Quasinilpotent operators on separable Hilbert spaces have nontrivial invariant subspaces

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Abstract

The invariant subspace problem is a well known unsolved problem in funtional analysis. While many partial results are known, the general case for complex, infinite dimensional separable Hilbert spaces is still open. It has been shown that the problem can be reduced to the case of operators which are norm limits of nilpotents. One of the most important subcases is the one of quasinilpotent operators, for which the problem has been extensively studied for many years. In this paper, we will prove that every quasinilpotent operator has a nontrivial invariant subspace. This will imply that all the operators for which the ISP has not been established yet are norm-limits of operators having nontrivial invariant subspaces.

1 Introduction

The invariant subspace problem is one of the most important unsolved problems in functional analsysis. It asks whether every bounded linear operator $T \in B(X)$ (X Banach space) has a nontrivial invariant subspace, i.e. a closed [1] linear subspace W different from X and $\{0\}$ such that $T(W) \subseteq W$. The finite dimensional case (with $\dim X \geq 2$) and the non separable (infinite dimensional) one are easy to settle. Enflo and Read were the first ones to construct counterexamples in the general setting. However, for reflexive Banach spaces (and in particular for Hilbert spaces) the problem is still open. We also note that the problem is open even in the real case, but here we will only deal with complex spaces.
A long-standing important subproblem of the ISP (which stands for invariant subspace problem) is the ISP for quasinilpotent operators, i.e. operators T such that $\sigma(T) = \{0\}$. It is clear that solving the ISP for quasinilpotent operators also establishes the problem for every operator whose spectrum is a singleton:

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[1] Throughout this paper, we always tacitly assume that the subspaces we are talking about are closed.

indeed, let T be such that $\sigma(T) = \{\lambda\}$. Then, $T - \lambda \, \mathrm{Id}$ is quasinilpotent. Consequently, if we can show that quasinilpotent operators have some nontrivial invariant subspace W, we can say that, if $x \in W$, then $Tx - \lambda x = y \in W$. But then, $Tx = y + \lambda x \in W$. Thus, whenever $x \in W$, $Tx \in W$, so that W is also a nontrivial invariant subspace for T. This subproblem has been subject of research in the last half century; we refer to the papers [7] and [19-23] for some results.
In the next section we will recall some notions and some results that will be used in the proof of our main result, which is shown in Section 4. The restrictions of the ISP obtained up to now and some consequences of our main Theorem are discussed in Section 3.
For more details on the ISP, we refer to [1-3]. Throughout the paper, H will always denote a complex, infinite dimensional, separable Hilbert space.

2 Preliminaries

The results in Section 3 and 4 involve some notions of (essential) spectra, which we recall here. First of all, the spectrum of an operator is defined as:

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \, \mathrm{Id} \, \text{is not invertible}\} \tag{2.1}$$

It is well know that this spectrum can be written as the union of three other spectra, namely the point spectrum $\sigma_p(T)$, which consists of the eigenvalues of T, the continuous spectrum $\sigma_c(T)$, consisting of the λ's such that $T - \lambda \, \mathrm{Id}$ is injective, has dense range but is not surjective, and the residual spectrum $\sigma_r(T)$, which contains the λ's such that $T - \lambda \, \mathrm{Id}$ is injective but does not have dense range. Anoher important kind of spectrum is the approximate point spectrum $\sigma_a(T)$, which contains the approximate eigenvalues of T. Approximate eigenvalues can also be characterised as those λ's for which $T - \lambda \, \mathrm{Id}$ is not bounded below. It is clear that:

$$\sigma_c(T) = \sigma_a(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) \tag{2.2}$$

In order to define some essential spectra of T, we need to define:

Definition 2.1. *An operator $T \in B(H)$ is Fredholm if its kernel and cokernel are finite-dimensional and its range is closed. This is equivalent to: T is such that $\dim \ker T, \dim \ker T^* < \infty$ and its range is closed. This is shown, for instance, in Exercise 8 in [4] or in Corollary 2.2 in [5]. The set of Fredholm operators is denoted by Φ. The index of a Fredholm operator is defined by:*

$$\mathrm{ind}(T) := \dim \ker T - \dim \mathrm{coker}\, T = \dim \ker T - \dim \ker T^*$$

The set of Fredholm operators with index 0 is denoted by Φ_0.

The Fredholm spectrum $\sigma_\Phi(T)$ is defined as follows:

$$\sigma_\Phi(T) := \{\lambda \in \mathbb{C} : T - \lambda \, \mathrm{Id} \notin \Phi\} \tag{2.3}$$

Moreover, the Weyl spectrum $\sigma_w(T)$ is:

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \operatorname{Id} \notin \Phi_0\} \tag{2.4}$$

The Weyl spectrum is invariant under compact perturbation, i.e. $\sigma_w(T) = \sigma_w(T + K)$ for every compact K. This easily follows from the following characterisation discovered by Schechter:

$$\sigma_w(T) = \bigcap_{K \in \mathcal{K}(H)} \sigma(T + K) \tag{2.5}$$

where $\mathcal{K}(H)$ is the ideal of compact operators. We have:

$$\sigma_\Phi(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$$

Furthermore, it is proved in Exercise 7 in [4] or Remark 4.3 in [5] that the Fredholm spectrum is always nonempty when H is, as in our case, infinite dimensional. When T is quasinilpotent, this implies that $\sigma_\Phi(T) = \sigma_w(T) = \sigma(T) = \{0\}$. For more on these topics, we refer to [6].
We will now recall some important results which will be used in our proof. In [7], Tcaciuc proved that:

Proposition 2.1 (Tcaciuc)**.** *Let $T \in B(H)$ be quasinilpotent. Then, the following are equivalent:*
(i) T has a nontrivial invariant subspace;
(ii) there exists a rank 1 operator F such that $T + \alpha F$ is quasinilpotent for all $\alpha \in \mathbb{C}$;
(iii) there exists a rank 1 operator F such that $T + \alpha F$ is quasinilpotent for $\alpha = 1$ and for some other $\alpha \neq 0, 1$.

Proof. This is Theorem 2.3 in [7] applied to the case of (complex, infinite dimensional, separable) Hilbert spaces. □

We will make use of this really useful characterisation together with the next proposition:

Proposition 2.2 (Tcaciuc)**.** *Let $T \in B(H)$ be quasinilpotent and F be a rank 1 operator. Then, exactly one of the following three possibilities happens:*
(i) $T + \alpha F$ is quasinilpotent for all $\alpha \in \mathbb{C}$;
(ii) for all nonzero $\alpha \in \mathbb{C}$, with possibly one exception, $\sigma_p(T + \alpha F)$ is countably infinite;
(iii) there is some natural number K such that for all nonzero $\alpha \in \mathbb{C}$, $0 < |\sigma_p(T + \alpha F) \setminus \{0\}| < K$.

Proof. This is Proposition 2.4 in [7] in the case of (complex, infinite dimensional, separable) Hilbert spaces. □

The last ingredient of our proof is the following conjecture of Herrero's, which has been proved in [8] by Jiang and Ji.

Proposition 2.3 (Herrero-Jiang-Ji Theorem)**.** *Let $T \in B(H)$ have a connected spectrum. Then, $T = K + S$, where K is compact and S is strongly irreducible.*

An operator is strongly irreducible if ATA^{-1} is irreducible for every invertible operator A, otherwise it is strongly reducible. This is Definition 2.23 in [11]; we refer to this book for more details and other equivalent characterisations of these operators. We recall that, if T has a disconnected spectrum, then $T + K$ is strongly reducible for every compact operator K (this has been noticed, for instance, in the introduction of [9]) [2]. The compact operator in Proposition 2.3 can be chosen to have a small norm: the case of operators with a singleton spectrum is Proposition 5.36 in [11], while the general case is proved in [24].
It will be important in our proof to note that every nonempty set A which is either finite (but not a singleton) or countably infinite is disconnected in $\mathbb{C} \cong \mathbb{R}^2$. This well known result can be intuitively seen to be true; we will not give a proof of this here.

3 Some reductions

In this section we will prove a useful reduction of the ISP for separable Hilbert spaces. We will first recall some related reductions obtained by other authors. It can be shown (see later) that if $\sigma_\Phi(T) \neq \sigma(T)$, then T has a nontrivial hyperinvariant subspace (i.e. a nontrivial subspace which is invariant under every operator S which commutes with T; clearly, such a space is also invariant under T). Moreover, it has been proved that every operator for which $\sigma_\Phi(T) = \sigma(T)$ is quasitriangular, i.e. there exists an increasing sequence $\{P_n\}$ of finite-rank projections converging strongly to Id such that

$$\|P_nTP_n - TP_n\| \to 0$$

The ISP is then equivalent to the ISP for biquasitriangular operators, i.e. operators T which are quasitriangular and whose adjoint T is quasitriangular. Actually, something more can be shown: we can consider just the operators with connected spectrum, since a disconnected spectrum is known to imply the existence of nontrivial (hyper)invariant subspaces. Moreover, we can assume that $0 \in \sigma(T)$ by translation by some scalar λ. Thus, we can define a class $C(H)$ consisting of the biquasitriangular operators with connected spectrum and connected Fredholm spectrum such that 0 belongs to $\sigma(T)$. By Theorem 1.1 in [13], $C(H) = \mathcal{N}^-$, the norm closure of the space of nilpotent operators. Actually, the problem can be reduced a bit more, as we will see later. This result shows that understanding the norm closure of nilpotents leads to interesting consequences in the theory of invariant subspaces. There are various papers in which the norm closure of $\mathcal{N}$ has been studied: among these, we refer to [12-15]. We refer to [16] for some results on invariant subspaces for quasitriangular operators. We now prove the following restriction:

[2]This means that there is no compact K such that $T + K$ is strongly irreducible. In fact, this was the reason why Herrero's conjecture only considered operators with connected spectrum.

Theorem 3.1. *The ISP for separable, complex, infinite dimensional Hilbert spaces is equivalent to the ISP for operators $T \in B(H)$ such that:*

$$\sigma(T) = \sigma_c(T) = \sigma_a(T) = \sigma_\Phi(T) = \sigma_w(T)$$

and for which the spectrum is connected, contains 0 and is not a singleton.

We will prove this result using the following extension of Lomonosov Theorem, obtained in [17]:

Theorem 3.2 (Kim - Pearcy - Shields)**.** *Let $T \in B(H)$ be nonscalar. If there exists a nonzero compact K such that* $\text{rank}(TK - KT) \leq 1$*, then T has a nontrivial hyperinvariant subspace.*

In [18], Kim, Pearcy and Shields defined the class $\Delta(H)$ of operators for which there is some nonzero K such that $\text{rank}(TK - KT) = 1$ and they proved some results. In particular, we recall the following ones:

Proposition 3.1. *$T \in \Delta(H)$ if and only if $(\alpha T + \beta \, \text{Id}) \in \Delta(H)$ for all $\alpha \in \mathbb{C} \setminus \{0\}$ and for all $\beta \in \mathbb{C}$.*

Proposition 3.2. *If $\sigma(T)$ is disconnected, then $T \in \Delta(H)$.*

Proposition 3.3. *If $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$, then $T \in \Delta(H)$.*

We can now prove the above restriction of the ISP.

Proof. We can consider $\sigma(T)$ to be connected by Proposition 3.2. Moreover, we can assume $\sigma_p(T) \cup \sigma_p(T^*) = \emptyset$ by Proposition 3.3. Suppose that $\sigma_\Phi(T) \neq \sigma(T)$. Then there exists some λ such that $T - \lambda \, \text{Id}$ is not invertible but it is Fredholm. Since $\sigma_p(T) \cup \sigma_p(T^*) = \emptyset$, $T - \lambda \, \text{Id}$ is injective, but since it is not invertible it cannot be surjective. The range is closed because it is Fredholm, so that the range is different from H. Since T is nonscalar, the range is not $\{0\}$, and hence it is a nontrivial invariant subspace for $T - \lambda \, \text{Id}$, which implies that it is a nontrivial invariant subspace also for T. Thus, since

$$\sigma_\Phi(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$$

we can assume the equality of these three spectra. Moreover, we can assume $\sigma_r(T) = \emptyset$, because otherwise the closure of the range is a nontrivial invariant subspace. Therefore, we obtain:

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) = \sigma_c(T)$$

By (2.2), the continuous spectrum is equal to the approximate point spectrum, so that the chain of equalities in the Theorem can be indeed assumed. By Theorem 4.1, operators whose spectrum is a singleton can be excluded. To conclude, note again that 0 can be assumed to be in the spectrum by translation by some scalar λ. □

We also note that Theorem 4.1 has the following corollary:

Corollary 3.1. *Every operator in the norm closure of $\mathcal{N}$ can be written as the norm limit of operators having nontrivial invariant subspaces.*

Proof. Since nilpotent operators are quasinilpotent, this is an obvious consequence of the fact that quasinilpotent operators have some nontrivial invariant subspace. □

Remark 3.1. This result can actually be easily shown just using the fact that nilpotent operators have nontrivial invariant subspaces. We have reported it here because this fact may be useful to contruct nontrivial invariant subspaces starting from the approximating sequence of nilpotents, as suggested in Question 1 at the end of [13].

4 Main result

In this section we will prove the main result of the paper, namely:

Theorem 4.1. *Let $T \in B(H)$ be such that $\sigma(T)$ is a singleton. Then, T has a nontrivial invariant subspace.*

Proof. As we noted in the introduction, we only need to show this in the case where T is quasinilpotent. We will prove that the unique possiblity that can occur in Proposition 2.2 is the first one. This will imply, via Proposition 2.1, that T has a nontrivial invariant subspace. Throughout the proof, F is any (fixed) rank 1 operator. Suppose that either (ii) or (iii) in Proposition 2.2 happens (so, $T + \alpha F$ is not quasinilpotent). As it is noticed at the beginning of the proof of Theorem 2.3 in [7], all the nonzero elements in $\sigma(T + \alpha F)$ are eigenvalues. i.e. they belong to the point spectrum. This means that

$$\sigma(T + \alpha F) = \sigma_p(T + \alpha F) \cup \{0\}$$

Actually, this is a particular case of the following result:

Lemma 4.1. *Let $T \in B(H)$. Then:*

$$\sigma(T) = \sigma_p(T) \cup \sigma_w(T)$$

Proof. This is (8.50) in [10]. We will give here a short proof. First, note that:

$$\sigma_p(T) \cup \sigma_w(T) \subseteq \sigma(T)$$

Suppose that $\sigma_p(T) \cup \sigma_w(T) \subsetneq \sigma(T)$. Then, there is some λ such that $T - \lambda \,\mathrm{Id}$ is Fredholm with index 0 and is also injective. Since it is Fredholm, the closure of $\mathrm{Im}(T - \lambda \,\mathrm{Id})$ is equal to $\mathrm{Im}(T - \lambda \,\mathrm{Id})$ itself. By injectivity together with non-invertibility, $T - \lambda \,\mathrm{Id}$ is not surjective, so $\mathrm{Im}(T - \lambda \,\mathrm{Id}) \neq H$ and hence the range is not dense in H. Thus, $T - \lambda \,\mathrm{Id}$ is injective but the range is not dense, which implies $\lambda \in \sigma_R(T)$ (the residual spectrum). It is well known that:

$$\overline{\sigma_R(T)} \subseteq \sigma_p(T^*) \tag{4.1}$$

where $\overline{A} := \{\overline{z}, z \in A\}$ (the set of complex conjugates of the elements of A). Using this, we conclude that $\overline{\lambda} \in \sigma_p(T^*)$. But then $T^* - \overline{\lambda}\,\mathrm{Id} = (T - \lambda\,\mathrm{Id})^*$ is not injective, so $\dim\ker((T - \lambda\,\mathrm{Id})^*) > 0$. But this integer is equal to $\dim\ker(T - \lambda\,\mathrm{Id})$ because this operator is Fredholm with index 0. Thus, $\lambda \in \sigma_p(T)$, a contradiction. Therefore, the above equality holds.
A simple way to prove (4.1) is by using Hahn-Banach Theorem. More precisely, let $\lambda \in \sigma_R(T)$. By definition, the range of $T - \lambda\,\mathrm{Id}$ is not dense. By the Hahn-Banach Theorem, there is some nonzero $z \in X^*$ (the dual of X) that vanishes on $\mathrm{Im}(T - \lambda\,\mathrm{Id})$. $\forall x \in H$, we have:

$$\langle z, (T - \lambda\,\mathrm{Id})x \rangle = \langle (T^* - \overline{\lambda}\,\mathrm{Id})z, x \rangle = 0$$

Thus, $(T^* - \overline{\lambda}\,\mathrm{Id})z = 0 \in X^*$ and hence $\overline{\lambda} \in \sigma_p(T^*)$. Note that this holds for any Banach space X, not only for Hilbert spaces. The above argument shows that:

$$\overline{\sigma_R(T)} \subseteq \sigma_p(T^*)$$

which is (4.1). □

Because of this result, our assumptions imply that $\sigma(T + \alpha F)$ is either finite (but not a singleton) or countably infinite. Thus, $\sigma(T + \alpha F)$ is not connected in $\mathbb{C}$. Since T has a connected spectrum, by Herrero-Jiang-Ji Theorem we know that $T = K + S$ for some compact K and strongly irreducible S. This implies that $T + \alpha K = K_1 + S$, where $K_1 = K + \alpha F$ is compact because it is the sum of compact operators. But the spectrum of $T + \alpha F$ is not connected, so such a K_1 cannot exist, because $T + \alpha F + K_2$ is strongly reducible for all compact K_2, as noticed in Section 2. Therefore, we have a contradiction, and consequently $\sigma(T + \alpha F)$ must be connected for all α. This implies that neither (ii) nor (iii) can happen, because for (almost) [3] all α the spectrum would be disconnected. Hence, the spectrum is a singleton, and since it contains $\sigma_w(T + \alpha F) =$ (by what we said in Section 2) $= \sigma_w(T) = \{0\}$ the operator is quasinilpotent. Thus, (i) holds, and this concludes the proof. □

5 Conclusion

In this paper we have established the invariant subspace problem for quasinilpotent operators. This leads to a restriction of the ISP to non-quasinilpotent operators belonging to the norm closure of the space of nilpotents. We note that Herrero-Jiang-Ji Theorem, together with the results obtained by Tcaciuc, turn out to be a really useful tool in our proof. If Herrero-Jiang-Ji Theorem could be extended to the case of reflexive, separable Banach spaces, then we could follow the same idea used in Section 4 to prove that quasinilpotent operators on such spaces have nontrivial invariant subspaces.

[3] By (iii) in Proposition 2.1, and by taking (if necessary) a multiple of the rank 1 operator F (which is still a rank 1 operator), it is clear that if this result holds for almost all α, then it holds for all α.

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