On the inconsistency between Fermat point and Fermat Least Time Principle

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Abstract

Fermat posed a challenge problem thus: Given three points find a fourth in such a way that the sum of its distances from the three given points is a minimum. The solution point is called Fermat Point (FP). The problem involved *three given points* and the minimization of sum of *three distances*. The solution contained some interesting special cases which involved the *three given points* but only *two distances* whose sum was a minimum. We found the special cases provide a simple method for exposing the inconsistency between FP and Fermat's least time principle (FLTP). The perfect setting for our finding was provided by the natural phenomena of reflection and refraction of light. In the application of FLTP to these processes also, we have the same conditions of three given points and two distances. The three points are: the end points of the broken line path and the point of incidence. The two distances are: the lengths of the two broken line segments - travelled before and after reflection or refraction. We show in this article that FP and FLTP lead to contradictory results about the point connecting the given points that provides the minimal sum of the distances. In optimization parlance this means that FP and FLTP give different points to locate a service facility catering to three given towns. Our result leads to the conclusion that FP and FLTP are mutually inconsistent. Simply put, we pitch FP against FLTP and show the inconsistency between the two.

Key Words/Phrases

Fermat Point, Torricelli point, Fermat's least time principle, minimal sum of distances, least time path, reflection, refraction, Snell's laws, Descartes.

Introduction

Fermat Point (FP)[1-6] and Fermat's Least Time Principle (FLTP)[7-16] are very well known. Each has a rich history. Even today they continue to be of great interest both in theory and application as seen from the publications in literature. Similarly, the phenomenon of reflection and refraction of light, wherein the two (FP, FLTP) play a role, has a rich history and everlasting interest in various branches of research. They have been analyzed and discussed by many scholars from philosophical, cultural, science points of view[3,8,9,13-16]. These principles are also topics of discussion in fundamental research works [3,8,9,13].

Fermat proposed his problem of minima, thus: Given three points find a fourth in such a way that the sum of its distances from the three given points is a minimum[1-6]. The solution point is called the Fermat Point (FP). The problem involved *three given points* and the minimization of sum of *three distances*. The solution when found, contained some interesting special cases. The special cases involved *the three given points* but only *two distances* whose sum was a minimum. We found the special cases provide a simple method of exposing the inconsistency between FP and another famous principle of Fermat, namely the Fermat's least time principle (FLTP)[7-16]. FLTP states that light takes the path that minimizes the time of travel between any two given points even when it suffers either reflection or refraction on the way, instead of going along the straight line connecting the two end points of the path. The ideal setting of three points and two distances is provided for us by the natural phenomena of reflection and refraction of light. The three points are: the two end points of the path of the light ray and the third point is the point where the ray bends on the way – the point of incidence. The two distances are: the lengths of the two broken line segments of the path - travelled before and after reflection or refraction as the case might be.

Using principles of geometry, we demonstrate in this article that FP and FLTP yield inconsistent results for the minimum distance or minimum time path when applied to the path of a light ray that undergoes reflection or refraction. This result leads us to conclude that FP and FLTP are mutually contradictory and that if one is valid the other is invalid or else both are invalid.

Statements of FP, FLTP, and Snell's Laws of reflection and refraction

Statement of FP

Fermat point (also referred to as Torricelli point sometimes) of a triangle ABC is defined as the point P such that the sum of the distances from P to the three vertices of the triangle, that is, (PA + PB + PC) is a minimum[1].

The original form in which Fermat posed his challenge regarding such a point reads as[14]:

Given three points, find a fourth in such a way that the sum of its distances from the three given points is a minimum.

Statement of FLTP

When light travels from a point A to another point B both A and B lying in the same medium or in different media, the time of travel from A to B (which is a sum of two time intervals, one before and the other after reflection or refraction) is a minimum. Stated differently, out of all different paths that it might take to get from one point to another, light takes the path which requires the shortest time[12]. In the case of reflection of light, this principle becomes Fermat's least distance principle since the speed of travel remains constant throughout the path of travel.

Statement of Snell's law of reflection and refraction

Law of reflection: When a ray of light passes from one point in a medium to another point in the same medium the incident ray, the reflected ray, the normal to the surface of reflection at the point of incidence all lie in the same plane. The angle of incidence is equal to the angle of reflection. That is, reflection occurs at equal angles to the surface (or to the normal).

Law of refraction: When a ray of light passes from a point in one medium to a point in another medium, the incident ray, the refracted ray, the normal to the surface of separation at the point of incidence all lie in the same plane. The sine of the angle of incidence bears a constant ratio to sine of the angle of refraction. The value of the ratio depends only on the two media

Application of FLTP for path of reflection

We find from the above statements of FP, FLTP, that they both deal with minimization of sum of either distances or the associated times of travel of those distances. For example, FLTP deals with minimization of the sum of two distances and/or of times of their travel by light, along a broken line path connecting two points, suffering either reflection or refraction on the way. While in the case of refraction FLTP minimizes the sum of times of travel (but not of distances), in the case of reflection FLTP minimizes the sum of distances (of times consequent upon it) of travel.

Analysis of reflection

Let two points A, B, and a line m (intersection of a plane reflecting surface and a plane perpendicular to it) be given (see Fig. 1). A, B do not lie on m but lie on the same side of m. We are required to find the point D on m at which a ray of light from A is reflected at D to pass through B. Let us assume the reflecting surface to be in the horizontal and the intersecting plane to be in the vertical direction. Then rays AD, DB and the normal to the reflecting surface at D are all in the same vertical plane. In the location of point D on m we are governed by Snell's laws of equal angles to the surface or the normal.

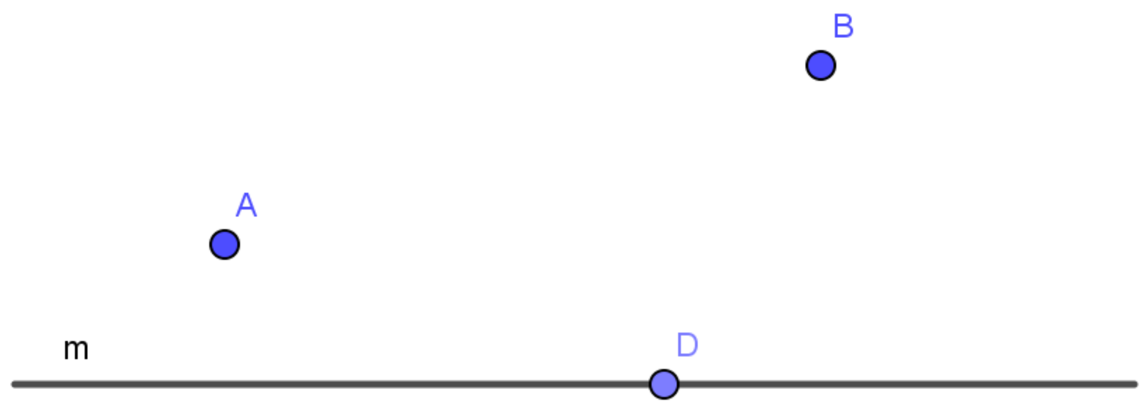

Fig.1 . Points A, B and the line m not passing through them, all in one plane are given. A, B lie on the same side of m, Locate point D on m such that a light ray from A gets reflected at D and passes through B.

Location of D, the point of incidence

A simple and well-known method of locating D on m is this: Reflect B in m to get B'. Join AB'. The intersection of AB' and m gives the location of D. It is the point that minimizes the sum of the two distances, (AD + DB). Since (AD + DB) = (AD + DB') = AB' is a minimum. We also note that if we draw the perpendicular DD' to m at D then angle ADD' = angle D'DB (see Fig. 2). In view of this equality of angles, a ray of light from A incident at D on m will get reflected to pass through B. Therefore, AD and DB form a reflection ray couple.

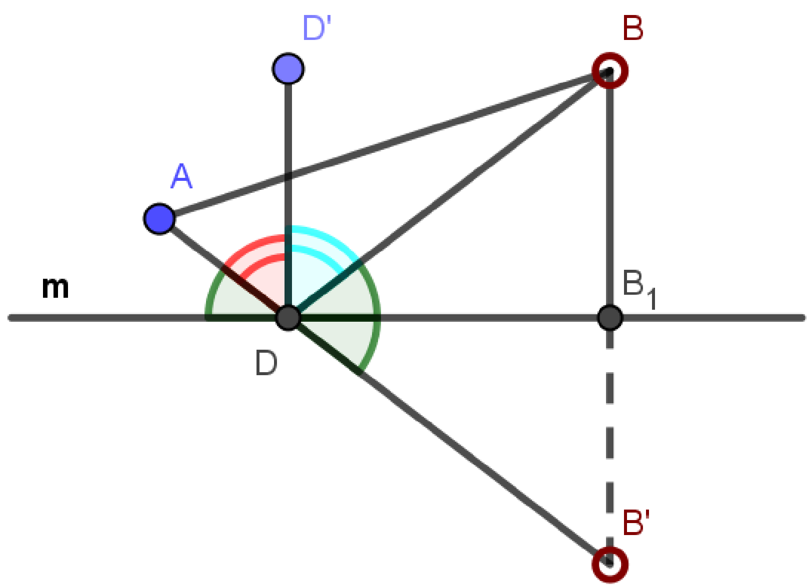

Fig.2. B' is the reflection of B in m. D is the intersection of AB' and m. DD' is the normal to m at D. $\angle ADm = \angle B'DB_1 = \angle BDB_1$, $\angle ADD' = \angle D'DB$.

According to FLTP as applied to this reflection, the sum of the two distances AD and DB i.e. (AD + DB) is a minimum (since the speed of travel throughout the path ADB is a constant).

Having obtained the reflection ray couple AD, DB, we proceed to locate the FP of the triangle ADB formed from the three points A, D, B.

Location of Fermat point (FP)

We draw the equilateral triangle ABG on AB so that G and D are on the opposite sides of AB (see Fig.3). Draw the circumcircle of triangle ABG. Join GD. Let it intersect the circumcircle at F.

F, then, is the Fermat point of triangle ABG. According to the definition of Fermat point of a triangle, it follows that the sum of the distances of F from the vertices of triangle ADB, that is (FA + FB + FD) is a minimum.

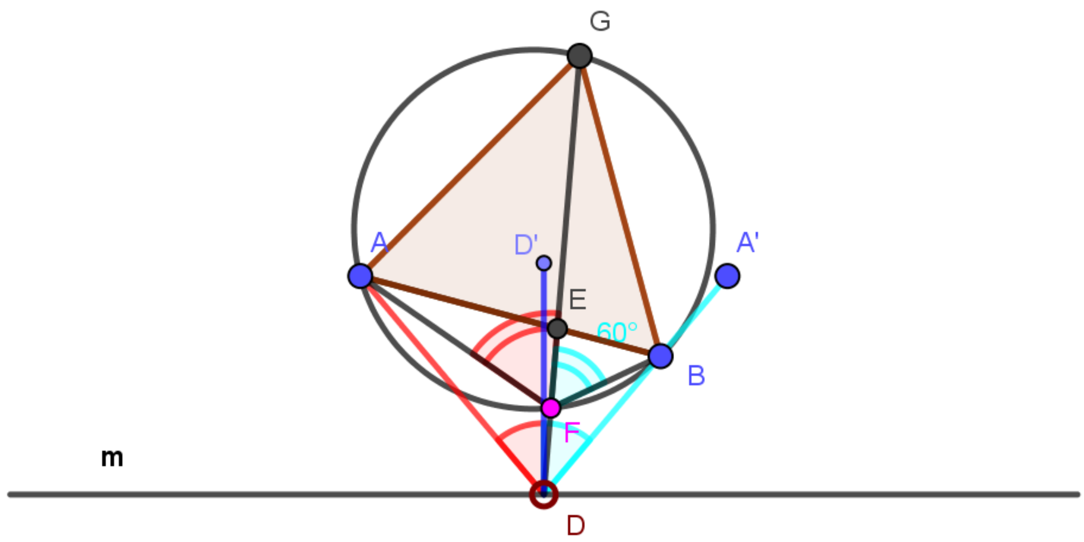

Fig. 3. D is the point on m at which a light ray from A is reflected at D to pass through B. An equilateral triangle ABG and its circumcircle AFBG are constructed. D, G are joined. DG intersects the circumcircle at F. F is the Fermat point of the triangle ABD. ∴ (AF + FB + FD) is a minimum. But (AD + DB) is minimum according to FLTP).

But, if a triangle contains an angle equal to or greater than 120° the vertex containing that angle itself becomes the Fermat point[1]. In such cases the sum of two distances becomes a minimum. Conversely, if we want the sum of two distances to be a minimum then the two distances must form sides of a triangle enclosing between them an angle equal to or greater than 120°.

The inconsistency between FP and FLTP

If (AD + DB) is to be a minimum, then the definition of FP demands that D be the FP of the triangle ABD. That is, FP demands that angle ADB ≥ 120°. But angle ADB < 120° since D lies outside the circumcircle. Consequently, D is not the FP of the triangle ABD. Therefore, FP asserts that (AD + DB) is not a minimum. But FLTP asserts that (AD + DB) is a minimum.

Thus, it is clear the definitions of FP and FLTP yield contradictory results about the point of minimal distance. Therefore, FP and FLTP are mutually inconsistent.

An easy way to appreciate this point is to look at it from the optimization-of-distances point of view. A common service facility that caters to *three* user entities such as schools, petrol bunks, bank branches etc., is to be located solely based on the criterion of minimal sum of the distances of the facility from the three users. FP recommends locating it at F but FLTP recommends locating it at D. Where exactly are we to locate the facility, then? The contradiction between FP and FLTP is apparent.

Thus, there arises an inconsistency between the results of FP and the results of FLTP when applied to the reflection path of light rays. If one is valid the other is invalid. Or else, both are invalid. Since Snell's law is connected only with the relative sizes of the angles ADD' and D'DB, it is valid for any point located on

AD and the other point located anywhere on DB. Minimization of the sum of distances has nothing to do with the equality of the angles of incidence and reflection.

Let us consider FLTP for refraction now.

Let two points A, B, and a line m (intersection of a plane surface of separation of two media and a plane perpendicular to it) be given (see Fig. 4). A lies in medium 1 and B lies in medium 2. We are required to find the point D on m at which a ray of light from A is refracted so as to pass through B. Let us assume the refracting surface to be in the horizontal and the intersecting plane to be in the vertical direction. Then rays AD, DB and the normal to the reflecting surface are all in the same vertical plane. In the location of point D on m we are governed by Snell's law of sines of refraction.

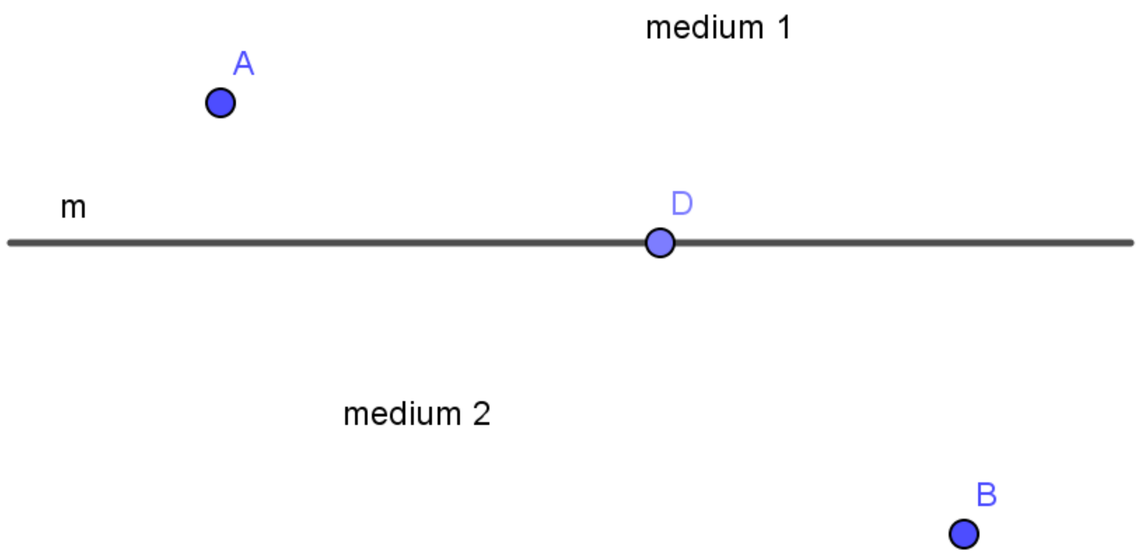

Fig.4 Points A,B and the line m, all lie in one plane, are given. Locate point D . on m such that a light ray from A gets refracted at D so as to pass through B.

Unlike in the case of reflection, there is no simple method of locating the point D on m. It was associated with a great deal of controversy in the 17th century. Lamborn[13] gives a detailed historical development of different aspects connected with it. It was mainly between Descartes, who is said to have independently discovered the law of refraction, and Fermat. Descartes tried to give a mechanical model/mechanism for the process of refraction of light. Initially Descartes used a model of motion of tennis ball going from the racket to the net and hitting the net its motion slows down and the direction changes as well. This model had given rise to results that were not appealing to commonsense. Later he gave a bent arm balance model[17] which satisfactorily accounted for the experimental observations and the sine law.

Fermat on the other hand based his theory on his method of maxima and minima. He tried to locate the point N on AB such that the time of travel along the broken line path MNH (see Fig. 5) is minimum. The principle underlying this method came to be known as Fermat's least time principle.

We show in Fig. 5 the diagram of refraction given by Fermat in his Method[7]. Since giving here the proof of the method in identifying the correct path of refraction takes us too far astray, we refer the reader to the original paper of Fermat. We give only the details needed for our purpose here.

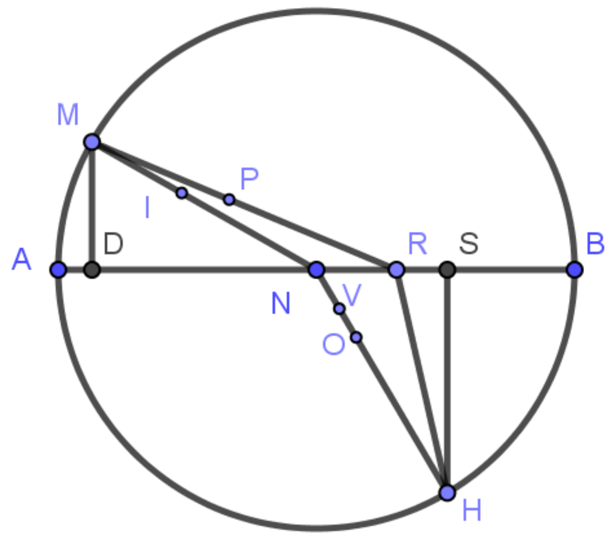

Fig. 5. Fermat's diagram to determine the location of the point of incidence N on the surface of separation AB of the two media - the upper and lower semicircles. MNH and MRH are two possible paths of refraction of light. Our object is to detrmine the point on AB that gives least travel time from M to H. D and S are the feet of perpendiculars from M and H on to AB. IN : NH = $t_1 : t_2$. PR : RH = $t'_1 : t'_2$. MN : DN = RN : NO, DN : NS = NO : NV.

M and H are two given points and AB a given surface of refraction. M is in one medium and H is in a different medium. AB is the surface of separation of the two media. A light ray from the point M travels to the point H refracting (bending) at a point on AB on the way. MN and NH and, MR and RH are two of the many candidate refraction ray couples connecting M and H. Since there can be only one refraction ray couple connecting two given points one needs to locate the correct point of incidence on AB. To this end, Fermat uses his least time principle in locating the correct point of incidence. He arrives at point N, the center of the circle with AB as diameter, to be the correct point of incidence.

Fermat identifies MNH to be the path of least time from M to H. IN : NH = $t_1 : t_2$ where t_1 and t_2 are the travel times in the medium of M and of H respectively. The sum of the travel times ($t_1 + t_2$) represented by the sum of the line segments IN and NH in Fig. 5 is a minimum.

For our demonstration, we deliberately choose a path of refraction such that the incident ray and refracted ray enclose an angle less than 120°. That is, angle MNH < 120° (see Fig. 6).

We will now find the FP of the triangle INH (see Fig 6) to find the point that minimizes the sum of its distances to the vertices of the triangle.

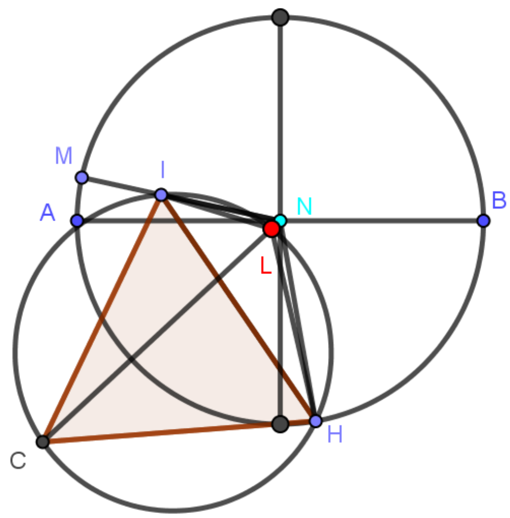

Fig.6. Equilateral triangle IHC is constructed on side IH so that the third vertex C and N lie on opposite sides of IH. Its circumcircle is constructed. Join C, N. CN intersects the circle at L. Therefore, L is FP of triangle INH. Therefore, FP says (IL + LN + LH) is a minimum and FLTP says (IN + NH) is a minimum.

We join IH to form the triangle INH. To find the Fermat point of this triangle, we construct an equilateral triangle IHC with side IH so that the third vertex C, and N lie on opposite sides of IH. We construct the circumcircle of triangle IHC. Join C, N. CN intersects the circumcircle at L. Therefore, L is the Fermat point of the triangle IHN. Therefore, it follows that the sum of the distances IL, LN, LH that is, (IL + LN + LH) is a minimum.

Now, ILHC is a cyclic quadrilateral. Angle ICH = 60°. Therefore, angle ILH = 120°. Therefore, L is the FP of triangle ILH containing the angle 120°. As seen earlier, we get from FP of this triangle, (IL + LH) is a minimum. But, IL < IN and LH < NH. Therefore, (IL + LH) < (IN + NH). While FP demands (IL + LH) is a minimum, FLTP demands (IN + NH) is a minimum.

Thus, the results given by FP and FLTP are contradictory in the case of refraction also. This demonstrates the inconsistency between FP and FLTP as applied to refraction of light.

Note: Strictly speaking, Fermat's method of maxima and minima cannot be applied to the minimization of sum of two or more time intervals – it can only be applied to the minimization of sum of two or more distances.

Thus, in both reflection and refraction of light FLTP and FP lead to mutually inconsistent results.

Conclusion

The inconsistency between the two well recognized fundamental principles of FP and FLTP calls for a fresh appraisal of Fermat's 'Method of Maxima and Minima' from which the two arise.

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Dedication

I dedicate this work with reverence to my parents Smt. Nagaratnamma Padyala and
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