A solution to the (hyper)invariant subspace problem

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Abstract

In this paper we will give an affirmative solution to the (hyper)invariant subspace problem for complex, separable, infinite dimensional reflexive Banach spaces. Our method of proof is based on an extension of Lomonosov Theorem proved in [4]: we will show that every nonscalar operator belongs to the class $\Delta(X)$ defined in [5], which will imply that every nonscalar $T \in B(X)$ has a nontrivial (hyper)invariant subspace. In the last section, we will discuss the relationship between our proof and the general case of the problem for Banach spaces (in particular, nonreflexive ones).

1 Introduction

The (hyper)invariant subspace problem (which will be often called here (H)ISP) is the simple-to-state question:

Question 1.1 ((Hyper)invariant subspace problem)**.** Let $T \in B(X)$ be a bounded linear operator from the complex Banach space X to itself. Does there always exist a nontrivial T-(hyper)invariant closed subspace?

Recall that a closed [1] subspace W is invariant for T if $T(W) \subseteq W$. Since $W = \{0\}$ and $W = X$ are always obviously invariant, we will call them 'trivial' and we will only look for nontrivial invariant subspaces. W is called hyperinvariant if $S(W) \subseteq W$ for all $S \in \{T\}' := \{S \in B(H) : ST = TS\}$ (clearly, every hyperinvariant subspace is also invariant). The first results on the existence of nontrivial invariat subspaces are due to von Neumann, who solved the ISP for compact operators in the 1930s, even though he did not publish his results. We refer to [1-2] and [14] for more details on the history of the results obtained, and for more details on the problem. We only recall that the problem for finite dimensional Banach spaces and for non separable Banach spaces has been easily

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[1] Throughout this paper, we always tacitly assume that the subspaces we are talking about are closed.

solved, so we are only interested in the infinite dimensional separable case. A fundamental result was obtained by Lomonosov in [3]. This spectacular result solves the problem for many operators:

Theorem 1.1 (Lomonosov Theorem). *Let $T \in B(X)$ be nonscalar (i.e. it is not a multiple of* Id*). If there exists a nonzero compact operator K such that $TK = KT$, then T has a nontrivial hyperinvariant subspace.*

Actually, the result obtained by Lomonosov was even more general, but here we will use this version. It has been shown that there exist operators which do not satisfy the hypothesis of Lomonosov Theorem. Furthermore, Enflo [6] and Read [7-8] were the first ones to construct counterexamples in non reflexive Banach spaces. An important extension of Lomonosov result is the following one (see [4]):

Theorem 1.2 (Kim - Pearcy - Shields). *Let $T \in B(X)$ be nonscalar. If there exists a nonzero compact operator K such that* $\text{rank}(TK - KT) \leq 1$*, then T has a nontrivial hyperinvariant subspace.*

In [5], Kim, Pearcy and Shields defined the class $\Delta(X)$ of nonscalar operators $T \in B(X)$ for which there exists some nonzero K such that $\text{rank}(TK - KT) = 1$. They also proved some results, which we report here:

Proposition 1.1. *$T \in \Delta(X)$ if and only if $(\alpha T + \beta \, \text{Id}) \in \Delta(X)$ for all $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$.*

Proposition 1.2. *If either T or T^* has an eigenvalue, i.e. $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$, then $T \in \Delta(X)$.*

Here T^* denotes the adjoint. The result holds for both the adjoint defined in Banach spaces and the adjoint in Hilbert spaces.
It is clear that, if we can show that every operator belongs to $\Delta(X)$, the (hyper)invariant subspace problem is solved. This is what we will do in Section 2. The result will be that the (H)ISP holds true for every infinite dimensional complex reflexive Banach space X. In Section 3 we will actually notice that the proof remains valid even when we consider nonreflexive Banach spaces. While this may seem at first a contradiction with the counterexamples constructed, there are other known results whch hold true when we work under some logical axioms, and for which instead we can construct counterexamples under some other axioms (e.g. Whitehead problem, see [10-11]). We will discuss this conclusion and propose new problems for future research.

2 Proof of the (H)ISP

Throughout this section, X is a complex infinite dimensional reflexive Banach space. We first need to recall some well known notions and facts (see [9,12-13] for Fredholm operators and for the essential spectrum), which actually also hold even when X is nonreflexive. First of all, the spectrum of an operator $T \in B(X)$ is defined as:

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \operatorname{Id} \text{is not invertible}\}$$

The point spectrum of T, denoted by $\sigma_p(T)$, is contained in $\sigma(T)$, and it consists of the eigenvalues of T. We will make use of a certain notion of essential spectrum, namely Fredholm spectrum. First, we need to recall:

Definition 2.1. *An operator $T \in B(X)$ is called Fredholm if* $\dim \ker T < \infty$ *and* $\dim \ker T^* < \infty$.

Remark 2.1. Sometimes, the definition is stated in a different way: T is Fredholm if $\operatorname{Im} T$ (the range of T) is closed and if $\dim \ker T < \infty$ and $\dim \operatorname{coker} T < \infty$. However, it is well known that the finiteness of the dimension of the cokernel implies that the range is closed, so that this assumption is redundant. Moreover, it is known that $\dim \operatorname{coker} T < \infty \Leftrightarrow \dim \ker T^* < \infty$ whichever is the infinite dimensional Banach space X, so the two definitions are actually equivalent.

The Fredholm spectrum is defined by:

$$\sigma_\Phi(T) := \{\lambda \in \mathbb{C} : T - \lambda \operatorname{Id} \notin \Phi\} \tag{2.1}$$

Here, Φ denotes the set of Fredholm operators. It is known that $\sigma_\Phi(T) \subseteq \sigma(T)$. Moreover, the following relation (together with the next proposition) will be crucial in our proof:

$$\sigma_\Phi(T) = \sigma_{B(X)/K(X)}(\pi(T)) \tag{2.2}$$

The RHS is the spectrum of the equivalence class of T in the Calkin algebra $B(X)/K(X)$, where $K(X)$ is the ideal of compact operators. For a proof, see Corollary 4.2 in [13]. It is well known (see, for instance, Exercise 7 in [9] or Remark 4.3 in [13]) that:

Proposition 2.1. *When X is an infinite dimensional Banach space, we have that:*

$$\sigma_{B(X)/K(X)}(\pi(T)) \neq \emptyset$$

Now we have all the ingredients to begin our proof. We first prove:

Proposition 2.2. *Let $T \in B(X)$ be nonscalar. Suppose that there exists $\lambda \in \mathbb{C}$ such that either* $\dim \ker(T - \lambda \operatorname{Id}) = \infty$ *or* $\dim \ker(T - \lambda \operatorname{Id})^* = \infty$ *(or both), then $T \in \Delta(X)$, and hence T has a nontrivial (hyper)invariant subspace.*

Proof. If T is nonscalar, $T - \lambda \operatorname{Id}$ is nonscalar, whichever is $\lambda \in \mathbb{C}$. Since the dimension of the kernel of either $T - \lambda \operatorname{Id}$ or $(T - \lambda \operatorname{Id})^*$ is infinity, we know that one of these two operators certainly has 0 as an eigenvalue. Consequently, by Proposition 1.2 we conclude that $(T - \lambda \operatorname{Id}) \in \Delta(X)$. But then, by Proposition 1.1 with $\alpha = 1$ and $\beta = \lambda$, we have that $T \in \Delta(X)$. By Theorem 1.2, T has a nontrivial (hyper)invariant subspace. □

We can finally prove:

Theorem 2.1. *Let $T \in B(X)$ be nonscalar. Then, T has a nontrivial (hyper)invariant subspace.*

Proof. If there exists $\lambda \in \mathbb{C}$ such that either $\dim \ker(T - \lambda \,\mathrm{Id}) = \infty$ or $\dim \ker(T - \lambda \,\mathrm{Id})^* = \infty$ (or both), then T has a nontrivial (hyper)invariant subspace by Proposition 2.2. So suppose that $\dim \ker(T - \lambda \,\mathrm{Id}) < \infty$ and $\dim \ker(T - \lambda \,\mathrm{Id})^* < \infty$ for all λ. Then, $T - \lambda \,\mathrm{Id}$ is a Fredholm operator $\forall \lambda$, which implies that $\sigma_\Phi(T) = \emptyset$. But this is impossible by Proposition 2.1 and (2.2), so this latter case never happens, and we have concluded the proof. □

This proof shows that both the invariant subspace problem and the hyperinvariant subspace problem has an affirmative solution on reflexive infinite dimensional complex Banach spaces.

3 Discussion of the obtained results

It is easy to see that the above proof also applies to nonreflexive Banach spaces. This fact, together with the counterexamples constructed, leads to the conclusion that, at least in the nonreflexive case, it is always important to specify **under which logical axioms we are working**. This is not the first time that something like this happens. Indeed, the first case not strictly related to the field of mathematical logic in which using different logical theories led to different results has been the Whitehead problem, as shown by Shelah in [10]. More precisely, in this case it has been shown that the axiom of constructibility $V = L$ implies that every Whitehead group is free, while using Martin's axiom together with the negation of the continuum hypothesis we can construct Whitehead groups which are not free, so that we have counterexamples to the problem. Since then, other problems have been shown to belong to this "class". It is clear that even the (H)ISP belongs to it. However, it is not clear to the author which logical axiom (or set of logical axioms) has been used in this proof. Thus, among the new arising questions, we have the following one (for the time being, we call this not-yet-determined axiom 'LM', from 'Lomonosov'):

Question 3.1. Which is the axiom LM, explicitely?

It is important to answer this question in order to understand what we need to assume to assure the existence of nontrivial (hyper)invariant subspaces in the general case.
Another fundamental question is:

Question 3.2. Is it possible to construct counterexamples (under some suitable logical axioms) even in the case where X is a complex infinite dimensional *reflexive* Banach space?

While we now know that for non reflexive infinite dimensional complex Banach spaces we need to specify which logical axioms are used, it is still possible that the case of reflexive Banach spaces, or at least Hilbert spaces, always leads to a positive solution. A negative answer to Question 3.2 would thus be good news, because at least in such cases we would be able to consider nontrivial (hyper)invariant subspaces without a dependence on the logical axioms used. Since the counterexamples by Enflo and Read work under ZFC, we can state:

Theorem 3.1. *Let X be an infinite dimensional complex Banach space. Then:*
(i) if we assume the axiom LM, the (hyper)invariant subspace problem holds true whichever is X;
(ii) if we assume ZFC, we can construct counterexamples in the non reflexive case (while it is not yet known if counterexamples can be constructed for the reflexive case, see Question 3.2).

4 Conclusion

In this paper we have settled the (hyper)invariant subspace problem for reflexive infinite dimensional complex Banach spaces by showing that every nonscalar operator in $B(X)$ belongs to $\Delta(X)$. Moreover, we have noticed that the proof also holds in the nonreflexive case. This result, together with the counterexamples by Enflo and Read, leads to new important open questions (see Section 3), which stimulate further research.

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