A NEW ASYMPTOTIC FORMULA FOR THE PARTIAL SUM OF THE LOGARITHMIC FUNCTION

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Abstract. In this short paper we estimate the size of the partial sum of the logarithmic function $\sum\limits_{n\leq x}\log n$ using a different method. We also use it to give an upper and lower bound for the factorial of any positive integer.

Introduction

Stirling's formula gives $n! \sim \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$ as $n \longrightarrow \infty$ [?]. Expressed as an inequality

$$\sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot e^{-n} \leq n! \leq \sqrt{2\pi}\cdot n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n}}.$$

The Ramanujan estimate [1] also gives

$$\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1}{30}\right)^{\frac{1}{6}}\left(1-\frac{11}{11520(n+a)^4}\right) < n!$$

$$\leq \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1}{30}\right)^{\frac{1}{6}}\left(1-\frac{11}{11520(n+b)^4}\right),$$

where $a = \dfrac{39}{54}$ and

$$b = \left(\frac{11}{11520\left(1-\left(\frac{30e^6}{391\pi^3}\right)^{\frac{1}{6}}\right)}\right)^{\frac{1}{4}} - 1 = 0.35499112666\ldots.$$

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1. Representation formula

Theorem 1.1. *If a is an odd positive integer, then*

$$\frac{a-1}{2} = \sum_{\substack{1\leq j\leq a\\(2,j)=1}}\left\lfloor\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\rfloor.$$

Proof. Consider an odd positive integer a. There are as many even integers as odd integers less than a. That is, for $1\leq m<a$, there are $\frac{a-1}{2}$ such possibilities. On the other hand, notice that the set of all positive even integers is the union of the sequences $\{m2^{\alpha}\}$, where m is distinct for each sequence and runs through the

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odd integers and α runs through the positive integers ($\alpha = 1, 2, \cdots$). Consider the sequence

$$2, 2^2, \ldots, 2^m.$$

Clearly there are $\lfloor \frac{\log a}{\log 2} \rfloor$ such terms in this sequence. Again consider those of the form

$$3 \cdot 2, 3 \cdot 2^2 \ldots, 3 \cdot 2^m.$$

Clearly there are $\lfloor \frac{\log(a/3)}{\log 2} \rfloor$ such number of terms in this sequence. We terminate the process, by considering the sequence

$$2 \cdot j, j \cdot 2^2, \ldots, j \cdot 2^m,$$

where $(2, j) = 1$ and there are $\left\lfloor \dfrac{\log(\frac{a}{j})}{\log 2} \right\rfloor$ such number of terms in this sequence. In comparison with the previous count, we obtain $\frac{a-1}{2} = \sum_{\substack{1 \leq j \leq a \\ (2,j)=1}} \left\lfloor \frac{\log(\frac{a}{j})}{\log 2} \right\rfloor$, which is what we set out to prove. □

Remark 1.2. Having establish this formula, we are ready to give a different estimate for the partial sum of the logarithmic function. Here is how it goes.

Corollary 1.1. For all positive integers a

$$\sum_{j \leq a} \log j = a \log a - a \log 2 + \log 2 - \log 2 \sum_{j \leq a} \left\{ \frac{\log\left(\frac{a}{j}\right)}{\log 2} \right\} + \mathcal{O}(1).$$

Remark 1.3. In the forthcoming proof we consider a to be an odd integer. And it is very easy to see that it will remain valid for all positive integers once the proof goes through for all odd integers.

Proof. Invoking theorem (**??**), we have that

$$\begin{aligned} \frac{a-1}{2} &= \sum_{\substack{j \leq a \\ (2,j)=1}} \frac{\log\left(\frac{a}{j}\right)}{\log 2} - \sum_{\substack{j \leq a \\ (2,j)=1}} \left\{ \frac{\log\left(\frac{a}{j}\right)}{\log 2} \right\} \\ &= \frac{1}{\log 2} \sum_{\substack{j \leq a \\ (2,j)=1}} \log\left(\frac{a}{j}\right) - \sum_{\substack{j \leq a \\ (2,j)=1}} \left\{ \frac{\log\left(\frac{a}{j}\right)}{\log 2} \right\}. \qquad (1.1) \end{aligned}$$

Also we observe that

$$\frac{a-1}{2} = \frac{1}{\log 2} \sum_{\substack{j \leq a \\ (2,j)\neq 1}} \log\left(\frac{a}{j}\right) - \sum_{\substack{j \leq a \\ (2,j)\neq 1}} \left\{ \frac{\log\left(\frac{a}{j}\right)}{\log 2} \right\} + \mathcal{O}(1). \qquad (1.2)$$

Combining equations (1) and (1.2), we have

$$a-1=\frac{1}{\log 2}\sum_{j\leq a}\log\left(\frac{a}{j}\right)-\sum_{j\leq a}\left\{\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\}+\mathcal{O}(1)$$

$$=\frac{1}{\log 2}\left(\sum_{j\leq a}\log a-\sum_{j\leq a}\log j\right)-\sum_{j\leq a}\left\{\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\}+\mathcal{O}(1). \tag{1.3}$$

It follows from (1.3) that

$$\log 2^{a-1}+\log 2\sum_{j\leq a}\left\{\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\}=\sum_{j\leq a}\log a-\sum_{j\leq a}\log j+\mathcal{O}(1)$$

$$=a\log a-\sum_{j\leq a}\log j+\mathcal{O}(1). \tag{1.4}$$

Arranging terms, we have

$$\sum_{j\leq a}\log j=a\log a-a\log 2+\log 2-\log 2\sum_{j\leq a}\left\{\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\}+\mathcal{O}(1),$$

where $\left\{\frac{\log\left(\frac{a}{j}\right)}{\log 2}\right\}=G(n)$ denotes the fractional part of $\frac{\log\left(\frac{a}{j}\right)}{\log 2}$. □

Remark 1.4. Stirling's formula gives an upper and lower bound for the factorial of any positive integer. We present here a slightly different version.

Theorem 1.5. *For all $n\geq 1$ and for some constant $C(n)>1$*

$$\frac{n^n}{2^{n-1+G(n)}}\leq n!\leq C(n)\frac{n^n}{2^{n-1+G(n)}},$$

where

$$G(n):=\sum_{m\leq n}\left\{\frac{\log\left(\frac{n}{m}\right)}{\log 2}\right\}.$$

Proof. Recall from Corollary 1.1

$$\sum_{m\leq n}\log m\geq n\log n-n\log 2+\log 2-\log 2\sum_{m\leq n}\left\{\frac{\log\left(\frac{n}{m}\right)}{\log 2}\right\}.$$

Recall $\sum_{m\leq n}\log m = \log n!$. Using this identity, it follows that

$$\log n! \geq n\log n - n\log 2 + \log 2 - \log 2G(n),$$

from which we see that

$$n! \geq \frac{n^n}{2^{n-1+G(n)}}. \tag{1.5}$$

Also it is clear from Corollary 1.1

$$n! = \frac{n^n}{2^{n-1+G(n)}}e^{\mathcal{O}_n(1)} \leq C(n)\frac{n^n}{2^{n-1+G(n)}}. \tag{1.6}$$

From (1.5) and (1.6), we obtain the inequality

$$\frac{n^n}{2^{n-1+G(n)}} \leq n! \leq C(n)\frac{n^n}{2^{n-1+G(n)}},$$

where $G(n) := \sum_{m\leq n}\left\{\frac{\log\left(\frac{n}{m}\right)}{\log 2}\right\}$. $\square$

References

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2. Robbins, Herbert, *A remark on Stirling's formula*, Amer. Math. Mon., vol. 62.1, 1955, pp.26–29

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