A SEQUENCE OF ELEMENTARY INTEGRALS RELATED TO INTEGRALS STUDIED BY GLAISHER THAT CONTAIN TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. We generalize several integrals studied by Glaisher. These ideas are then applied to obtain an analog of an integral due to Ismail and Valent.

1. Introduction

The following integral
$$\int_0^\infty \frac{\sin x\sinh(x/a)}{\cos(2x)+\cosh(2x/a)}\frac{dx}{x}=\frac{\tan^{-1}a}{2} \tag{1.1}$$
can be deduced as a particular case of entry 4.123.6 from [2]. The case $a=1$ of this integral can be found in an old paper by Glaisher [1]. We are going to generalize the above integral as

Theorem 1. *Let n be an odd integer. Then*
$$\int_0^1 \frac{\sin\big(n\sin^{-1}t\big)\sinh\big(n\sinh^{-1}(t/a)\big)}{\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)}\frac{dt}{t\sqrt{1-t^2}\sqrt{1+t^2/a^2}}=\frac{\tan^{-1}a}{2}. \tag{1.2}$$
When n is large, then the main contribution to the integral 1.2 comes from a small neighbourhood around $t=0$ and the integral reduces to 1.1.

Another integral by Glaisher reads (equation 24 in [1])
$$\int_0^\infty \frac{\cos x\cosh x}{\cos(2x)+\cosh(2x)}\,x\,dx=0.$$
It would be generalized as

Theorem 2. *Let n be an even integer. Then*
$$\int_0^1 \frac{\cos(n\sin^{-1}t)\cosh\left(n\sinh^{-1}t\right)}{\cos(2n\sin^{-1}t)+\cosh\left(2n\sinh^{-1}t\right)}\frac{tdt}{\sqrt{1-t^4}}=0. \tag{1.3}$$
Unfortunately there doesn't seem to be any nice parametric extensions similar to that in Theorem 1.

A particularly interesting integral is this one due to Ismail and Valent [4]
$$\int_{-\infty}^\infty \frac{dt}{\cos(K\sqrt{t})+\cosh\left(K'\sqrt{t}\right)}=1,$$
where $K=K(k)$ and $K'=K(\sqrt{1-k^2})$ are elliptic integrals of the first kind. Berndt [3] gives a generalization of this formula and as an intermediate result proves that (see Corollary 3.3)
$$\int_0^\infty \frac{x^{4k+1}dx}{\cos x+\cosh x}=(-1)^k\frac{\pi^{4k+2}}{2^{2k+1}}\sum_{j=0}^\infty(-1)^j\frac{(2j+1)^{4k+1}}{\cos\frac{\pi(2j+1)}{2n}}. \tag{1.4}$$
The next theorem gives an elementary analog of this formula:

Theorem 3. *Let k and n be a positive integers such that $k<\left[\frac{n}{2}\right]$. Then*
$$\int_0^1 \frac{t^{2k}}{\cos(2n\sin^{-1}\sqrt{t})+\cosh\left(2n\sinh^{-1}\sqrt{t}\right)}\frac{dt}{\sqrt{1-t^2}}$$
$$=\frac{\pi(-1)^k}{2^{2k+1}n}\sum_{j=1}^{n/2}\frac{(-1)^{j-1}\tan\frac{\pi(2j-1)}{2n}}{\cosh\left(n\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}\right)}\left(\frac{\sin^2\frac{\pi(2j-1)}{2n}}{\cos\frac{\pi(2j-1)}{2n}}\right)^{2k}. \tag{1.5}$$
In the last section 5, it will be explained why this form of the integral in Theorem 3 has been chosen.

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2. Proof of Theorem 1

We break the proof into a series of lemmas.

Lemma 4. *Let n be an odd integer. Then we have the partial fractions expansion*

$$\frac{\sin\big(n\sin^{-1}t\big)\sinh\big(n\sinh^{-1}(t/a)\big)}{\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)}\frac{2n}{t^2}$$
$$=\sum_{j=1}^{n}\frac{i(-1)^{j-1}}{\sin\frac{\pi(2j-1)}{2n}}\cdot\frac{\left(a\cos\frac{\pi(2j-1)}{2n}+i\right)\left(a+i\cos\frac{\pi(2j-1)}{2n}\right)}{t^2\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)-a^2\sin^2\frac{\pi(2j-1)}{2n}}. \tag{2.1}$$

Proof. When n is an odd integer, the expressions $2n\sin\big(n\sin^{-1}t\big)\sinh\big(n\sinh^{-1}(t/a)\big)/t^2$ and $\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)$ are polynomials in t^2 of degrees $n-1$ and n, respectively:

$$\frac{\sin\big(n\sin^{-1}t\big)\sinh\big(n\sinh^{-1}(t/a)\big)}{\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)}\frac{2n}{t^2}=\frac{P_{n-1}(t^2)}{Q_n(t^2)}.$$

Let's find the n roots of the denominator polynomial $Q_n(x)$. $Q_n(x)$ can be written as

$$Q_n(x)=\cos\big(n\sin^{-1}\sqrt{x}+in\sinh^{-1}(\sqrt{x}/a)\big)\cos\big(n\sin^{-1}\sqrt{x}-in\sinh^{-1}(\sqrt{x}/a)\big),$$

and thus its roots can be found from the equations

$$\sin^{-1}\sqrt{x}\pm i\sinh^{-1}(\sqrt{x}/a)=\frac{\pi(2j-1)}{2n},\quad j=1,2,...,n,$$

or equivalently from the equations

$$\sqrt{x}\sqrt{1+\frac{x}{a^2}}\pm\frac{i\sqrt{x}}{a}\sqrt{1-x}=\sin\frac{\pi(2j-1)}{2n},\quad j=1,2,...,n,$$

One can get rid of the radical expressions to come to a quadratic equation wrt x:

$$x^2\left((1-a^2)^2+4a^2\cos^2\frac{\pi(2j-1)}{2n}\right)+2xa^2(1-a^2)\sin^2\frac{\pi(2j-1)}{2n}+\sin^4\frac{\pi(2j-1)}{2n}=0,\quad j=1,2,...,n.$$

One can easily deduce from this that the n roots of the denominator polynomial are thus

$$x_j=\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)^{-1}a^2\sin^2\frac{\pi(2j-1)}{2n},\quad j=1,2,...,n. \tag{2.2}$$

Now we can find the partial fractions expansion

$$\frac{P_{n-1}(t^2)}{Q_n(t^2)}=\sum_{j=1}^{n}\frac{P_{n-1}(x_j)}{Q_n'(x_j)}\frac{1}{t^2-x_j}. \tag{2.3}$$

A simple calculation shows that

$$\frac{Q_n'(x_j)}{P_{n-1}(x_j)}=\sqrt{\frac{x_j}{a^2+x_j}}\frac{\cosh\big(n\sinh^{-1}(\sqrt{x_j}/a)\big)}{\sin\big(n\sin^{-1}\sqrt{x_j}\big)}-\sqrt{\frac{x_j}{1-x_j}}\frac{\cos\big(n\sin^{-1}\sqrt{x_j}\big)}{\sinh\big(n\sinh^{-1}(\sqrt{x_j}/a)\big)}.$$

The equation $\cos\big(2n\sin^{-1}\sqrt{x_j}\big)+\cosh\big(2n\sinh^{-1}(\sqrt{x_j}/a)\big)=0$ implies

$$\cosh\big(n\sinh^{-1}(\sqrt{x_j}/a)\big)=\mu_j\sin\big(n\sin^{-1}\sqrt{x_j}\big),\qquad\cos\big(n\sin^{-1}\sqrt{x_j}\big)=i\nu_j\sinh\big(n\sinh^{-1}(\sqrt{x_j}/a)\big)$$

where $\mu_j=\pm$, $\nu_j=\pm$. To determine the signs μ_j,ν_j, one can consider the limiting case $a>>1$. We have

$$\sqrt{x_j}=\sin\frac{\pi(2j-1)}{2n}-\frac{i}{a}\sin\frac{\pi(2j-1)}{2n}\cos\frac{\pi(2j-1)}{2n}+O(a^{-2}).$$

This means

$$\sin^{-1}\sqrt{x_j}=\frac{\pi(2j-1)}{2n}-\frac{i}{a}\sin\frac{\pi(2j-1)}{2n}+O(a^{-2}).$$

From this it follows that $\mu_j = \nu_j = (-1)^{j-1}$ and thus

$$\frac{Q_n'(x_j)}{P_{n-1}(x_j)} = (-1)^{j-1}\left(\sqrt{\frac{x_j}{a^2+x_j}} - i\sqrt{\frac{x_j}{1-x_j}}\right)$$
$$= i(-1)^j \frac{\sin\frac{\pi(2j-1)}{2n}\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)}{\left(a\cos\frac{\pi(2j-1)}{2n}+i\right)\left(a+i\cos\frac{\pi(2j-1)}{2n}\right)}.$$

Substituting this into 2.3 we get the desired result. □

Lemma 5.

$$\int_0^1 \frac{1}{t^2\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)-a^2\sin^2\frac{\pi(2j-1)}{2n}}\frac{t\,dt}{\sqrt{1-t^2}\sqrt{1+t^2/a^2}} \tag{2.4}$$

$$= \frac{\tan^{-1}a + i\tanh^{-1}\cos\frac{\pi(2j-1)}{2n}}{i\left(a\cos\frac{\pi(2j-1)}{2n}+i\right)\left(a+i\cos\frac{\pi(2j-1)}{2n}\right)}. \tag{2.5}$$

Proof. Composition of two substitutions $t^2 = 1-(1+1/a^2)\sin^2\phi$, $(0<\phi<\tan^{-1}a)$ and $\tan\phi = s$, $(0<s<a)$ reduces this integral to an integral of a rational function. □

Lemma 6. *For n odd, one has*

$$\sum_{j=1}^n \frac{(-1)^{j-1}}{\sin\frac{\pi(2j-1)}{2n}} = n.$$

Proof. Put $t=1$, $a=i$ in Lemma 4. □

From the three lemmas above it follows immediately that

$$\int_0^1 \frac{\sin\left(n\sin^{-1}t\right)\sinh\left(n\sinh^{-1}(t/a)\right)}{\cos\left(2n\sin^{-1}t\right)+\cosh\left(2n\sinh^{-1}(t/a)\right)}\frac{dt}{t\sqrt{1-t^2}\sqrt{1+t^2/a^2}}$$
$$= \frac{\tan^{-1}a}{2} + \frac{i}{2n}\sum_{j=1}^n \frac{(-1)^{j-1}}{\sin\frac{\pi(2j-1)}{2n}}\tanh^{-1}\cos\frac{\pi(2j-1)}{2n}.$$

To finish the proof, note that the sum in this formula is 0 because (since n is odd) j-th and $(n+1-j)$-th terms cancel each other out.

3. Proof of Theorem 2

Lemma 7. *Let n be an even integer. Then*

$$\frac{\cos(n\sin^{-1}t)\cosh\left(n\sinh^{-1}(t/a)\right)}{\cos(2n\sin^{-1}t)+\cosh\left(2n\sinh^{-1}(t/a)\right)} = \frac{(-1)^{n/2}}{2}\frac{a^n}{1+a^{2n}}$$
$$+\sum_{j=1}^n \frac{(-1)^j a^2\sin\frac{\pi(2j-1)}{2n}}{2n\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)}\cdot\frac{\left(a\cos\frac{\pi(2j-1)}{2n}+i\right)\left(a+i\cos\frac{\pi(2j-1)}{2n}\right)}{t^2\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)-a^2\sin^2\frac{\pi(2j-1)}{2n}}.$$

Proof. When n is even, the functions $\cos(n\sin^{-1}t)$ and $\cosh\left(n\sinh^{-1}(t/a)\right)$ are a polynomials in t^2 of degree $n/2$. This means we can write

$$\frac{\cos(n\sin^{-1}t)\cosh\left(n\sinh^{-1}(t/a)\right)}{\cos(2n\sin^{-1}t)+\cosh\left(2n\sinh^{-1}(t/a)\right)} = C + \frac{R_{n-1}(t^2)}{Q_n(t^2)}$$

where R_{n-1} is a polynomial of order $n-1$ and Q_n was defined in the proof of the Lemma 4. $Q_n(x)$ has n roots given by 2.2.

To find the constant C consider the limit $t\to+\infty$ assuming that $a>0$. In this case

$$\sin^{-1}t = \frac{\pi}{2} - i\ln(2t) + O(t^{-1}), \qquad \sinh^{-1}(t/a) = \ln(2t/a) + O(t^{-1}),$$

and we get

$$C = \frac{(-1)^{n/2}}{2}\frac{a^n}{1+a^{2n}}.$$

Since the order of the polynomial R_{n-1} is smaller than the order of the polynomial Q_n we can write the partial fractions expansion

$$\frac{R_{n-1}(t^2)}{Q_n(t^2)} = \sum_{j=1}^{n} \frac{R_{n-1}(x_j)}{Q_n'(x_j)} \frac{1}{t^2 - x_j}.$$

A calculation similar to that in Lemma 4 shows that

$$\begin{aligned}
\frac{Q_n'(x_j)}{R_{n-1}(x_j)} &= \frac{2n}{x_j}\left(\sqrt{\frac{x_j}{a^2+x_j}}\frac{\sinh\left(n\sinh^{-1}(\sqrt{x_j}/a)\right)}{\cos\left(n\sin^{-1}\sqrt{x_j}\right)} - \sqrt{\frac{x_j}{1-x_j}}\frac{\sin\left(n\sin^{-1}\sqrt{x_j}\right)}{\cosh\left(n\sinh^{-1}(\sqrt{x_j}/a)\right)}\right) \\
&= (-1)^{j-1}\frac{2n}{x_j}\left(\sqrt{\frac{x_j}{a^2+x_j}} - i\sqrt{\frac{x_j}{1-x_j}}\right) \\
&= \frac{2n(-1)^j\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)^2}{a^2\sin\frac{\pi(2j-1)}{2n}\left(a\cos\frac{\pi(2j-1)}{2n}+i\right)\left(a+i\cos\frac{\pi(2j-1)}{2n}\right)}.
\end{aligned}$$

This completes the proof of the lemma. □

Using Lemmas 5 and 7 we find

$$\begin{aligned}
&\int_0^1 \frac{\cos\left(n\sin^{-1}t\right)\cosh\left(n\cosh^{-1}(t/a)\right)}{\cos\left(2n\sin^{-1}t\right)+\cosh\left(2n\sinh^{-1}(t/a)\right)}\frac{t\,dt}{\sqrt{1-t^2}\sqrt{1+t^2/a^2}} \\
&= \frac{(-1)^{n/2}}{2}\frac{a^{n+1}}{1+a^{2n}}\tan^{-1}(1/a) + a^2\sum_{j=1}^{n}(-1)^j\frac{\tanh^{-1}\cos\frac{\pi(2j-1)}{2n} - i\tan^{-1}a}{2n\left(a^2-1+2ia\cos\frac{\pi(2j-1)}{2n}\right)}\sin\frac{\pi(2j-1)}{2n},
\end{aligned}$$

and in particular when $a = 1$

$$\begin{aligned}
\int_0^1 \frac{\cos(n\sin^{-1}t)\cosh\left(n\sinh^{-1}t\right)}{\cos(2n\sin^{-1}t)+\cosh\left(2n\sinh^{-1}t\right)}\frac{tdt}{\sqrt{1-t^4}} &= \frac{\pi}{16}(-1)^{n/2} - \sum_{j=1}^{n}(-1)^j\frac{\pi+4i\tanh^{-1}\cos\frac{\pi(2j-1)}{2n}}{16n\cot\frac{\pi(2j-1)}{2n}} \\
&= \frac{\pi}{16n}\left((-1)^{n/2}n - \sum_{j=1}^{n}\frac{(-1)^j}{\cot\frac{\pi(2j-1)}{2n}}\right).
\end{aligned}$$

To calculate the sum in this expression we use Lemma 7 with $t = 1$ and $a \to \infty$ to get

$$\sum_{j=1}^{n}\frac{(-1)^j}{\cot\frac{\pi(2j-1)}{2n}} = (-1)^{n/2}n.$$

This completes the proof of the theorem.

4. Proof of Theorem 3

Here we restrict the consideration to the case $a = 1$.

Lemma 8. *The following partial fractions expansion holds for positive integers k and n such that $k < \left[\frac{n}{2}\right]$*

$$\begin{aligned}
&\frac{t^{2k}}{\cos\left(2n\sin^{-1}\sqrt{t}\right)+\cosh\left(2n\sinh^{-1}\sqrt{t}\right)} \\
&\quad= \frac{(-1)^k}{2^{2k}n}\sum_{j=1}^{n/2}\frac{1}{4t^2+\frac{\sin^4\frac{\pi(2j-1)}{2n}}{\cos^2\frac{\pi(2j-1)}{2n}}}\frac{(-1)^{j-1}\tan\frac{\pi(2j-1)}{2n}}{\cosh\left(n\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}\right)}\frac{1+\cos^2\frac{\pi(2j-1)}{2n}}{\cot^2\frac{\pi(2j-1)}{2n}}\left(\frac{\sin^2\frac{\pi(2j-1)}{2n}}{\cos\frac{\pi(2j-1)}{2n}}\right)^{2k}.
\end{aligned}$$

Proof. From consideration of the limit $t \to +\infty$ once can see (similarly to that in Lemma 7) that the leading coefficient of the polynomial $Q_n(t) = \cos\left(2n\sin^{-1}\sqrt{t}\right) + \cosh\left(2n\sinh^{-1}\sqrt{t}\right)$ is $2^{2n-1}(1+(-1)^n)$ and thus that $Q_n(t)$ is an even polynomial of degree $2\left[\frac{n}{2}\right]$. Its roots are (see 2.2)

$$x_j = -\frac{i\sin^2\frac{\pi(2j-1)}{2n}}{2\cos\frac{\pi(2j-1)}{2n}}, \quad y_j = \frac{i\sin^2\frac{\pi(2j-1)}{2n}}{2\cos\frac{\pi(2j-1)}{2n}}, \quad j = 1, 2, ..., \left[\frac{n}{2}\right].$$

For further calculations, we will need explicit values of $\sin^{-1}\sqrt{x_j}$ and $\sinh^{-1}\sqrt{x_j}$, where the principal branches of the multivalued functions are implied. First, one can write

$$\sin^{-1}\sqrt{x_j} = \xi_j - i\eta_j, \quad \sinh^{-1}\sqrt{x_j} = \varphi_j - i\psi_j,$$

with $\xi_j, \eta_j, \varphi_j, \psi_j > 0$. Further, from elementary identities $1 - 2t = \cos(2\sin^{-1}\sqrt{t})$ and $1 + 2t = \cosh(2\sinh^{-1}\sqrt{t})$ one can see that

$$\cos(2\xi_j)\cosh(2\eta_j) = \cosh(2\varphi_j)\cos(2\psi_j) = 1,$$

$$\sin(2\xi_j)\sinh(2\eta_j) = \sinh(2\varphi_j)\sin(2\psi_j) = \frac{\sin^2\frac{\pi(2j-1)}{2n}}{\cos\frac{\pi(2j-1)}{2n}}.$$

These equations can be easily solved to yield

$$\xi_j = \psi_j = \frac{\pi(2j-1)}{4n}, \quad \eta_j = \varphi_j = \frac{1}{2}\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}.$$

Thus

$$\sin^{-1}\sqrt{x_j} = \frac{\pi(2j-1)}{4n} - \frac{i}{2}\sinh^{-1}\tan\frac{\pi(2j-1)}{2n},$$
$$\sinh^{-1}\sqrt{x_j} = \frac{\pi(2j-1)}{4ni} + \frac{1}{2}\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}.$$

Similarly

$$\sin^{-1}\sqrt{y_j} = \frac{\pi(2j-1)}{4n} + \frac{i}{2}\sinh^{-1}\tan\frac{\pi(2j-1)}{2n},$$
$$\sinh^{-1}\sqrt{y_j} = -\frac{\pi(2j-1)}{4ni} + \frac{1}{2}\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}.$$

For $k < \left[\frac{n}{2}\right]$ we have the partial fractions expansion

$$\frac{t^{2k}}{Q_n(t)} = \sum_{j=1}^{n/2}\left(\frac{x_j^{2k}}{Q_n'(x_j)}\frac{1}{t-x_j} + \frac{y_j^{2k}}{Q_n'(y_j)}\frac{1}{t-y_j}\right).$$

Calculations using the formulas above yield

$$Q_n'(x_j) = -Q_n'(y_j) = \frac{4ni(-1)^j\cos^2\frac{\pi(2j-1)}{2n}}{\sin\frac{\pi(2j-1)}{2n}\left(1+\cos^2\frac{\pi(2j-1)}{2n}\right)}.$$

Now substitute this into the formula above. □

Using Lemma 8 and the following consequence of Lemma 5

$$\int_0^1 \frac{1}{4t^2 + \frac{\sin^4\frac{\pi(2j-1)}{2n}}{\cos^2\frac{\pi(2j-1)}{2n}}}\frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}\frac{\cot^2\frac{\pi(2j-1)}{2n}}{1+\cos^2\frac{\pi(2j-1)}{2n}},$$

one can easily complete the proof of Theorem 3.

5. Discussion

Let's introduce the notation

$$\alpha_z = n\sinh^{-1}\sin\frac{\pi z}{2n},$$

where we assume the principal branches of the multivalued functions of complex variable. With this definition one can rewrite the integral in Theorem 1 with $a = 1$ as

$$\int_0^1 \frac{\sin\left(n\sin^{-1}t\right)\sinh\left(n\sinh^{-1}t\right)}{\cos\left(2n\sin^{-1}t\right)+\cosh\left(2n\sinh^{-1}t\right)}\frac{dt}{t\sqrt{1-t^4}} = \frac{\pi}{2n}\int_0^n \frac{\sin\frac{\pi x}{2}\sinh\frac{\alpha_x}{2}}{\cos\pi x + \cosh\alpha_x}\frac{dx}{\sinh\frac{\alpha_x}{n}}.$$

As we will now show, the last integral has an interesting symmetry.

Let us define y_* by the equation

$$\alpha_{iy_*} = \pi i n,$$

and consider the integral over an interval on the imaginary axes

$$J = \int_{iy_*}^0 \frac{\sin\frac{\pi z}{2}\sinh\frac{\alpha_z}{2}}{\cos\pi z + \cosh\alpha_z}\frac{dz}{\sinh\frac{\alpha_z}{n}} = \int_0^{y_*} \frac{\sinh\frac{\pi y}{2}\sin\frac{\alpha_{iy}}{2i}}{\cos\pi y + \cos(\alpha_{iy}/i)}\frac{dy}{\sin\frac{\alpha_{iy}}{in}}.$$

When y is real, then α_{iy} is purely imaginary, so we make change of variables $\alpha_{iy} = \pi i s$. We get

$$\sin\frac{\pi s}{2n} = \sinh\frac{\pi y}{2n},$$

which implies that

$$\pi y = \alpha_s.$$

Also it is easy to show that

$$\frac{dy}{\sin\frac{\pi s}{n}} = \frac{ds}{\sinh\frac{\alpha_s}{n}}.$$

Thus the integral under consideration becomes

$$J = \int_0^n \frac{\sin\frac{\pi s}{2}\sinh\frac{\alpha_s}{2}}{\cos\pi s + \cosh\alpha_s}\frac{ds}{\sinh\frac{\alpha_s}{n}}.$$

To recap what we have just showed:

$$\int_{iy_*}^0 \frac{\sin\frac{\pi z}{2}\sinh\frac{\alpha_z}{2}}{\cos\pi z + \cosh\alpha_z}\frac{dz}{\sinh\frac{\alpha_z}{n}} = \int_0^n \frac{\sin\frac{\pi s}{2}\sinh\frac{\alpha_s}{2}}{\cos\pi s + \cosh\alpha_s}\frac{ds}{\sinh\frac{\alpha_s}{n}}.$$

The integral in theorem 1 has been chosen to have the same kind of symmetry:

$$\int_0^1 \frac{t^{2k}}{\cos(2n\sin^{-1}\sqrt{t}) + \cosh\left(2n\sinh^{-1}\sqrt{t}\right)}\frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{n}\int_0^n \frac{\left(\sin\frac{\pi x}{2n}\right)^{4k+2}}{\cos\pi x + \cosh\alpha_x}\frac{dx}{\sinh\frac{\alpha_x}{n}},$$

$$\int_{iy_*}^0 \frac{\left(\sin\frac{\pi z}{2n}\right)^{4k+2}}{\cos\pi z + \cosh\alpha_z}\frac{dz}{\sinh\frac{\alpha_z}{n}} = \int_0^n \frac{\left(\sin\frac{\pi x}{2n}\right)^{4k+2}}{\cos\pi x + \cosh\alpha_x}\frac{dx}{\sinh\frac{\alpha_x}{n}}.$$

We mention without proof an alternative representation for the sum in Theorem 3 with $k = 0$:

$$\sum_{j=1}^{n/2} \frac{(-1)^{j-1}\tan\frac{\pi(2j-1)}{2n}}{\cosh\left(n\sinh^{-1}\tan\frac{\pi(2j-1)}{2n}\right)} = \sum_{y=1}^{n} \frac{\coth\left(n\sinh^{-1}\sin\frac{\pi(2y-1)}{2n}\right)}{\coth\left(\sinh^{-1}\sin\frac{\pi(2y-1)}{2n}\right)} - \frac{n}{2}$$

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