Nonextensive belief entropy

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Abstract

The belief entropy has high performance in handling uncertain information, which is the extension of information entropy in Dempster-shafer evidence theory. The Tsallis entropy is an extent of information entropy, which is a nonextensive entropy. However, how to applied the idea of belief entropy to improve the Tsallis entropy is still an open issue. This paper proposes the nonextensive belief entropy(NBE), which consists of belief entropy and Tsallis entropy. If the extensive constant of the proposed model equal to 1, then the NBE will degenerate into classical belief entropy. Furthermore, When the basic probability assignment degenerates into probability distribution, then the proposed entropy will be degenerated as classical Tsallis entropy. Meanwhile, if NBE focus on the probability distribution and the extensive constant equal to 1, then the NBE is equate the classical information entropy. Numerical examples are applied to prove the efficiency of the proposed entropy. The experimental results show that the proposed entropy can combine the belief entropy and Tsallis entropy effectively and successfully.

Keywords: Belief entropy, Tsallis entropy, Nonextensive belief entropy, Mass function

1. Introduction

There are a lot of uncertainties in the real world [1, 2]. In order to deal with the uncertain information, many mathematical models and theories have been proposed [3, 4], such as belief function [5], networks science [6], Choquet integral [7], quantum theory [8]. Qin et al [9] proposed a new total uncertainty measure to solve the issues of decision making in evidential environment. Liu et al. [10] used the physarum polycephalum assignment to equilibrate fuzzy user. Deng and Jiang [11] applied the maximum uncertainty allocation to improve Dempster–Shafer belief structure. Jaunzemis et al [12] used the judical evidential reasoning to gather evidence information of hypothesis resolution. Fei et

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al [13, 14] applied the soft likelihood functions to solve decision making problems. Khan and Anwar [15] applied the weighted evidence and Dempster–Shafer combination rule to improve time-domain data fusion and applied the proposed model to classify objects. How to measure the uncertainty degree of uncertain issues is a vital research spot [16]. To measure the uncertain information, many mathematical methods and theories have been presented [17]. Garg et al. [18, 19] used entropy theory to improve measurement methods under Pythagorean fuzzy environment. Vilasini and Colbeck [20] applied the Tsallis entropy to analyse causal structures. Dragan and Alexandru [21] proposed the pseudo-entropic model, which is a reliability model. Ren et al. [22] applied a large cognitive experiment of Mechanical Turk to estimate the entropy rate of english. Among these entropies, a belief entropy, named as Deng entropy, is proposed to measure the uncertainty of BPA [23]. The belief entropy is the most studied entropy at present [24], which is an extend of information entropy and can evaluate uncertainties more flexible than information entropy [25]. The belief entropy is based on the evidence theory, which means that the belief entropy can represent the uncertainties under the frame of discernment effectively [26]. When the basic probability assignment degenerate into the probability distribution, the belief entropy is degenerated as classical information entropy [27]. Relying on the advantages on representing uncertainty, the belief entropy has been widely studied by scholars at home and abroad [28]. Prajapati and Saha [29] applied the entropy theory to predict next word in the text with the aid of language model. Abellan [30] analyzed the properties of belief entropy in evidential environment. Zhu [31] proposed the maximum value dimension and power law of belief distribution of the maximum belief entropy. Kang and Deng proposed the maximum belief entropy [32], which is a meaningful model. The maximum belief entropy can obtain the maximum value of belief entropy of basic probability assignment. Then, Gao and Deng [33] proposed the Pseudo-Pascal Triangle form for the maximum belief entropy. Zheng and Tang [34] applied the deng entropy to obtain weighted risk priority number and used it into failure mode and effects analysis field.

Tsallis proposed the Tsallis entropy, which can measure the extensibility of system. Due to the high performance in representing uncertainties, the Tsallis entropy has been studied widely [35]. Gao et al. [36] proposed a new uncertainty measure with the aid of Tsallis entropy and applied the proposed model into evidential environment. Sholehkerdar [37] analyzed the theories of Tsallis entropy under the image fusion environment. Campos et al [38] extended Tsallis entropy from the research of probability space to the parameter space for pp and collisions. However, how to combine the belief entropy with the Tsallis entropy is still an open issue.

This paper proposes the NBE, which has the advantages of the belief entropy and Tsallis entropy. When the extension constant of the proposed model is 1, the non-extension belief entropy will degenerate into the classical belief entropy. In addition, when the mass fucntion is reduced to probability distribution, the proposed entropy becomes the classical Tsallis entropy. At the same time, if the NBE is concentrated in the probability distribution, and the extensive constant is 1, then the NBE is equal to the classical information entropy. In this paper, some meaningful theorems and proofs of the proposed entropy are given.

The remain of this paper is structured as follows. Section 2 introduces the preliminary. Section 3 presents the NBE. Section 4 illustrates the flexibility of NBE. Section 5 summarizes the whole paper.

2. Preliminaries

To handling the uncertainties everywhere, many methods have been presented [39, 40], which were applied in the applications of decision making [41, 42], pedestrian detection [43], statistical analysis [44], medical diagnosis [45], Emergency alternative evaluation citechen2020emergency. In this section, frame of discernment [46, 47], mass function [48, 49], maximum belief entropy [50, 51] are briefly introduced.

2.1. Frame of Discernment

Frame of discernment is an extent of classical probability space. When the frame of discernment focuses on the single subsets, the frame of discernment is degenerated as classical probability space [52]. Given a frame of discernment $\Omega = \{x_1, x_2, \dots, x_n\}$, the power set of frame of discernment is defined as follows:

Definition 2.1. *(Power Set of Frame of Discernment) [53]*

$$2^{\Omega} = \{\emptyset, \{x_1\}, \{x_2\}, \dots, \{x_n\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_i\}, \dots, \Omega\} \tag{1}$$

2.2. Mass Function

Given a frame of discernment $\Omega = \{x_1, x_2, \dots, x_n\}$, the mass function, m, on 2^{Ω} is defined as follows:

Definition 2.2. *(Mass Function) [53]*

$$m : 2^{\Omega} \to [0, 1] \tag{2}$$

Where, $m(\emptyset) = 0$ and $\Sigma_{B \in 2^{\Omega}} m(B) = 1$ with a focal element, B, of 2^{Ω}. The mass function is also called basic probability assignment, which was improved with the aid of complex number [54, 55].

2.3. Belief Entropy

Entropy theory has the good performance in measuring the uncertainty degree of a given system [56]. As we know, the belief entropy is the most effective entropy in evidential environment. Given a mass function m on a given frame of discernment $\Omega = \{x_1, x_2, \dots, x_n\}$. The definition of belief entropy under m is as follows:

Definition 2.3. *(Belief entropy) [23]*

$$E_d = -\sum_{B\in 2^{\Omega}} m(B)\log_2 \frac{m(B)}{2^{|B|}-1} \tag{3}$$

When the mass function is degenerated as a classical probability distribution, belief entropy will be degenerated into Shannon entropy.

Theorem 2.1. *When the $m(B) = \frac{2^{|B|}-1}{\sum_{B\in 2^{\Omega}} 2^{|B|}-1}$, then the belief entropy will obtain the maximum value [32].*

2.4. Tsallis Entropy

The definition of Tsallis entropy under a probability distribution p_i is as follows:

Definition 2.4. *(Tsallis entropy) [57]*

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \qquad q \in \mathbb{R} \tag{4}$$

When k is a conventional positive constant and $\sum_{i=1}^{W} p_i = 1$. It is obvious that $\lim_{q\to 1} S_q = -k\sum_{i=1}^{W} p_i \ln p_i$. In this way, the Tsallis entropy is degenerated as classical information entropy.

Example 2.1. *When the $N = 32$, $q = 2$ and the Tsallis entropy conforms to uniform distribution, the Tsallis entropy as follow:*

$$\begin{aligned} S &= \frac{1}{q-1}(1-\sum_{i} p(i)^q) \\ &= \frac{1}{q-1}(1-32*(\frac{1}{32})^q) = \frac{1}{q-1}(1-(\frac{1}{32})^{q-1}) \\ &= 1-32*(\frac{1}{32})^q \\ &= \frac{31}{32} \end{aligned}$$

3. The proposed model

3.1. Nonextensive belief entropy

Given a frame of discernment $\Theta = \{x_1, x_2, \ldots, x_n\}$, the definition of NBE as follows:

Definition 3.1. *(Nonextensive belief entropy)*

$$S_q(A) = \frac{1}{q-1}(1-\sum_{B\in 2^{\Theta}}(\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}})) \tag{5}$$

Theorem 3.1. *If the basic probability assignment degenerate into probability distribution, then the NBE will degenerate into classical Tsallis entropy.*

Proof 3.1. *Relying on the Eq.(4.3), we can obtain the following equation:*

$$S_q(A) = \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}}))$$

Since the basic probability assignment degenerates into the classical probability distribution, then we can get that $2^{|B|} - 1 = 1$.

Then, we can obtain the follow ing equation:

$$\begin{aligned} S_q(A) &= \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}})) \\ &= \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (m(B))^q) \\ &= \frac{1}{q-1}(1 - \sum_{i \in \Theta} (p(i))^q) \end{aligned}$$

Hence, the NBE will degenerate into classical Tsallis entropy.

Theorem 3.2. *If the extensive constant* $q = 1$*, then the NBE will degenerate into classical belief entropy.*

Proof 3.2. *Relying on the Eq.(4.3), we can obtain the following equation:*

$$S_q(A) = \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}}))$$

Since extensive constant $q = 1$*, then we can get that* $2^{|B|} - 1 = 1$.

Then, we can obtain the follow ing equation:

$$\begin{aligned} \lim_{q \to 1} S_q(A) &= \lim_{q \to 1} \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}})) \\ &= \lim_{q \to 1} \frac{(1 - \sum_{B \in 2^{\Theta}} \frac{m(B)}{2^{|B|}-1} \exp((q-1) \ln \frac{m(B)}{2^{|B|}-1})(2^{|B|}-1)}{q-1} \\ &= \lim_{q \to 1} \frac{(1 - \sum_{B \in 2^{\Theta}} m(B) \exp((q-1) \ln \frac{m(B)}{2^{|B|}-1})}{q-1} \\ &= \sum_{B \in 2^{\Theta}} -m(B) \ln \frac{m(B)}{2^{|B|}-1}) \end{aligned}$$

Hence, the NBE will degenerate into classical belief entropy.

Theorem 3.3. *When the probability distribution is uniformly distributed, then the NBE obtains the maximum value.*

Proof 3.3. *Relying on the Eq.(4.3), we can obtain the following equation:*

$$S_q(A) = \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}}))$$

Now, the Lagrange function can be defined as:

$$S_0 = \frac{1}{q-1}(1 - \sum_{B \in 2^{\Theta}} (\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}})) + \lambda(\sum_{B \in 2^{\Theta}} m(B) - 1)$$

$$\frac{\partial S_0}{\partial m(B)} = \frac{1}{q-1} \frac{(-q(m(B))^{q-1})}{(2^{|B|}-1)^{q-1}} + \lambda$$

Then

$$\frac{1}{q-1} \frac{(-q(m(B))^{q-1})}{(2^{|B|}-1)^{q-1}} = -\lambda$$

In this way, we can find that:

$$m(B_1) = m(B_2) = \ldots = m(B_n) = \ldots = m(B_{2^n}) = \frac{1}{2^n}$$

Then the NBE obtains the maximum value.

3.2. Discussion

Unknown information processing is an important issue in all fields, which has been a long-term concern by scholars [58, 59]. Entropy theory is an effective tool to handle uncertain information, which has been studied by a lot of scholars [60, 61]. The belief entropy has high performance in representing uncertainty. This paper proposes the NBE, which is combined with belief entropy and Tsallis entropy. When the extensive constant equal to 1, then the NBE degenerates into classical belief entropy. In addition, if the basic probability assignment degenerates into probability distribution, then the NBE will be degenerated as classical Tsallis entropy. In the case when the extensive constant equal to 1 and NBE focus on the probability distribution, then the NBE is the same as the classical information entropy. The relationship of NBE, belief entropy, Tsallis entropy and information entropy can be shown in Fig. 1.

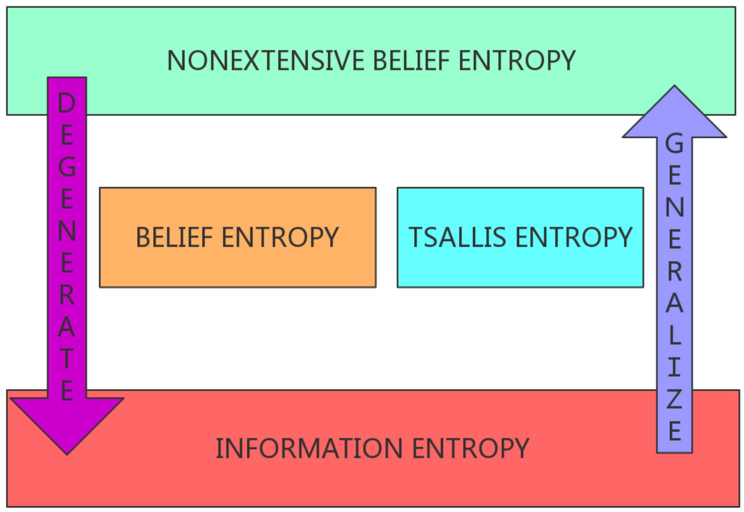

Figure 1: The relationship of four entropies

4. Numerical examples

In this section, plenty of numerical examples will be given to prove the effectiveness of NBE.

Example 4.1. *Assume a mass function $m(A) = 1$, the associated information entropy H, belief entropy E, Tsallis entrpy S and NBE NB as follows:*

$$H = -1 * log_2 1 = 0$$

$$E = -1 * log_2 \frac{1}{2^1 - 1} = 0$$

$$S = \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta}((m(B))^q)) = \frac{1}{q-1}(1-1) = 0$$

$$NB = \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta}(\frac{(m(B))^q}{(2^{|B|}-1)^{q-1}})) = \frac{1}{q-1}(1-1) = 0$$

Example 4.2. *Assume a mass function $m(A) = m(B) = m(C) = m(D) = 1/4$, the associated information entropy H, belief entropy E, Tsallis entropy S and NBE NB as follows:*

$$H = -\frac{1}{4} * log_2 \frac{1}{4} - \frac{1}{4} * log_2 \frac{1}{4} - \frac{1}{4} * log_2 \frac{1}{4} - \frac{1}{4} * log_2 \frac{1}{4} = 2$$

$$E = -\frac{1}{4} * log_2 \frac{1/4}{2^1 - 1} - \frac{1}{4} * log_2 \frac{1/4}{2^1 - 1} - \frac{1}{4} * log_2 \frac{1/4}{2^1 - 1} - \frac{1}{4} * log_2 \frac{1/4}{2^1 - 1} = 2$$

$$S = \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} ((m(B))^q)) = \frac{1}{q-1}(1 - 4 * (\frac{1}{4})^q))$$

$$NB = \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} (\frac{(m(B))^q}{(2^{|B|} - 1)^{q-1}})) = \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} ((m(B))^q)) = S$$

In this case, the NBE degenerates into the Tsallis entropy. Assume the extensive constant $q = 1$, the S and NB can be shown as follows:

$$\lim_{q=1} S = \lim_{q=1} \frac{1}{q-1}(1 - 4 * (\frac{1}{4})^q)) = \log_2 4 = 2$$

$$\lim_{q=1} NB = \lim_{q=1} \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} ((m(B))^q)) = \log_2 4 = 2$$

In this case, the NBE degenerates into the belief entropy and classical information entropy.

Example 4.3. *Assume a mass function $m(A, B, C, D) = 1$, the associated belief entropy E and NBE NB with $q = 1$ as follows:*

$$E = -1 * log_2 \frac{1}{2^4 - 1} = log_2 15$$

$$\begin{aligned} NB &= \lim_{q \to 1} \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} (\frac{(m(B))^q}{(2^{|B|} - 1)^{q-1}})) \\ &= \lim_{q \to 1} \frac{1}{q-1}(1 - \sum_{B \in 2^\Theta} (\frac{1}{(15)^{q-1}})) \\ &= log_2 15 \end{aligned}$$

In this case, the NBE degenerates into the belief entropy.

Example 4.4. *Assume a mass function under a given frame of discernment with 18 elements, $m(2, 3, 5) = 0.1$, $m(7) = 0.1$, $m(B) = 0.6$, $m(\Theta) = 0.2$. Table 1 lists various NBEs under $q = 2$ with B changing, which is graphically shown in Fig. 2.*

Table 1: Nonextensive belief entropy with B

Cases	Nonextensive belief entropy
$B = \{1\}$	0.6286
$B = \{1, 2\}$	0.8686
$B = \{1, 2, 3\}$	0.9371
$B = \{1, \ldots, 4\}$	0.9646
$B = \{1, \ldots, 5\}$	0.9770
$B = \{1, \ldots, 6\}$	0.9829
$B = \{1, \ldots, 7\}$	0.9857
$B = \{1, \ldots, 8\}$	0.9872
$B = \{1, \ldots, 9\}$	0.9879
$B = \{1, \ldots, 10\}$	0.9882
$B = \{1, \ldots, 11\}$	0.9884
$B = \{1, \ldots, 12\}$	0.9885
$B = \{1, \ldots, 13\}$	0.9885
$B = \{1, \ldots, 14\}$	0.9885
$B = \{1, \ldots, 15\}$	0.9886
$B = \{1, \ldots, 16\}$	0.9886
$B = \{1, \ldots, 17\}$	0.9886

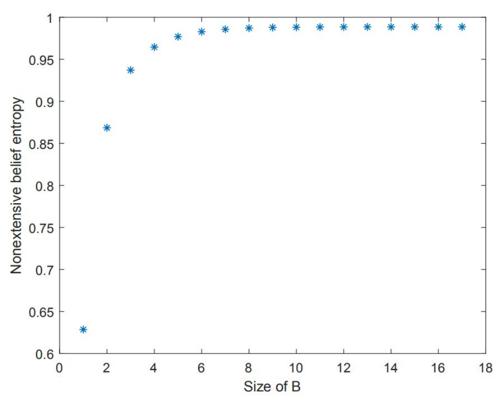

Figure 2: The nonextensive belief entropies as a function of the size of B

The results shows that the NBE of m increases monotonically as the size of subset B increases. When the proportion of unknown information in a mass function increases, entropy will increase, which is consistent with human cognition. Fig. 3 shows that the uncertainty degree of basic probability assignment with four uncertainty measurement models.

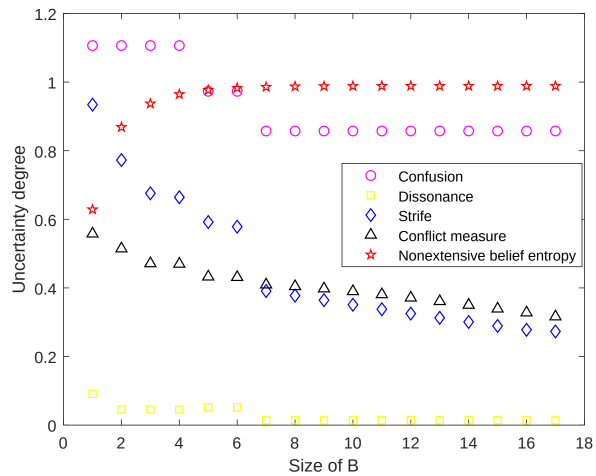

Figure 3: The comparison between nonextensive belief entropy and four methods

We clearly see that NBE increases monotonically as the size of B increases. It shows that the proposed entropy is more stable than other models.

5. Conclusion

The belief entropy and Tsallis entropy are the research hotspots at present, which have been studied by many scholars. To applying the idea of belief entropy to the Tsallis entropy, this paper proposed the NBE, which has the properties of Tsallis entropy and belief entropy. When the extensive constant of the proposed model is 1, then the NBE will degenerate into the classical belief entropy. In addition, when the basic probability assignment is degraded to probability distribution, the proposed entropy becomes the classical Tsallis entropy. At the same time, if NBE is concentrated in the probability distribution and the extensive constant is 1, then the NBE is equal to the classical information entropy. Some theorems and proofs of the proposed entropy has been proposed in this paper. Numerical examples are applied to prove the efficiency of the proposed entropy by comparing the proposed model and other models. The experimental results show that the proposed entropy can combine the belief entropy and Tsallis entropy effectively and that the proposed entropy is more stable than other models.

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