DERIVING FORMULA FOR VOLUME OF SPHERES IN "HIGHER DIEMENSIONS"

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ABSTRACT

In this short paper, I will expresss and prove the volume of n-dimensional Spherical balls. The evaluation corely is expressed via multivariate calculus, factorials and special function.

AIMS :

1. Proof of formular for volume of sphere in N dimensions
2. Amazing Results relating to this formulae in real analysis

METHODOLOGY

The volume of sphere in N dimension is given as[1] :

$$V_N(R) = \frac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!} R^N$$

Now for N = 2, we have

$$V_2(R) = \pi R^2$$

For N = 3

$$V_3(R) = \frac{4\pi}{3} R^3$$

Now we would prove this formulae but to proof this formulae we will prove that

$$V_R(N) = V_N(R) = V_N(1) \times R^N$$

Now this means volume in N dimensions equal to the volume in a unit sphere times R^N

Introducing change in variable $\frac{x_i}{R} = y_i$

By the law of multivariate calculus

$$V_N(R) = \oint d\, x_1 dx_2 dx_3 \dots \dots \dots . dx_n$$

Then

$$\oint d\, x_1 dx_2 dx_3 \dots \dots \dots . dx_n = \oint J(x_1) dy_1 dy_2 \dots \dots \dots . dy_n$$

Where J(x) is the **JACOBIAN MATRIX** of variable x

Then the determinant will be :

$$detJ(x_i) = trace\left(\frac{dxi}{dy_i}\right)_{ij}$$

$\int dy_1 dy_2 \dots \dots \dots . dy_n = \frac{1}{R^N} \oint d\, x_1 dx_2 dx_3 \dots \dots \dots . dx_n$

Since

$$V_N(R) = \int \prod_{i=0}^{n} dx_i$$

Hence we proved that

$$\frac{V_N(R)}{R^N} = V_N(1)$$

So finally we proved the claim that $V_N(R) = R^N\, V_N(1)$

Recall by the basic rule of small increment

$$V(x) = \int A(x)dx$$

For a unit circle $R = \sqrt{(}1 - x^2)$ then we have

$$V_N(R) = \int_{-1}^{1} A(x)dx$$

But since $A(x) = V_{N-1}(R)$ we are going to have

$$V_N(R) = \int_{-1}^{1} V_{N-1}(R)dx = \int_{-1}^{1} R^{N-1} V_{N-1}(1)dx$$

So we have

$$\int_{-1}^{1} (\sqrt{(}1 - x^2))^{N-1} V_{N-1}(1)dx$$

By changing variable we have

$$\int_{-1}^{1} (\sqrt{(}1 - x^2))^{N-1} V_{N-1}(1)dx = 2\int_{0}^{1} (\sqrt{(}1 - x^2))^{N-1} V_{N-1}(1)dx$$

Let $x^2 = t$, $2xdx = dt$

Thus by symmetry we have [2]

$$\int_{-1}^{1} (\sqrt{(1-t)})^{N-1} t^{\frac{-1}{2}} V_{N-1}(1)dt = 2\int_{0}^{1} (\sqrt{(1-t)})^{N-1} t^{\frac{-1}{2}} V_{N-1}(1)dt$$

In which the left hand side is a **SPECIAL FUNCTION(BETA)**

By the relationship between beta functions and factorials[3]

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

From the integrand $m = \frac{-1}{2}\, and\ n = \frac{N-1}{2}$

Thus

$$B\left(\frac{-1}{2},\frac{N-1}{2}\right) = \frac{(\frac{-1}{2}-1)!\,(\frac{N-1}{2}-1)!}{\left(\frac{-1}{2}+\frac{N-1}{2}-1\right)!}$$

Since gamma(0.5) = $\sqrt{\pi}$ = -0.5![1]

Thus the expression gives

$$B\left(\frac{-1}{2},\frac{N-1}{2}\right) = \frac{\sqrt{\pi}\,(\frac{N-1}{2}-1)!}{\left(\frac{N}{2}\right)!}$$

Now putting back in the definition we have

$$V_N(1) = \frac{\sqrt{\pi}\,(\frac{N-1}{2}-1)!}{\left(\frac{N}{2}\right)!} V_{N-1}(1)$$

Since $V_N(R) = V_{n-1}(R) \times V_{n-2}(R) \ldots\ldots\ldots . V_1(R)$

Where $V_N(R)\ is\ the\ volume\ in\ space$

And $V_{n-1}(R) \times V_{n-2}(R) \ldots\ldots\ldots . V_1(R)\ is\ the\ product\ of\ smaller\ volumes$

Like telescoping terms the factorials cancels out over several terms

Hence we have

$$\frac{V_N(R)}{R^N} = \frac{0!\,\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!} = \frac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$$

FINALLY proved

$$V_N(R) = \frac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!} R^N$$

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REFERENCES

[1] Arfken, George B. Mathematical Methods for Physicists. Orlando: Academic Press, 1985. Print

[2] Parks, Harold. "The Volume of the Unit n-Ball." Mathematical Association of America. 86.4 (2013): 270-274. Web. 23 Feb. 2014.

[3] http://en.wikipedia.org/wiki/Gamma_function