A New Theorem in Biquaternion Field and Its Applications in Quantum Mechanics.

Elio Conte

School of Advanced International Studies on Applied Theoretical and non Linear Methodologies of Physics, Bari, Italy.

Abstract: *A new theorem is demonstrated. There is a class of biquaternions for which the power of the biquaternions at the order n is so that* $R^{n-1} \neq 0$ *and* $R^{n} = 0$*.This theorem is of interest not only in the mathematical tool of the biquaternion field but it is of particular interest also in the applications of biquaternion algebra in quantum mechanics.*

Key words: a new theorem in biquaternion algebra, Hamilton algebra, Clifford algebra, biquaternions, quantum mechanics.

Corresponding author: elio.conte@fastwebnet.it

Introduction

Clifford's preliminary pattern of biquaternions contains an outline of a calculus devised for the analytical treatment of his theory. Clifford's field was to extend Hamilton's quaternion calculus where Hamilton's biquaternion was a quantity as quaternion with complex coefficients [1]. Presently, Clifford and Hamilton algebra have become a very fertile ground for physics. Clifford’s algebra has become a useful language for physics because it maximally exploits geometric properties and symmetries. It is known to physicists mainly as the algebras of Pauli spin matrices and of Dirac gamma matrices, but its utility goes far beyond the applications to quantum theory and spin for which these matrix forms were introduced.

We retain that the most interesting property of the biquaternions is their non commutativity that of course finds a direct correspondence in noncommutativity of linear operators in quantum mechanics. As P.M. Dirac said:

I saw that non commutation was really the dominant characteristics of Heisenberg's new theory. It was really more important than Heisenberg's idea of building up the theory in terms of quantities closely connected with experimental results. So I was led to concentrate on the idea of non commutatitivity and to see how the ordinary dynamics which people has been using until then should be modified to include it [2].

The biquaternions have their peculiar nature of being non commutativity and for this reason they must be profoundly studied in their mathematical algebraic structure and in the light of application to quantum physics.

The aim of the present paper is to demonstrate a new basic theorem on biquaternions. It is of extreme interest in the field itself of biquaternions, thus in the mathematical body, but also, as we shall see, in the field of application of the biquaternions to physics and, in particular, to quantum mechanics.

Generalities on Biquaternions

According to our previous results, let us develop some basic features of the algebra of the biquaternions on the basis of our precedent literature [3-16].

Let us introduce four basic elements e_0 and e_i ($i = 1, 2, 3$) that we call the basic unities of the biquaternions.

We have to make two basic assumptions:

– the first assumption is that it exists the scalar square of each element e_0 and e_i as

$$e_0 e_0 = 1, \qquad e_1 e_1 = k_1, \qquad e_2 e_2 = k_2, \qquad e_3 e_3 = k_3 \tag{1}$$

with k_i ($i = 1, 2, 3$) real numbers and $k_i \neq 0$;

– the second assumption is their anticommutativity:

$$e_i e_0 = e_0 e_i = e_i; \qquad e_i e_j = - e_j e_i; \quad i = 1, 2, 3; \qquad j = 1, 2, 3; \quad i \neq j \qquad (2)$$

Let us deduce the algebra of the biquaternions : write the multiplication of the basic unities e_i in the following manner[4]

$$\begin{aligned} e_1 e_2 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \\ e_2 e_3 &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 \\ e_3 e_1 &= \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 \end{aligned} \qquad (3)$$

and let us impose left and right alternation

$$\begin{aligned} e_1 e_1 e_2 &= (e_1 e_1) e_2; \qquad & e_1 e_2 e_2 &= e_1 (e_2 e_2); \\ e_2 e_2 e_3 &= (e_2 e_2) e_3; \qquad & e_2 e_3 e_3 &= e_2 (e_3 e_3); \\ e_3 e_3 e_1 &= (e_3 e_3) e_1; \qquad & e_3 e_1 e_1 &= e_3 (e_1 e_1); \end{aligned} \qquad (4)$$

By using the (3) and the (4), we obtain that

$$\alpha_1 = \alpha_2 = \beta_2 = \beta_3 = \gamma_1 = \gamma_3 = 0;$$

$$\alpha_3 \neq 0; \qquad \beta_1 \neq 0; \qquad \gamma_2 \neq 0; \qquad (5)$$

$$k_1 = - \gamma_2 \alpha_3; \quad k_2 = - \beta_1 \alpha_3; \quad k_3 = - \beta_1 \gamma_2$$

If we select $k_i = + 1$, we finally obtain that

$$e_1 e_1 = e_2 e_2 = e_3 e_3 = 1; \qquad e_0 e_i = e_i e_0 ; \qquad (6)$$

$$e_1 e_2 = - e_2 e_1 = i\, e_3 ; \qquad e_2 e_3 = - e_3 e_2 = i\, e_1; \qquad e_3 e_1 = - e_1 e_3 = i\, e_2$$

We have how characterized the four basic elements, e_0 and e_i, of this algebra.

For any given complex number $z_\mu = x_\mu + i\, y_\mu$ ($\mu = 0, 1, 2, 3$), we have a biquaternion

$$Z = \sum_\mu z_\mu e_\mu \qquad (7)$$

with the iperconjugate biquaternion Z^* given as it follows

$$Z^* = z_0 - z_1 e_1 - z_2 e_2 - z_3 e_3$$

The complex conjugate Z^+ is given as

$$Z^+ = \sum_\mu z_\mu^* e_\mu \qquad (8)$$

and the conjugate $\overline{Z}$ is given as

$$\overline{Z} = z_0^* - z_1^* e_1 - z_2^* e_2 - z_3^* e_3$$

The norm of the biquaternion Z is

$$N(Z) = ZZ^* = Z^*Z = z_0^2 - z_1^2 - z_2^2 - z_3^2 \qquad (9)$$

and the inverse of Z is

$$Z^{-1} = [N(Z)]^{-1}Z \qquad (10)$$

and $N(Z) \neq 0$. It follows that we have not inverse when $N(Z) = 0$.

Given two biquaternions A and B we have that

$$N(AB) = N(A)\, N(B) \qquad (11)$$

while a biquaternion zero divisor has $Z \neq 0$ and $N(Z) = 0$.

We have also that $e_i^{-1} = e_i$, and

$$(e_i)^{1/2} = \frac{1 + i\, e_i}{\sqrt{2i}} \;; \qquad (-e_i)^{1/2} = \frac{1 - i\, e_i}{\sqrt{2i}} \;; \qquad i = 0, 1, 2, 3 \qquad (12)$$

Given the biquaternion Z, biquaternion transformations are given by

LBT(U, A)

$$Z' = U\, Z\, U^{+} + A \qquad (13)$$

where we may consider U and A biquaternions having unitary norm

$N(U) = 1$; $\quad N(A) = \pm 1$

General LBT may be written as $Z'=UZB+A$. $A = 0$ gives Linear Homogeneous Biquaternion Transformations, LHBT and LBT form a group. This is the group of the Biquaternion Linear Transformations.

This is the algebra of the biquaternions.

We have now the problem to fix a correspondence with the algebra of matrices. For $n= 2$ an isomorphic representation of biquaternions having basic unities e_i ($i=1, 2, 3$) is obtained by Pauli matrices. We have that

$$e_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\; e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\; e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (14)$$

with the generic biquaternion Z having matrix representation

$$Z = \begin{pmatrix} z_0 + z_3 & z_1 - i\, z_2 \\ z_1 + i\, z_2 & z_0 - z_3 \end{pmatrix} \qquad (15)$$

and $N(Z) \equiv \det.(Z)$.

This is the basic representation of the biquaternions at the order *n = 2*.

We may see that in (6) the multiplication of the basic unities $e_i e_j$ ($i = 1, 2, 3; j = 1, 2, 3; i \neq j$) is determined according to the cyclic permutation (*i, j, k*) of (*1, 2, 3*). We will extend this basic rule to the cases *n = 4, 8*, ... At the moment, let us remain still at the case *n=2* and let us continue to explore some other basic features of the biquaternions. We have established that in the case *n = 2*, e_i are represented by Pauli's matrices. We have also admitted in (1) and (6) that $e_i^2 = +1$. $e_i^2 = 1$ is the first important result that characterizes the extraordinary properties of the biquaternions.

We have the following statement.

Since $e_i^2 = 1$, the basic unities e_i ($i = 1, 2, 3$) of the biquaternions are expression of an intrinsic indetermination that such hypercomplex numbers may exhibit.

This is the first time that indetermination is connected with a proper number field in mathematics; consequently, we have to express in detail how such statement is realized.

Since $e_i^2 = 1$, we have that the basic unities e_i may assume only two possible numerical values, or *+ 1* or *-1*.

It follows that we have mean values such that

$$-1 \leq \langle e_i \rangle \leq +1 \tag{16}$$

Such mean values $\langle e_i \rangle$ determine two probabilities, *p*(*+1*) and *p*(*-1*), that such basic unities have respectively the value + *1* or the value *-1* . We have in fact that

$$\langle e_i \rangle = (-1)\, p(-1) + (+1)\, p(+1)\ ; \quad i = 1, 2, 3 \tag{17}$$

with

$$p(+1) + p(-1) = 1$$

We derive that

$$p(+1) = \frac{1}{2} + \frac{\langle e_i \rangle}{2}$$

and $$i = 1, 2, 3 \tag{18}$$

$$p(-1) = \frac{1}{2} - \frac{< e_i >}{2}$$

We have obtained the first indication that the biquaternions must be conceived as a number field expressing an intrinsic indetermination.

Let us go on explaining in more detail the reason for such intrinsic indetermination in biquaternions.

Let us consider the biquaternion

$$Z = x_1 e_1 + x_2 e_2 + x_3 e_3 \tag{19}$$

with x_i ($i = 1, 2, 3$) three real numbers such that

$$K = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2} \tag{20}$$

We have that

$$Z^2 = (x_1 e_1 + x_2 e_2 + x_3 e_3) (x_1 e_1 + x_2 e_2 + x_3 e_3) = K^2 \tag{21}$$

and thus Z may assume only two numerical values, or $+ K$ or $- K$. With regard to the mean value of Z, we have that

$$- K \leq < Z > \leq + K$$

or (22)

$$- K \leq x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \leq + K$$

The mean value of Z must be between $- K$ and $+ K$. The (22) must hold for any real numbers x_i ($i = 1, 2, 3$) and thus also for $x_i = < e_i >$. In this case we obtain that $K^2 \leq K$, $K \leq 1$, $K^2 \leq 1$, and

$$< e_1 >^2 + < e_2 >^2 + < e_3 >^2 \leq 1 \tag{23}$$

The (23) represents an important result for the algebra of the biquaternions. It indicates the impossibility for the basic unities e_i ($i = 1, 2, 3$) of the biquaternions to assume a unique and definite numerical value simultaneously. Here it is the origin of the intrinsic indetermination of the biquaternions.

Consider the case in which $< e_1 > = 1$. On the basis of (17) and (18), we have that e_1 assumes the numerical value $+ 1$ with $p(+1) = 1$, $p(-1) = 0$. In this case we have from the (23) that $< e_2 > = < e_3 > = 0$, and this implies, on the basis of the (18), that both e_2 and e_3 cannot have a definite numerical value, or $+1$ or -1. Similarly, it follows for $e_1 >=-1$. If, instead, $< e_2 >$ is either $+1$ or -1, it follows that $< e_1 >= < e_3 >= 0$ and e_1 and e_3 cannot assume a definite value; finally, if $< e_3 >$ is either $+1$ or -1, it follows that $< e_1 >= < e_2 >= 0$, and e_1, e_2 cannot assume a definite numerical value.

We may say that such basic unities only conserve the role of mathematical objects responding to the basic features of the algebra of the biquaternions while their possibility to assume a definite numerical value remains to be only of potential kind.

Let us examine the biquaternion algebra at the orders $n = 4,8,......$.

For the $n = 2$, we use basic unities e_i ($i = 1, 2, 3$; $n = 2$) with the permutation (i, j, k) of ($1, 2, 3$). When we consider the order $n = 4$, we must have a basic set of unities $E_{0\,i}$ ($i = 1, 2, 3$) , and the basic set of unities $E_{i\,0}$ ($i = 1, 2, 3$). For one basic unities , we must have, in fact, E_{01}, E_{02}, E_{03}, with multiplication rule given by cyclic permutation of ($1, 2, 3$). For the other basic unity, we must have E_{10}, E_{20}, E_{30} with the same multiplication rule and cyclic permutation. In addition, we must have also that $E_{0\,i} E_{j\,0} = E_{j\,0} E_{0\,i}$ ($i = 1, 2, 3; j = 1, 2, 3$) in order to assure that such basic unities may assume definite values together, each independently from the other. In this case they can be taken in consideration together. Owing to this basic requirement, we cannot use unities e_i at the order $n = 2$ since it may be easily shown that any unity commuting with basic unities e_i ($i = 1, 2, 3; n = 2$) may be only a scalar quantity .Thus we require basic unities, and correspondingly, basic matrices, $E_{0\,i}$ and E_{i0} at least at order $n= 4$.

They are introduced in the following manner

$$E_{0\,i} = I^{1} \otimes e_{i}; \qquad\qquad E_{i\,0} = e_{i} \otimes I^{2} \tag{24}$$

The notation $\otimes$ denotes direct product of matrices, and I^{i} is the ith 2x2 unit matrix. Thus, following the previous steps of the order $n = 2$, in the case of $n= 4$ we have two distinct sets of biquaternion basic unities, $E_{0\,i}$ and $E_{i\,0}$, with

$E_{0i}^2 = 1$; $E_{i0}^2 = 1$ $i = 1, 2, 3;$ (25)

$E_{0\,i}\, E_{0\,j} = i\, E_{0\,k}$; $E_{i\,0}\, E_{j\,0} = i\, E_{k\,0}$ $j = 1, 2, 3;$ $i \neq j$

and

$E_{i0}\, E_{0\,j} = E_{0\,j}\, E_{i\,0}$ (26)

with (i, j, k) cyclic permutation of ($1, 2, 3$).

In matrix notation we have the following scheme

$E_{0\,i} = I^{\,1} \otimes e_{\,i}$; $E_{i\,0} = e_{\,i} \otimes I^{\,2}$ (27)

where the notation $\otimes$ denotes direct product of matrices, and $I^{\,i}$ is the ith 2x2 unit matrix

Let us examine now the following result

$(I^{\,1} \otimes e_i)\,(e_j \otimes I^{\,2}) = E_{0\,i}\, E_{j\,0} = E_{j\,i}$ (28)

It is obtained according to our basic rule on cyclic permutation required for basic unities of biquaternions. We have that $E_{0\,i}\, E_{j0} = E_{j\,i}$ with $i = 1, 2, 3$ and $j=1, 2, 3,$ with $E_{ji}^2 = 1$, $E_{i\,j}\, E_{k\,m} \neq E_{k\,m}\, E_{i\,j}$, and $E_{i\,j}\, E_{k\,m} = E_{p\,q}$ where p results from the cyclic permutation (i, k, p) of ($1, 2, 3$) and q results from the cyclic permutation (j, m, q) of ($1, 2, 3$).

In Conclusion : in the case $n = 4$ we have two distinct basic matrices $E_{0\,i}$, $E_{i\,0}$ and, in addition, basic sets of unities ($E_{i\,j}$, $E_{i\,p}$, $E_{0\,m}$) with (j, p, m) basic permutation of ($1, 2, 3$). Similarly, we may realize other basic sets of biquaternion unities using ($E_{j\,i}$, E_{pi}, $E_{m\,0}$).

This is the biquaternion approach to consider biquaternions at the order $n = 4$. In the case $n= 8$ we have the possibility to introduce three sets of biquaternion basic unities. We will have $E_{0\,0\,i}$, $E_{0\,i\,0}$, and $E_{i\,0\,0}$, $i = 1, 2, 3$ and

$E_{0\,0\,i} = I^{\,1} \otimes I^{\,1} \otimes e_i$; $E_{0\,i0} = I^{\,2} \otimes e_i \otimes I^{\,2}$; $E_{i\,0\,0} = e_i \otimes I^{\,3} \otimes I^{\,3}$

and

$$(I^{\,1} \otimes I^{\,1} \otimes e_i) \cdot (I^{\,2} \otimes e_i \otimes I^{\,2}) \cdot (e_i \otimes I^{\,3} \otimes I^{\,3}) = e_i \otimes e_i \otimes e_i =$$

$$= E_{0\,0\,i}\, E_{0\,i\,0}\, E_{i\,0\,0} = E_{\,i\,i\,i} \qquad (29)$$

Still we have that

$E_{0\,0\,i}\, E_{0\,i\,0} = E_{0\,i\,0}\, E_{i\,0\,0}$; $E_{0\,0\,i} E_{i\,0\,0} = E_{i\,0\,0} E_{0\,0\,i}$; $E_{0\,i\,0}\, E_{i\,0\,0} = E_{i\,0\,0} E_{0\,i0}$ (30)

In the case $n = 8$ we have three distinct basic unities and, in addition, we have other basic unities, as example (E_{ijk}, E_{iji}, E_{ijj}). Other cases are obviously possible.

Generally speaking, fixed the order n of the biquaternion basic unities, we will have that

$$\begin{aligned}
&\Gamma_1 = \Lambda_n \\
&\Gamma_{2m} = \Lambda_{n-m} \otimes e_2^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n \qquad (31)\\
&\Gamma_{2m+1} = \Lambda_{n-m} \otimes e_3^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n \\
&\Gamma_{2n} = e_2 \otimes I^{(2)} \otimes \ldots\ldots\ldots \otimes I^n
\end{aligned}$$

with

$\Lambda_n = e_1^{(1)} \otimes e_1^{(2)} \otimes \ldots\ldots \otimes e_1^{(n)} =$

$(e_1 \otimes I^{(1)} \otimes \ldots\ldots \otimes I^n)\cdot(\ldots\ldots\ldots)\cdot(I^{(1)} \otimes I^{(2)} \ldots \otimes I^{(n)} \otimes e_1)$; $\quad m = 1, \ldots\ldots, n - 1$

according to the n-possible dispositions of e_1 and $I^1, I^2, \ldots, I^n$ in the distinct direct products.

Basic unities are determined by the number of dispositions possible for e_i and $I^{(n)}$.

In this manner we have established that biquaternions exist at different orders $n = 2, 4, 8, \ldots\ldots$ We may now give the explicit expressions of E_{0i}, E_{i0}, and E_{ij} in matrix form:

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \; E_{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \; E_{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad (32)$$

$$E_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \; E_{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; E_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad (33)$$

$$E_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \; E_{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; E_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$E_{12}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix};E_{13}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix};E_{21}=\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix};$$

$$E_{31}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix};\quad E_{23}=\begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix};\quad E_{32}=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{pmatrix}.$$

We have two different basic unities

$$(E_{0\,0}, E_{0\,i}), (E_{0\,0}, E_{i0}) \tag{34}$$

with

$$E_{00}^2=1; \qquad E_{0i}^2=E_{i0}^2=1; \qquad E_{0\,i}E_{0\,j}=i\,E_{0\,k};$$

$$E_{i\,0}\,E_{j\,0}=i\,E_{k\,0}; \qquad E_{0\,i}E_{j\,0}=E_{j\,0}E_{0\,i}.$$

In the first case a biquaternion is given to be

$$Z=z_0E_{0\,0}+z_1E_{0\,1}+z_2E_{0\,2}+z_3E_{0\,3} \tag{35}$$

while in the second case we have that

$$Q=z_0E_{0\,0}+z_1E_{1\,0}+z_2E_{2\,0}+z_3E_{3\,0} \tag{36}$$

The (35) and the (36) reconduce ourselves to all the basic statements that we just discussed in the previous section for biquaternions at order $n = 2$. All the basic features of biquaternions that we enunciated in the case $n = 2$ still remain valid also in the case $n = 4$ and in the following orders $n = 8, \ldots$ of biquaternions.

In particular, we continue to have that

$$<E_{0\,1}>^2+<E_{0\,2}>^2+<E_{0\,3}>^2\le 1$$

and (37)

$$<E_{10}>^2+<E_{2\,0}>^2+<E_{3\,0}>^2\le 1$$

In this manner also the general indeterminacy principle of the biquaternions remains valid and their probability expressions.

The New Theorem

There is a class of biquaternions for which the power of the biquaternions at the order n is so that $R^{n-1} \neq 0$ *and* $R^n = 0$.

Proof.

As class of these biquaternions let us select the following two biquaternions at the order $n = 2$:

$$R = a\left(\frac{\sqrt{1}}{2}e_1 + \frac{\sqrt{1}}{2i}e_2\right); \qquad S = a\left(\frac{\sqrt{1}}{2}e_1 - \frac{\sqrt{1}}{2i}e_2\right) \tag{38}$$

where a is an arbitrary scalar.

We have that

$R^0 = 1$; $\quad R^1 = R$; $\quad R^2 = 0$;

$S^0 = 1$; $\quad S^1 = S$; $\quad S^2 = 0$

Consider the same class of biquaternions at the order *n=4.*

We have the following two biquaternions

For *n = 4* we have that

$$R = a\left[\sqrt{1}\left[q_0 \otimes \left(\frac{e_1}{2} - \frac{ie_2}{2}\right)\right] + \sqrt{2}\left[\left(\frac{e_1}{2} - \frac{ie_2}{2}\right) \otimes \left(\frac{e_1}{2} + \frac{ie_2}{2}\right)\right] + \sqrt{3}\left[q_1 \otimes \left(\frac{e_1}{2} - \frac{ie_2}{2}\right)\right]\right]$$

(39)

$$S = a\left[\sqrt{1}\left[q_0 \otimes \left(\frac{e_1}{2} + \frac{ie_2}{2}\right)\right] + \sqrt{2}\left[\left(\frac{e_1}{2} + \frac{ie_2}{2}\right) \otimes \left(\frac{e_1}{2} - \frac{ie_2}{2}\right)\right] + \sqrt{3}\left[q_1 \otimes \left(\frac{e_1}{2} + \frac{ie_2}{2}\right)\right]\right]$$

where q_0 and q_1 are the following biquaternions

$$q_0 = \frac{1}{2}(1 + e_3); \qquad q_1 = \frac{1}{2}(1 - e_3)$$

We have

$$R = a\left[\left(\frac{\sqrt{3}+\sqrt{1}}{4}\right)(E_{01} - iE_{02}) + \left(\frac{\sqrt{1}-\sqrt{3}}{4}\right)(E_{31} - iE_{32}) + \frac{\sqrt{2}}{4}(E_{11} + iE_{12} - iE_{21} + E_{22})\right]$$

(40)

$$S = a\left[\left(\frac{\sqrt{3}+\sqrt{1}}{4}\right)(E_{01} + iE_{02}) + \left(\frac{\sqrt{1}-\sqrt{3}}{4}\right)(E_{31} + iE_{32}) + \frac{\sqrt{2}}{4}(E_{11} - iE_{12} + iE_{21} - E_{22})\right]$$

At the order $n = 4$, we have $R^0 = 1, R^1 = R$, and it results

$$R^2 = a\left[\sqrt{1}\sqrt{2}\left[\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\otimes q_0\right]+\sqrt{2}\sqrt{3}\left[\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\otimes q_1\right]\right] \quad (41)$$

and

$$R^3 = a^{3/2}\sqrt{1}\sqrt{2}\sqrt{3}\left[\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\otimes\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\right] \quad (42)$$

Finally

$$R^4=R^3R= a^2\sqrt{1}\sqrt{2}\sqrt{3}\left[\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\otimes\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\right]\bullet$$

$$(43)$$

$$\bullet\left[\sqrt{1}\left[q_0\otimes\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\right]+\sqrt{2}\left[\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\otimes\left(\frac{e_1}{2}+\frac{ie_2}{2}\right)+\sqrt{3}\left[q_1\otimes\left(\frac{e_1}{2}-\frac{ie_2}{2}\right)\right]\right]\right]=0$$

We have also that $S^0 = 1,$ $S^1 = S,$ and

$$S^2 = a\left[\sqrt{1}\sqrt{2}\left[\left(\frac{e_1}{2}+\frac{ie_2}{2}\right)\otimes q_0\right]+\sqrt{2}\sqrt{3}\left[\left(\frac{e_1}{2}+\frac{ie_2}{2}\right)\otimes q_1\right]\right] \quad (44)$$

$$S^3 = a^{3/2}\sqrt{1}\sqrt{2}\sqrt{3}\left[\left(\frac{e_1}{2}+\frac{ie_2}{2}\right)\otimes\left(\frac{e_1}{2}+\frac{ie_2}{2}\right)\right] \quad (45)$$

$$S^4 = S^3 S = 0 \quad (46)$$

Generally speaking, we may say that at any fixed order n we have this general property of these R, S biquaternions. Fixed the order n and calculated the corresponding biquaternions expressing R and S, we have that

$$R^n S^n = 0 \qquad \text{and} \qquad R^{n-1}S^{n-1} \neq 0 \quad (47)$$

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AN APPLICATION OF THE THEOREM TO THE CASE OF THE HARMONIC OSCILLATOR IN QUANTUM MECHANICS.

Generally speaking, in quantum mechanics some physical quantities are expressed in the following manner

$$A = f(N, a, b, ...) \quad (48)$$

where $a, b, ...$ may be constants, while N may assume only discrete values $0, 1, 2, ...$.

Let us calculate the case $n = 2$ of an harmonic oscillator. As we know, we have

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

and

$$Q = \frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2} e_1; \qquad P = \frac{1}{2}(2m\omega\hbar)^{1/2} e_2$$

and

$$R = Q - \frac{i}{m\,\omega}P \;; \qquad S = Q + \frac{i}{m\,\omega}P$$

$$H = \frac{1}{2}m\,\omega^2\,R\,S + \frac{1}{2}\hbar\omega\,e_3 \quad R\,S - S\,R = -\,\frac{2\hbar}{m\omega}\,e_3\,; \tag{49}$$

Where P and Q are biquaternions representing the momentum and the position respectively. N is given in the following manner

$$N = \frac{m\omega}{2\hbar}RS$$

with

$$H\,S = \frac{1}{2}m\,\omega^2\left[SR - \frac{2\hbar}{m\omega}e_3\right] + \frac{1}{2}\,\hbar\,\omega\,e_3\,S$$

On the other hand, we have that

$$0 = R^2\,S^2 = R\,(R\,S)\,S = R\left(\frac{2H - \hbar\omega\,e_3}{m\omega^2}\right)S = \frac{2}{m\omega^2}R\,H\,S - \frac{\hbar}{m\omega}\,R\,e_3\,S =$$

$$= R\,S\left(\frac{2H}{m\omega^2}\right) - \frac{\hbar}{m\omega}\,R\,S\,e_3 - \frac{2\hbar}{m\omega}\,R\,e_3\,S$$

Finally we obtain that

$$R\,S\left(\frac{2H}{m\omega^2} - \frac{\hbar}{m\omega}e_3 - \frac{2\hbar}{m\omega}\right) = 0$$

since $e_3\,S = S$. Considering $<e_3>=1$, $e_3 \rightarrow +\,1$, we obtain that

$$E = \hbar\,\omega + \frac{1}{2}\,\hbar\,\omega \quad ; \quad H \rightarrow E \tag{50}$$

Considering instead $<e_3>=-1$, $e_3 \rightarrow -\,1$, we have that

$$E = \frac{1}{2}\,\hbar\,\omega \tag{51}$$

as it is required for the case of the quantized harmonic oscillator considering only $n = 0, 1$.

In the same manner one shows that the general results is

$$E = n\ \hbar\,\omega + \frac{1}{2}\hbar\omega \tag{52}$$

and thus

$$E = \hbar\ \omega\,(N + ½) \tag{53}$$

owing to the (47) in the case at order *n.*

CONCLUSIONS

In this manner we have given a direct proof of the universal character of the biquaternions. We have evidenced that the theorem enters directly in the (49) that expresses the quantized energy levels of an harmonic oscillator. In this manner we have also confirmed the correctness of employing biquaternion algebra in the analysis of quantum systems. Our proof has enabled us to calculate the quantized values that the oscillator energy can have and it has given a definitive demonstration that biquaternions are necessary to express the intrinsic quantization that quantum systems exhibit.

We have to add still another comment.

Finally, let us observe the importance of the property $R^{n-1}S^{n-1} \neq 0$ and $R^{n}S^{n} = 0$. It has a general validity in the biquaternion algebra as mathematical tool and it may be used in general different cases to show also quantization of other quantum systems, as example in the theory of angular momentum or in the energy levels of the hydrogen atom.

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