aligned}
|I_1|\le\ &C\|\nabla^{m+3}\sigma\|_{L^2}(\|\nabla\rho\|_{L^\infty}\|\nabla^{m+1}\omega\|_{L^2}+\|\nabla^{m+2}\rho\|_{L^2}\|\omega\|_{L^\infty}\\
&+\|\rho\|_{L^\infty}\|\nabla^{m+2}\omega\|_{L^2}+\|\nabla^{m+1}\rho\|\|\nabla\omega\|_{L^\infty})\\
\le\ &\frac{\gamma\varepsilon}{2}\|\nabla^{m+3}\sigma\|_{L^2}^2+C(\|\rho\|_{H^{m+2}}^2\|\omega\|_{H^{m+1}}^2+\|\rho\|_{H^{m+1}}^2\|\omega\|_{H^{m+2}}^2).
\end{aligned}$$

Since $(\rho,\omega,e)\in\Gamma_R$, we have

$$\begin{aligned}
|I_2|,|I_3|&\le C\|\nabla^{m+3}u\|_{L^2}(\|\nabla^2u\|_{H^{m+1}}\|\nabla\rho\|_{L^\infty}+\|\nabla^2u\|_{L^\infty}\|\rho\|_{H^{m+1}})\\
&\le C\|\rho\|_{H^{m+1}}\|\nabla^{m+3}u\|_{L^2}^2\\
&\le CR\|\nabla^{m+3}u\|_{L^2}^2.
\end{aligned}$$

$$\begin{aligned}
|I_4|,|I_5|&\le C\|\nabla\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\|\nabla^{m+2}u\|_{L^2}\\
&\le CR\|\nabla^{m+3}u\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
|I_6|&\le C\|\nabla^{m+3}u\|_{L^2}(\|\omega\|_{L^\infty}\|\nabla^{m+2}\omega\|_{L^2}+\|\nabla^{m+1}\omega\|_{L^2}\|\nabla\omega\|_{L^\infty})\\
&\le C\|\omega\|_{H^{m+1}}\|\nabla^{m+2}\omega\|_{L^2}\|\nabla^{m+3}u\|_{L^2},
\end{aligned}$$

$$\begin{aligned}
|I_7|&\le C\|\nabla^{m+3}u\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^{m+2}\rho\|_{L^2}+\|\nabla^{m+1}\rho\|_{L^2}\|\nabla\rho\|_{L^\infty})\\
&\le C\|\nabla^{m+2}\rho\|_{L^2}\|\rho\|_{H^{m+1}}\|\nabla^{m+3}u\|_{L^2},
\end{aligned}$$

$$|I_8|\le C\|\nabla^{m+1}f\|_{L^2}\|\nabla^{m+3}u\|_{L^2},$$

$$\begin{aligned}
|I_9|&\le C\|\nabla^{m+3}u\|_{L^2}(\|e^T\|_{L^\infty}\|\nabla^{m+2}e\|_{L^2}+\|\nabla^{m+1}e^T\|_{L^2}\|\nabla e\|_{L^\infty})\\
&\le C\|e\|_{H^{m+1}}\|\nabla^{m+2}e\|_{L^2}\|\nabla^{m+3}u\|_{L^2},
\end{aligned}$$

$$
\begin{aligned}
|I_{10}| &\le C\|\nabla^{m+3}E\|_{L^2}(\|\nabla e\|_{L^\infty}\|\nabla^{m+1}\omega\|_{L^2} + \|\nabla^{m+2}e\|_{L^2}\|\omega\|_{L^\infty} + \|e\|_{L^\infty}\|\nabla^{m+2}\omega\|_{L^2}\\
&\quad + \|\nabla^{m+1}e\|_{L^2}\|\nabla\omega\|_{L^\infty})\\
&\le \frac{\varepsilon}{2}\|\nabla^{m+3}E\|_{L^2}^2 + C_\varepsilon(\|e\|_{H^{m+2}}^2\|\omega\|_{H^{m+1}}^2).
\end{aligned}
$$

Then, choosing R sufficient small and combining the estimates I_1-I_{10}, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \gamma|\nabla^{m+2}\sigma|^2 + |\nabla^{m+2}u|^2 + |\nabla^{m+2}E|^2 dx + \int_\Omega \varepsilon(\gamma|\nabla^{m+3}\sigma|^2 + |\nabla^{m+3}E|^2)dx\\
&\quad + \int_\Omega \frac{\mu}{2\bar\rho}|\nabla^{m+3}u|^2 dx + \int_\Omega \frac{\mu+\lambda}{\bar\rho}|\nabla^{m+2}\mathrm{div}u|^2 dx\\
&\le C(\|\rho\|_{H^{m+2}}^2 + \|e\|_{H^{m+2}}^2)\|\omega\|_{H^{m+1}}^2 + C(\|\rho\|_{H^{m+1}}^2 + \|e\|_{H^{m+1}}^2)\|\omega\|_{H^{m+2}}^2\\
&\quad + C\|\omega\|_{H^{m+1}}^2\|\omega\|_{H^{m+2}}^2 + C\|e\|_{H^{m+1}}^2\|e\|_{H^{m+2}}^2 + C\|\rho\|_{H^{m+1}}^2\|\rho\|_{H^{m+2}}^2\\
&\quad + C\|\nabla^{m+1}f\|_{L^2}^2.
\end{aligned}
\tag{2.13}
$$

Then integrating (2.13) over $[0,T]$, we obtain

$$
\begin{aligned}
&\int_0^T \varepsilon(\gamma\|\nabla^{m+3}\sigma\|_{L^2}^2 + \|\nabla^{m+3}E\|_{L^2}^2) + \frac{\mu}{2\bar\rho}\|\nabla^{m+3}u\|_{L^2}^2 dt\\
&\le C\sup_{0\le t\le T}\|(\rho,\omega,e)\|_{H^{m+1}}^2\int_0^T \|(\rho,\omega,e)\|_{H^{m+2}}^2 dt + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt \qquad (2.14)\\
&= K. \qquad (2.15)
\end{aligned}
$$

Then, there exists a time $t^* \in (0,T)$ such that

$$
\varepsilon(\gamma T\|\nabla^{m+3}\sigma(t^*)\|_{L^2}^2 + T\|\nabla^{m+3}E(t^*)\|_{L^2}^2) + \frac{\mu}{2\bar\rho}\|\nabla^{m+3}u(t^*)\|_{L^2}^2 \le K.
$$

So, using the Poincaré inequality yields

$$
\gamma\|\nabla^{m+2}\sigma(t^*)\|_{L^2}^2 + \|\nabla^{m+2}E(t^*)\|_{L^2}^2 + \|\nabla^{m+2}u(t^*)\|_{L^2}^2 \le CK.
$$

Integrating (2.13) from t^* to t for $t \in [0,T]$, we have

$$
\gamma\|\nabla^{m+2}\sigma(t)\|_{L^2}^2 + \|\nabla^{m+2}E(t)\|_{L^2}^2 + \|\nabla^{m+2}u(t)\|_{L^2}^2 \le CK. \tag{2.16}
$$

Combing (2.16) with (2.14) and the Poincaré inequality, we obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}(\gamma\|\sigma\|_{H^{m+2}}^2 + \|u\|_{H^{m+2}}^2 + \|E\|_{H^{m+2}}^2) + \int_0^T \varepsilon(\gamma\|\sigma\|_{H^{m+3}}^2 + \|E\|_{H^{m+3}}^2) + \frac{\mu}{2\bar\rho}\|u\|_{H^{m+3}}^2 dt\\
&\le C\sup_{0\le t\le T}\|(\rho,\omega,e)\|_{H^{m+1}}^2\int_0^T \|(\rho,\omega,e)\|_{H^{m+2}}^2 dt + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt.
\end{aligned}
\tag{2.17}
$$

Applying ∇^{m+1} to (2.5), multiplying the resulting equations by $(\nabla^{m+1}\sigma)_t$, $(\nabla^{m+1}u)_t$, and $(\nabla^{m+1}E)_t$, respectively, and integrating it over $Q_T = \Omega\times[0,T]$ yields

$$
\begin{aligned}
&\int_0^T (||(\nabla^{m+1}\sigma)_t||_{L^2}^2 + \|(\nabla^{m+1}u)_t\|_{L^2}^2 + \|(\nabla^{m+1}E)_t\|_{L^2}^2)dt\\
&\le C\sup_{0\le t\le T}\|(\rho,\omega,e)\|_{H^{m+1}}^2\int_0^T \|(\rho,\omega,e)\|_{H^{m+2}}^2 dt + C\int_0^T \|e\|_{H^{m+2}}^2 dt\\
&\quad + C\int_0^T \|\omega\|_{H^{m+2}}^2 dt + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt + C\sup_{0\le t\le T}\|\rho\|_{H^{m+1}}^2\int_0^T \|\nabla^{m+1}f\|_{H^{m+2}}^2 dt\\
&\quad + C(\sup_{0\le t\le T}\|\rho\|_{H^{m+1}}^2 + \sup_{0\le t\le T}\|e\|_{H^{m+1}}^2)\sup_{0\le t\le T}\|(\rho,\omega,e)\|_{H^{m+1}}^2\int_0^T \|(\rho,\omega,e)\|_{H^{m+2}}^2 dt,
\end{aligned}
\tag{2.18}
$$

We get by virtue of (2.17) and (2.18) that $\mathcal{G}$ is a compact operator. The proof of present lemma is complete. $\square$

Then, the continuous of $\mathcal{G}$ is showed in the following lemma.

Lemma 2.6. *If R is small enough, then the operator $\mathcal{G}$ is continuous.*

Proof. Assume that $(\rho_n, \omega_n, e_n) \in \Gamma_R$, $\tau_n \in [0,1]$, $(\rho, \omega, e) \in \Gamma_R, \tau \in [0,1]$, and

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|(\rho_n - \rho, \omega_n - \omega, e_n - e)\|^2_{H^{m+1}} + \int_0^T \|(\rho_n - \rho, \omega_n - \omega, e_n - e)(t)\|^2_{H^{m+2}} dt = 0$$

and $\lim\limits_{n\to\infty} \tau_n = \tau$. Denote $(\sigma_n, u_n, E_n) = \mathcal{G}((\rho_n, \omega_n, e_n), \tau_n)$, $(\sigma, u, E) = \mathcal{G}((\rho, \omega, e), \tau)$. Let $\tilde{\sigma} = \sigma_n - \sigma$, $\tilde{u} = u_n - u$, $\tilde{E} = E_n - E$. Then $(\tilde{\sigma}, \tilde{u}, \tilde{E})$ is the periodic solution of the following equations

$$\begin{cases} \tilde{\sigma} - \varepsilon\Delta\tilde{\sigma} + \bar{\rho}\mathrm{div}\tilde{u} = H_1(\rho_n, \omega_n, e_n, \tau_n, \rho, \omega, e, \tau), \\ \tilde{u}_t - \dfrac{\mu}{\bar{\rho} + \tau_n\rho_n}\Delta\tilde{u} - \dfrac{\mu+\lambda}{\bar{\rho} + \tau_n\rho_n}\nabla\mathrm{div}\tilde{u} + \gamma\bar{\rho}\nabla\tilde{\sigma} + \mathrm{div}E = H_2(\rho_n, \omega_n, e_n, \tau_n, \rho, \omega, e, \tau) \\ \qquad +(\tau_n, \tau)f, \\ \tilde{E}_t - \Delta\tilde{E} - \nabla\tilde{u} = H_3(\rho_n, \omega_n, e_n, \tau_n, \rho, \omega, e, \tau), \end{cases} \tag{2.19}$$

$$\begin{aligned} H_1 &= (\tau - \tau_n)\mathrm{div}(\rho\omega) - \tau_n\mathrm{div}((\rho_n - \rho)\omega + (\omega_n - \omega)\rho_n), \\ H_2 &= \left(\frac{1}{\bar{\rho} + \tau_n\rho_n} - \frac{1}{\bar{\rho} + \tau\rho_n}\right)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) \\ &\quad + \left(\frac{1}{\bar{\rho} + \tau\rho_n} - \frac{1}{\bar{\rho} + \tau\rho}\right)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) \\ &\quad - (\tau_n - \tau)(\omega_n \cdot \nabla)\omega_n - \tau[((\omega_n - \omega)\cdot\nabla)\omega_n + (\omega\cdot\nabla)(\omega_n - \omega)] \\ &\quad - g(\tau_n\rho_n)(\nabla\rho_n - \nabla\rho) + \frac{1}{\bar{\rho} + \tau\rho}(p'(\bar{\rho} + \tau\rho) - p'(\bar{\rho} + \tau_n\rho))\nabla\rho \\ &\quad + \frac{1}{\bar{\rho} + \tau\rho}(P'(\bar{\rho} + \tau_n\rho) - P'(\bar{\rho} + \tau_n\rho_n))\nabla\rho + P'(\bar{\rho} + \tau_n\rho_n)\left(\frac{1}{\bar{\rho} + \tau\rho} - \frac{1}{\bar{\rho} + \tau_n\rho}\right)\nabla\rho \\ &\quad + P'(\bar{\rho} + \tau_n\rho_n)\left(\frac{1}{\bar{\rho} + \tau_n\rho} - \frac{1}{\bar{\rho} + \tau_n\rho_n}\right)\nabla\rho \\ &\quad + (\tau_n - \tau)(e_n\cdot\nabla)e_n^T - \tau[((e_n - e)\cdot\nabla)e_n^T + (e\cdot\nabla)(e_n - e)^T], \\ H_3 &= -(\tau_n - \tau)(\omega_n\cdot\nabla)e_n - \tau[(\omega_n - \omega)\cdot\nabla]e_n - \tau(\omega\cdot\nabla)(e_n - e) \\ &\quad + (\tau_n - \tau)\nabla\omega_n e_n + \tau(\nabla(\omega_n - \omega)e_n + \nabla\omega(e_n - e)). \end{aligned}$$

with periodic boundary condition. Similar to the method in the proof of the compactness of the operator $\mathcal{G}$ in Lemma 2.5, we obtain that

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|(\sigma_n - \sigma, u_n - u, E_n - E)(t)\|^2_{H^{m+1}} + \int_0^T \|(\sigma_n - \sigma, u_n - u, E_n - E)(t)\|^2_{H^{m+2}} dt = 0.$$

Thus, the continuity of the operator $\mathcal{G}$ is proved. $\square$

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section, we are devoted to studying the existence of periodic solutions to the problem (1.1). For this goal, we shall first focus on study of the reformulated problem (1.4) stated in the first section, which is equivalent to the problem (1.1), the desired solution of the problem (1.4) will be obtained by an approaching process for the regularized problem (2.4). We first show the existence of solutions for (2.4) by virtue of the topological degree theory.

3.1. **The existence of approximated solution.**

Proposition 3.1. *Under the condition Theorem 1.1, the regularized problem (2.4) admits a solution* $(\sigma, u, E) \in \Gamma_R$.

Proof. To solve problem (2.4) $(\sigma, u, E) \in \Gamma_R$ in (2.4) is equivalent to solve the equation

$$U - \mathcal{G}(U, 1) = 0, \; U = (\sigma, u, E) \in \Gamma_R.$$

In order to apply the topological degree theory, we only have to show that there exists a $R > 0$, which is to be determined as below, such that

$$(I - \mathcal{G}(\cdot, \tau))(\partial B_R(0)) \neq 0, \;\; for \; any \; \tau \in [0, 1], \tag{3.1}$$

where $B_R(0)$ is a ball of radius R centered at the origin in Γ. If (3.1) holds, then to prove the existence of solution, we only need to prove that

$$deg(I - \mathcal{G}(\cdot, 1), B_R(0), 0) \neq 0,$$

For this purpose, we are going to show that there exists $R > 0$ such that (3.1) holds, We prove it by contradiction, let $((\sigma, u, E), \tau)$ be a solution of (3.1) for some small $R > 0$ by replacing (ρ, ω, e), then $((\sigma, u, E), \tau)$ satisfies

$$\begin{cases} \sigma_t - \varepsilon\Delta\sigma + \bar{\rho}\mathrm{div}u = -\tau\mathrm{div}(\sigma u), \\ u_t - \dfrac{\mu}{\bar{\rho} + \tau\sigma}\Delta u - \dfrac{\mu + \lambda}{\bar{\rho} + \tau\sigma}\nabla\mathrm{div}u + \gamma\bar{\rho}\nabla\sigma - \mathrm{div}E = \left(\dfrac{P'(\bar{\rho})}{\bar{\rho}} - \dfrac{P'(\bar{\rho} + \tau\sigma)}{\bar{\rho} + \tau\sigma}\right)\nabla\sigma \\ \quad -\tau(u \cdot \nabla)u + \tau E^T \cdot \nabla E + \tau f, \\ E_t - \nabla u - \varepsilon\Delta E = \tau(-u \cdot \nabla E + \nabla u E). \end{cases} \tag{3.2}$$

Applying ∇^{m+2} to the (3.2) then multiplying the resulting equations by $\gamma\nabla^{m+2}$, $\nabla^{m+2}u$, and $\nabla^{m+2}E$, respectively, and summing the resultant equations and integrating it over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+2}\sigma|^2 + |\nabla^{m+2}u|^2 + |\nabla^{m+2}E|^2)dx + \int_\Omega \varepsilon(\gamma|\nabla^{m+3}\sigma|^2 + |\nabla^{m+3}E|^2)dx \\ &\quad + \int_\Omega \left(\frac{\mu}{\bar{\rho} + \tau\sigma}|\nabla^{m+3}u|^2 + \frac{\mu + \lambda}{\bar{\rho} + \tau\sigma}|\nabla^{m+2}\mathrm{div}u|^2\right) dx \\ &= -\tau\gamma\int_\Omega \nabla^{m+2}\mathrm{div}(\sigma u)\nabla^{m+2}\sigma dx + \int_\Omega \sum_{1 \leq l \leq m+2} C^l_{m+2}\nabla^l\frac{\mu}{\bar{\rho} + \tau\sigma}|\nabla^{m+2-l}\Delta u\nabla^{m+2}u|dx \\ &\quad + \int_\Omega \sum_{1 \leq l \leq m+2} C^l_{m+2}\nabla^l\frac{\mu + \lambda}{\bar{\rho} + \tau\sigma}\nabla^{m+2-l}\nabla\mathrm{div}u\nabla^{m+2}udx - \int_\Omega \nabla\frac{\mu}{\bar{\rho} + \tau\sigma}\nabla^{m+3}u\nabla^{m+2}udx \\ &\quad - \int_\Omega \nabla\frac{\mu + \lambda}{\bar{\rho} + \tau\sigma}\nabla^{m+2}\mathrm{div}u\nabla^{m+2}udx + \tau\int_\Omega \nabla^{m+2}((E^T \cdot \nabla)E - (u \cdot \nabla)u)\nabla^{m+2}udx \\ &\quad - \int_\Omega \nabla^{m+2}(g(\tau\sigma)\nabla\sigma)\nabla^{m+2}udx + \tau\int_\Omega \nabla^{m+2}f\nabla^{m+2}udx \\ &\quad + \tau\int_\Omega \nabla^{m+2}(-u \cdot \nabla E + \nabla u E)\nabla^{m+2}Edx. \end{aligned} \tag{3.3}$$

Using Lemmas 2.1-2.3, Cauchy inequality and Sobolev inequality, let R is small enough, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\sigma|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}E|^2)dx+\int_\Omega\varepsilon(\gamma|\nabla^{m+3}\sigma|^2+|\nabla^{m+3}E|^2)dx\\&\quad+\int_\Omega\left(\frac{\mu}{\bar\rho+\tau\sigma}|\nabla^{m+3}u|^2+\frac{\mu+\lambda}{\bar\rho+\tau\sigma}|\nabla^{m+2}\mathrm{div}u|^2\right)\\&\le C\|(\sigma,u,E)\|_{H^{m+1}}^2\|(\sigma,u,E)\|_{H^{m+2}}^2+C_1R(\|\nabla^{m+2}\sigma\|_{L^2}^2+\|\nabla^{m+2}E\|_{L^2}^2)\\&\quad+C\|\nabla^{m+1}f\|_{L^2}^2. \qquad (3.4)\end{aligned}$$

Now, we turn to estimate the dissipation $\|\nabla^{m+2}\sigma\|_{L^2}$. Noticing that the condition $\mathrm{div}(\rho F^T)=0$ which gives

$$\mathrm{divdiv}[(1+\sigma)(E+I)^T]=0,$$

thus we have

$$\begin{aligned}\frac{\partial^2(E^{ij})}{\partial_{x_i}\partial_{x_j}}&=\mathrm{divdiv}(E^T)\\&=\mathrm{divdiv}[(1+\sigma)(E+I)^T]-\mathrm{divdiv}(\sigma I+\sigma E^T)\\&=-\Delta\sigma-\mathrm{divdiv}(\sigma E^T). \qquad (3.5)\end{aligned}$$

Thus by applying div to the second equation of (3.2), we obtain

$$(\mathrm{div}u)_t-\mathrm{div}\left(\frac{\mu}{\bar\rho+\tau\sigma}\Delta u\right)-\mathrm{div}\left(\frac{\mu+\lambda}{\bar\rho+\tau\sigma}\nabla\mathrm{div}u\right)+(\gamma\bar\rho+1)\Delta\sigma=\mathrm{div}g_1, \qquad (3.6)$$

where

$$g_1=g(\tau\sigma)\nabla\sigma+\tau\mathrm{div}(u\cdot\nabla)u+\tau E^T\cdot\nabla E+\tau f-\mathrm{div}(\sigma E).$$

Applying ∇^{m+1} to (3.6), multiplying the resulting equation by $\nabla^{m+1}\sigma$, and then integrating them over Ω, we obtain

$$\begin{aligned}&(\gamma\bar\rho+1)\int_\Omega|\nabla^{m+2}\sigma|^2dx\\&\le\int_\Omega\nabla^{m+1}(\mathrm{div}u)_t\nabla^{m+1}\sigma dx+\|\nabla^{m+1}(\frac{\mu}{\bar\rho+\tau\sigma}\Delta u+\frac{\mu+\lambda}{\bar\rho+\tau\sigma}\nabla\mathrm{div}u)\|_{L^2}\|\nabla^{m+2}\sigma\|_{L^2}\\&\quad+\|\nabla^{m+1}\mathrm{div}(\sigma E)\|_{L^2}\|\nabla^{m+1}\nabla\sigma\|_{L^2}+\|\nabla^{m+1}\left(\tau(E^T\cdot\nabla E-u\cdot\nabla u)\right)\|_{L^2}\|\nabla^{m+2}\sigma\|_{L^2}\\&\quad+\|\nabla^{m+1}\left(g(\tau\sigma)\nabla\sigma)+\tau f\right)\|_{L^2}\|\nabla^{m+2}\sigma\|_{L^2}. \qquad (3.7)\end{aligned}$$

Using equation $(3.2)_1$ and integrating by parts, we have

$$\begin{aligned}\int_\Omega\nabla^{m+1}\mathrm{div}u_t\nabla^{m+1}\sigma dx&=-\int_\Omega\nabla^{m+1}u_t\nabla^{m+1}\nabla\sigma dx\\&=-\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\sigma dx-\int_\Omega\nabla^{m+1}\mathrm{div}u\nabla^{m+1}\sigma_t dx\\&=-\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\sigma dx+\bar\rho\|\nabla^{m+1}\mathrm{div}u\|_{L^2}^2+\varepsilon\int_\Omega\nabla^{m+1}\nabla\sigma\nabla^{m+1}\Delta u dx\\&\quad+\int_\Omega\tau\nabla^{m+1}\mathrm{div}(\sigma u)\nabla^{m+1}\mathrm{div}u dx\\&\le-\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\sigma dx+\bar\rho\|\nabla^{m+1}\mathrm{div}u\|_{L^2}^2+\varepsilon\|\nabla^{m+2}\sigma\|_{L^2}\|\nabla^{m+3}u\|_{L^2}\\&\quad+\tau\|\nabla^{m+1}\mathrm{div}(\sigma u)\|_{L^2}\|\nabla^{m+1}\mathrm{div}u\|_{L^2}. \qquad (3.8)\end{aligned}$$

We obtain

$$\begin{aligned}&\frac{\gamma\bar{\rho}+1}{2}\int_\Omega|\nabla^{m+2}\sigma|^2dx+\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\sigma dx\\&\le C_2\|\nabla^{m+3}u\|_{L^2}^2+C\|\sigma\|_{H^{m+1}}^2\|u\|_{H^{m+2}}^2+C\|u\|_{H^{m+1}}^2\|u\|_{H^{m+2}}^2+C\|\sigma\|_{H^{m+1}}^2\|\sigma\|_{H^{m+2}}^2\\&\quad+\|\sigma\|_{H^{m+1}}^2\|E\|_{H^{m+2}}^2+C\|E\|_{H^{m+2}}^2+C\|\nabla^{m+1}f\|_{L^2}^2.\end{aligned}\tag{3.9}$$

Taking the transpose of $(3.2)_3$ and then minusing $(3.2)_3$, we obtain

$$(E^T-E)_t+V-\varepsilon\Delta(E^T-E)=h^T-h-u\cdot\nabla(E^T-E),\tag{3.10}$$

where $V=\nabla u-(\nabla u)^T=\mathrm{curl}u,\ h=\nabla uE$. Noting the condition $F^{lk}\nabla_lF^{ij}=F^{lj}\nabla_lF^{ik}$ for all $t\ge0$, which implies

$$\nabla_kE^{ij}+E^{lk}\nabla_lE^{ij}=\nabla_jE^{ik}+E^{lj}\nabla_lE^{ik}\ t\ge0.\tag{3.11}$$

Thus we have

$$\begin{aligned}&\nabla_j\nabla_kE^{ik}-\nabla_i\nabla_kE^{jk}\\&=\nabla_k\nabla_jE^{ik}-\nabla_k\nabla_iE^{jk}\\&=\nabla_k\nabla_kE^{ij}-\nabla_k\nabla_kE^{ji}+\nabla_k(E^{lk}\nabla_lE^{ij}-E^{lj}\nabla_lE^{ik})\\&\quad-\nabla_k(E^{lk}\nabla_lE^{ji}-E^{li}\nabla_lE^{jk})\\&=\Delta(E^{ij}-E^{ji})+\nabla_k(E^{lk}\nabla_lE^{ij}-E^{lj}\nabla_lE^{ik})-\nabla_k(E^{lk}\nabla_lE^{ji}-E^{li}\nabla_lE^{jk}).\end{aligned}\tag{3.12}$$

Thus by applying curl to $(3.2)_2$, we have

$$V_t-\mu\Delta V+\Delta(E^T-E)=\mathrm{curl}g_2+S,\tag{3.13}$$

where g_2 and the antisymetric matrix S are defined as

$$g_2=\tau(E\cdot\nabla E^T+f)-\left(\frac{1}{\bar{\rho}+\tau\sigma}-\bar{\rho}\right)(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u)-\tau u\cdot\nabla u-\left(\frac{P(\tau\sigma+\bar{\rho})}{(\bar{\rho}+\tau\sigma)P'(\bar{\rho})}-1\right)\nabla\sigma,$$

$$S^{ij}=\nabla_k(E^{lk}\nabla_lE^{ij}-E^{lj}\nabla_lE^{ik})-\nabla_k(E^{lk}\nabla_lE^{ji}-E^{li}\nabla_lE^{jk}).$$

Applying ∇^{m+1} to (3.13), multiplying the resulting equation by $-\nabla^{m+1}(E^T-E)$, then integrating it over Ω, we have

$$\begin{aligned}&\int_\Omega|\nabla^{m+2}(E^T-E)|^2dx\\&\le\int_\Omega(\nabla^{m+1}V)_t\nabla^{m+1}(E^T-E)dx+\mu\|\nabla^{m+2}V\|_{L^2}\|\nabla^{m+2}(E^T-E)\|_{L^2}\\&\quad+\|\nabla^m\mathrm{curl}g_2\|_{L^2}\|\nabla^{m+2}(E-E^T)\|_{L^2}+\|\nabla^mS\|_{L^2}\|\nabla^{m+2}(E^T-E)\|_{L^2}.\end{aligned}\tag{3.14}$$

$$
\begin{aligned}
&\int_\Omega (\nabla^{m+1}V)_t \nabla^{m+1}(E^T-E)dx\\
&=\frac{d}{dt}\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)dx-\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)_t dx\\
&=\frac{d}{dt}\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)dx+\int_\Omega |\nabla^{m+1}V|^2dx-\varepsilon\int_\Omega \nabla^{m+1}V\nabla^{m+1}\Delta(E^T-E)dx\\
&\quad-\tau\int_\Omega \nabla^{m+1}V\nabla^{m+1}(h^T-h)dx+\tau\int_\Omega \nabla^{m+1}V\nabla^{m+1}(u\cdot(E^T-E))dx\\
&\le\frac{d}{dt}\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)dx+\|\nabla^{m+1}V\|_{L^2}^2+\varepsilon\|\nabla^{m+2}V\|_{L^2}\|\nabla^{m+2}(E^T-E)\|_{L^2}\\
&\quad+\|\nabla^{m+2}V\|_{L^2}\|\nabla^m(h^T-h)\|_{L^2}+\|\nabla^{m+2}V\|_{L^2}\|\nabla^m(u\cdot\nabla(E^T-E))\|_{L^2}\\
&\le\frac{d}{dt}\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)dx+\|\nabla^{m+1}V\|_{L^2}^2+\varepsilon\|\nabla^{m+2}V\|_{L^2}\|\nabla^{m+2}(E^T-E)\|_{L^2}\\
&\quad+\|\nabla^{m+2}V\|_{L^2}\|u\|_{H^{m+1}}\|E\|_{H^m}+\|\nabla^{m+2}V\|_{L^2}\|u\|_{H^m}\|E\|_{H^{m+1}}.
\end{aligned}
\tag{3.15}
$$

We have

$$
\begin{aligned}
&\frac12\int_\Omega |\nabla^{m+1}(E^T-E)|^2dx\\
&\le\frac{d}{dt}\int_\Omega \nabla^{m+1}V\nabla^{m+1}(E^T-E)dx+C_3\|\nabla^{m+2}V\|_{L^2}^2+C\|E\|_{H^m}^2\|u\|_{H^{m+1}}^2+C\|E\|_{H^{m+1}}^2\|u\|_{H^m}^2\\
&\quad+C\|u\|_{H^{m+1}}^2\|u\|_{H^{m+2}}^2+C\|E\|_{H^{m+1}}^2\|E\|_{H^{m+2}}^2+C\|\sigma\|_{H^{m+1}}^2\|\sigma\|_{H^{m+2}}^2+C\|\nabla^{m+1}f\|_{L^2}^2.
\end{aligned}
\tag{3.16}
$$

From (3.5) and (3.12), we arrive at

$$
\begin{aligned}
\Delta\mathrm{div}E&=\nabla\mathrm{div}\mathrm{div}E+\mathrm{div}\mathrm{curl}\mathrm{div}E\\
&=-\Delta\nabla\sigma-\nabla\mathrm{div}\mathrm{div}(\sigma E^T)-\Delta\mathrm{div}(E-E^T)-\mathrm{div}S.
\end{aligned}
\tag{3.17}
$$

Applying ∇^{m+1} on (3.17), then using the property of the Riesz potential, we have

$$
\begin{aligned}
\|\mathrm{div}\nabla^{m+1}E\|_{L^2}^2&\le C(\|\nabla^{m+2}\|_{L^2}^2+\|\nabla^{m+2}(E^T-E)\|_{L^2}^2+\|\nabla^{m+2}(\sigma E)\|_{L^2}^2+\|\nabla^{m+1}(E\nabla E)\|_{L^2}^2)\\
&\le C\|\nabla^{m+2}(\sigma,E^T-E)\|_{L^2}^2+CR\|\nabla^{m+2}E\|_{L^2}^2.
\end{aligned}
\tag{3.18}
$$

From the above estimate, utilizing the (3.11), we have

$$
\begin{aligned}
\|\nabla^{m+1}\nabla E\|_{L^2}^2&\le\|\nabla^{m+1}\mathrm{div}E\|_{L^2}^2+\|\nabla^{m+1}\mathrm{curl}E\|_{L^2}^2\\
&\le C\|\nabla^{m+2}(\sigma,E^T-E)\|_{L^2}^2+CR\|\nabla^{m+2}E\|_{L^2}^2+\|\nabla^{m+1}(E\nabla E)\|_{L^2}^2\\
&\le C\|\nabla^{m+2}(\sigma,E^T-E)\|_{L^2}^2+CR\|\nabla^{m+2}E\|_{L^2}^2,
\end{aligned}
$$

which implies

$$
\|\nabla^{m+2}E\|_{L^2}^2\le C\|\nabla^{m+2}(\sigma,E^T-E)\|_{L^2}^2.
\tag{3.19}
$$

Therefore, multiplying (3.9) and (3.16) by $\frac{6C_1R}{\gamma\bar\rho+1}$ and $4C_1R$, respectively, then taking R sufficiently small with $\frac{6C_1C_2R}{\gamma\bar\rho+1}<\frac{\mu}{8\bar\rho}$ and $4C_1RC_3\le\frac{\mu}{8\bar\rho}$. Then, adding the resulting equations with (3.4) to yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\sigma|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}E|^2+\frac{6C_1R}{\gamma\bar\rho}\nabla^{m+1}\nabla^{m+1}\nabla\sigma+4C_1R\nabla^{m+1}V\nabla^{m+1}(E^T-E))dx\\
&\quad+\int_\Omega\left(C_1R|\nabla^{m+1}\sigma|^2+\varepsilon\gamma|\nabla^{m+3}\sigma|^2+\frac{\mu}{8\bar\rho}|\nabla^{m+3}u|^2+C_1R|\nabla(E^T-E)|^2+\varepsilon|\nabla(E^T-E)|^2\right)dx\\
&\le C\|(\sigma,u,E)\|_{H^{m+1}}^2\|(\sigma,u,E)\|_{H^{m+2}}^2+C\|\nabla^{m+1}f\|_{L^2}^2.
\end{aligned}
\tag{3.20}
$$

Let

$$\Psi(t)=\int_\Omega\Big(\gamma|\nabla^{m+2}\sigma|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}E|^2+\frac{6C_1R}{\gamma}\nabla^{m+1}u\nabla^{m+1}\nabla h$$
$$+4C_1R\nabla^{m+1}V\nabla^{m+1}(E^T-E)\Big)\,dx.$$

It is easy to see that there exist constants $\underline{C}$, $\bar{C}$ such that

$$\underline{C}\|\nabla^{m+2}(\sigma,u,E)(t)\|_{L^2}^2\le\Psi(t)\le\bar{C}\|\nabla^{m+2}(\sigma,u,E)(t)\|_{L^2}^2,$$

if R is suitable small. Note that we also have

$$\int_\Omega\Big(C_1R|\nabla^{m+2}\sigma|^2+\frac{\mu}{8}|\nabla^{m+3}u|^2+C_1R|\nabla^{m+2}E|^2\Big)\,dx$$
$$\ge\underline{M}\int_\Omega(|\nabla^{m+2}\sigma|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}E|^2)dx,$$

for some positive constant $\underline{M}$. Integrating (3.20) from 0 to T over t yields

$$\begin{aligned}\int_0^T\|\nabla^{m+2}(\sigma,u,E)\|_{L^2}^2dt&\le C\sup_{0\le t\le T}\|(\sigma,u,E)\|_{H^{m+1}}^2\int_0^T\|(\sigma,u,E)\|_{H^{m+2}}^2dt\\&\quad+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt\\&\le CR^4+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt,\end{aligned}\tag{3.21}$$

where we have used the fact of time periodicity of (σ,u,E). By using the mean value theorem, there exists a time $\varsigma\in(0,T)$ such that

$$\|\nabla^{m+2}(\sigma,u,E)(\varsigma)\|_{L^2}^2\le CR^4+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.$$

Then, integrating (3.20) from ς to t for any $t\in(\varsigma,T]$ yields

$$\Psi(t)\le CR^4+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2.$$

Since σ, u, E are periodic, then it yields

$$\Psi(0)=\Psi(T)\le CR^4+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.$$

Thus, integrating (3.20) from 0 to t for $t\in[0,T]$, we have

$$\sup_{0\le t\le T}\Psi(t)\le CR^4+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.$$

This together with (3.21) and the Poincaré inequality

$$\begin{aligned}&\sup_{0\le t\le T}\|(\sigma,u,E)(t)\|_{H^{m+1}}^2+\int_0^T\|(\sigma,u,E)(t)\|_{H^{m+2}}^2dt\\&\le\sup_{0\le t\le T}\|(\sigma,u,E)(t)\|_{H^{m+2}}^2+\int_0^T\|(\sigma,u,E)\|_{H^{m+2}}^2dt\\&\le C_4R^4+C_5\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt,\end{aligned}$$

which implies

$$R^2\le C_4R^4+C_5\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.$$

Choose R and let $C_5 \int_0^T \|\nabla^{m+1} f\|_{L^2}^2 dt < \frac{R^2}{4}$, then the above inequality is a contradiction. Thus, (3.1) holds. Now, we will show that $\mathcal{G}(\cdot, 0) = 0$. In fact, when $\tau = 0$, similar to the proof of (2.10), we can easily obtain $(\sigma, u, E) = 0$ by the Poincaré inequality. Hence, we have

$$deg(I - \mathcal{G}(\cdot, 1), B_R(0), 0) = deg(I - \mathcal{G}(\cdot, 0), B_R(0), 0) = deg(I, B_R(0), 0) = 1.$$

Consequently, we have proved (3.1) which implies that (2.4) admits a solution $(\sigma, u, E) \in \Gamma_R$. The proof of proposition 3.1 is completed. □

Now we are devoted to proving the existence of periodic solution in (1.4), which is the main result of this section.

3.2. **Proof of the Theorem 1.1(existence).**

Proof. Let $(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)$ be the time periodic solution of the regularized problem (2.4). By the proof of Proposition 3.1, it holds that

$$\sup_{0 \le t \le T} \|(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)\|_{H^{m+2}}^2 + \int_0^T (\|\sigma_\varepsilon\|_{H^{m+2}}^2 + \|u_\varepsilon\|_{H^{m+3}}^2 + \|E_\varepsilon\|_{H^{m+2}}^2) dt \le C R^2, \tag{3.22}$$

where the constant C is independent of ε. Moreover, integrating (3.20) from t to $t + \delta$,then integrating it from 0 to T to obtain

$$\int_0^T (\|\nabla^{m+2}(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)(t + \delta)\|_{L^2}^2 - \|\nabla^{m+2}(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)(t)\|_{L^2}^2) dt \le C\delta,$$

where C is independent of ε. Moreover, we will show that $\sigma_\varepsilon \in C^{\alpha,\beta}(\Omega \times (0, T))$. Precisely, applying the fact $\sigma_\varepsilon \in L^\infty(0, T; H^{m+2}(\Omega))$ with $m > [\frac{n}{2}] + 1$, we have $\sigma_\varepsilon(x, t) \in C^\alpha(\Omega)$ for any $\alpha \in (0, 1)$ for any t. Obviously, we only need to prove that there exists $\beta \in (0, 1)$ such that $\sigma_\varepsilon(x, t) \in C^\beta[0, T]$, namely,

$$|\sigma_\varepsilon(x, t_1) - \sigma_\varepsilon(x, t_2)| \le C|t_1 - t_2|^\beta$$

for any $t_1, t_2 \in (0, T)$, $x \in \Omega$.

Take a ball B_r of radius r centered at x, with $r = |t_1 - t_2|^\iota$, $\iota = \frac{1}{2\alpha+n}$. Utilizing the (2.18) and the Poincaré inequality, we have

$$\begin{aligned}
\int_{B_r} |\sigma_\varepsilon(y, t_1) - \sigma_\varepsilon(y, t_2)| dy &= \int_{B_r} |\int_{t_1}^{t_2} \frac{\partial \sigma_\varepsilon(y, t)}{\partial t} dt| dy \\
&\le C \left(\int_{t_1}^{t_2} \int_{B_r} \left| \frac{\partial \sigma_\varepsilon(y, t)}{\partial t} \right|^2 dy dt \right)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}} r^{\frac{n}{2}} \\
&\le C|t_1 - t_2|^{\frac{1}{2}} r^{\frac{n}{2}}.
\end{aligned}$$

By mean value theorem, there exists $\tilde{x} \in B_R$ such that

$$|\sigma_\varepsilon(\tilde{x}, t_1) - \sigma_\varepsilon(\tilde{x}, t_2)| \le C|t_1 - t_2|^{\frac{1}{2}} r^{-\frac{n}{2}} \le C|t_1 - t_2|^{\frac{1 - n\iota}{2}}.$$

This together with the fact $\sigma_\varepsilon \in C^\alpha(\Omega)$ gives

$$\begin{aligned}
|\sigma_\varepsilon(x, t_1) - \sigma_\varepsilon(x, t_2)| &\le |\sigma_\varepsilon(x, t_1) - \sigma_\varepsilon(\tilde{x}, t_1)| + |\sigma_\varepsilon(\tilde{x}, t_1) - \sigma_\varepsilon(\tilde{x}, t_2)| + |\sigma_\varepsilon(\tilde{x}, t_2) - \sigma_\varepsilon(x, t_2)| \\
&\le C(|t_1 - t_2|^{\iota\alpha} + |t_1 - t_2|^{\frac{(1 - n\iota)}{2}}) \\
&\le C|t_1 - t_2|^{\frac{\alpha}{(2\alpha+n)}}.
\end{aligned}$$

Taking $\beta = \frac{\alpha}{2\alpha+n}$, we have

$$|\sigma_\varepsilon(x_1, t_1) - \sigma_\varepsilon(x_2, t_2)| \le C(|x_1 - x_2|^\alpha + |t_1 - t_2|^\beta),$$

where C is independent of ε. By the same argument, we have

$$u_\varepsilon \in C^{\alpha_1,\beta_1}(\Omega \times (0, T)), \ E_\varepsilon \in C^{\alpha_2,\beta_2}(\Omega \times (0, T)),$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1)$. By virtue of (3.22) and the Arzela-Ascoli Theorem, there exists a subsequence of $(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon)$, such that

$$(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon) \to (\sigma, u, E) \ uniformly,$$

$$(\sigma_\varepsilon, u_\varepsilon, E_\varepsilon) \overset{*}{\rightharpoonup} (\sigma, u, E) \ in \ L^\infty(0,T;H^{m+2}),$$

$$u_\varepsilon \rightharpoonup u \ in \ L^2(0,T;H^{m+3}),$$

$$(\sigma_\varepsilon, E_\varepsilon) \to (\sigma, E) \ in \ L^2(0,T;H^{m+1}),$$

$$u_\varepsilon \to u \ in \ L^2(0,T;H^{m+2}).$$

Thus, $(\sigma, u, E) \in \Theta \cap \Gamma_R$ is a time-periodic solution (1.4). The existence of Theorem 1.1 is complete. □

3.3. **The uniqueness of periodic solutions.** In this section, we are devoted to investigating the uniqueness of time-periodic solutions. Let (σ_1, u_1, E_1), $(\sigma_2, u_2, E_2) \in \Theta \cap \Gamma_R$ be the time-periodic solution of (1.4). Let $\sigma = \sigma_1 - \sigma_2$, $u = u_1 - u_2$, $E = E_1 - E_2$, then (σ, u, E) satisfies the following equations

$$\begin{cases} \sigma_t + \bar\rho \mathrm{div} u = -\sigma \mathrm{u}_1 - \sigma_2 \mathrm{u}, \\ u_t - \dfrac{\mu}{\bar\rho}\Delta u - \dfrac{\mu+\lambda}{\bar\rho}\nabla \mathrm{div} u + \gamma\bar\rho\nabla\sigma - \mathrm{div}E = \left(\dfrac{1}{\bar\rho+\sigma_1} - \dfrac{1}{\bar\rho+\sigma_2}\right)(\mu\Delta u_1 + (\mu+\lambda)\nabla\mathrm{div}u_1) \\ + \left(\frac{1}{\bar\rho+\sigma_2} - \frac{1}{\bar\rho}\right)(\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u) - (u\cdot\nabla)u_1 - (u_2\cdot\nabla)u + \left(\frac{P'(\bar\rho)}{\bar\rho} - \frac{P'(\bar\rho+\sigma_1)}{\bar\rho+\sigma_1}\right)\nabla\sigma \\ + \left(\frac{P'(\bar\rho+\sigma_2)}{\bar\rho+\sigma_2} - \frac{P'(\bar\rho+\sigma_1)}{\bar\rho+\sigma_1}\right)\nabla\sigma_2 + (E^T\cdot\nabla)E_1 + (E_2^T\cdot\nabla)E, \\ E_t - \nabla u + (u\cdot\nabla)E_1 + (u_2\cdot\nabla)E = \nabla u E_1 + \nabla u_2 E. \end{cases} \tag{3.23}$$

with periodic boundary condition. Now can apply the energy method as the subsection 3.1 to prove the uniqueness. By applying ∇^{m+1} to (3.23), and multiplying the resulting equations by $\gamma\nabla^{m+1}\sigma$, $\nabla^{m+1}u$ and $\nabla^{m+1}E$ respectively, summing up, then integrating over Ω yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+1}\sigma|^2 + |\nabla^{m+1}u|^2 + |\nabla^{m+1}E|^2)dx + \int_\Omega \left(\frac{\mu}{\bar\rho}|\nabla^{m+2}u|^2 + \frac{\mu+\lambda}{\bar\rho}|\nabla^{m+1}\mathrm{div}u|^2\right)dx \\ &= -\gamma\int_\Omega \nabla^{m+1}\mathrm{div}(\sigma u_1)\nabla^{m+1}\sigma dx - \gamma\int_\Omega \nabla^{m+1}\sigma_2 \mathrm{u}\nabla^{m+1}\sigma dx \\ &\quad + \int_\Omega \nabla^{m+1}\left(\left(\frac{1}{\bar\rho+\sigma_1} - \frac{1}{\bar\rho+\sigma_2}\right)(\mu\Delta u_1 + (\mu+\lambda)\nabla\mathrm{div}u_1)\right)\nabla^{m+1}u dx \\ &\quad + \int_\Omega \nabla^{m+1}\left(\left(\frac{1}{\bar\rho+\sigma_2} - \frac{1}{\bar\rho}\right)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u)\right)\nabla^{m+1}u dx - \int_\Omega \nabla^{m+1}((u\cdot\nabla)u_1)\nabla^{m+1}u dx \\ &\quad - \int_\Omega ((u_2\cdot\nabla)u)\nabla^{m+1}u dx + \int_\Omega \nabla^{m+1}\left(\left(\frac{P'(\bar\rho)}{\bar\rho} - \frac{P'(\bar\rho+\sigma_1)}{\bar\rho+\sigma_1}\right)\nabla\sigma\right)\nabla^{m+1}u dx \\ &\quad + \int_\Omega \nabla^{m+1}\left(\left(\frac{P'(\bar\rho+\sigma_2)}{\bar\rho+\sigma_2} - \frac{P'(\bar\rho+\sigma_1)}{\bar\rho+\sigma_1}\right)\nabla\sigma_2\right)\nabla^{m+1}u dx \\ &\quad + \int_\Omega \nabla^{m+1}((E^T\cdot\nabla)E_1)\nabla^{m+1}u dx + \int_\Omega \nabla^{m+1}((E_2^T\cdot\nabla)E)\nabla^{m+1}u dx \\ &\quad - \int_\Omega \nabla^{m+1}(u\cdot\nabla E)\nabla^{m+1}E dx \\ &\quad - \nabla^{m+1}(\nabla u E_1)\nabla^{m+1}E dx + \int_\Omega \nabla^{m+1}(\nabla u_2 E)\nabla^{m+1}E dx. \end{aligned} \tag{3.24}$$

Noticing that (σ_1, u_1, E_1), $(\sigma_2, u_2, E_2) \in \Theta \cap \Gamma_R$, using the same method in subsection 3.2 to supplement the dissipation term $\int_0^T \|\sigma\|^2_{H^{m+1}} dt$ and $\int_0^T \|E\|^2_{H^{m+1}} dt$, then letting R is suitably small, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\gamma|\nabla^{m+1}\sigma|^2 + |\nabla^{m+1}u|^2 + |\nabla^{m+1}E|^2 + CR\nabla^m u \nabla^m \nabla\sigma + CR\nabla^m u \nabla^m \nabla E\big)\, dx\\ &\qquad + \underline{M}_1 \int_\Omega (|\nabla^{m+1}\sigma|^2 + |\nabla^{m+2}u|^2 + |\nabla^{m+1}E|^2) dx \le 0.\end{aligned}$$

Integrating the above inequality from 0 to T, then choosing small R, we obtain

$$\int_0^T (\|\sigma(\cdot,t)\|^2_{H^{m+1}} + \|\nabla^{m+2}u(\cdot,t)\|^2_{H^{m+2}} + \|\nabla^{m+1}E(\cdot,t)\|^2_{H^{m+1}}) dt \le 0,$$

which means that $\sigma = u = E = 0$ a.e. in Q_T. The proof of uniqueness is complete.

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