Differential Quotients and Division by Zero

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Abstract: In this very short note, a pleasant relation of the basic idea of differential quotients dy/dx of Leipniz and division by zero $1/0 = 0$. This will give a natural interpretation of the important result $\tan(\pi/2) = 0$. Based on the preprint survey paper ([25]), we gave in [30] a fundamental relation among the basic concepts of division by zero calculus and derivatives as a direct extension of the preprint ([29]) which gave the generalization of the division by zero calculus to differentiable functions.

Key Words: Division by zero, differential quotient, division by zero calculus, differentiable, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, $[(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Background of division by zero calculus

In order to state the new viewpoint for the differential quotient dy/dx in a self-contained way, we do not need so much materials, but we will recall the simple background on the division by zero calculus for differentiable functions based on ([30]). For the basic references on the division by zero and the division by zero calculus, see the papers cited in the references.

For a function $y = f(x)$ which is $n (n > 0)$ order differentiable at $x = a$, we will **define** the value of the function

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{1.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as** $1/0 = 0$ **and** $0/0 = 0$**, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for $1/0 = 0$ and $0/0 = 0$. See, for example, [26].

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 **represents impossibility**. In particular, the value of the function $y = |x|/x$ at $x = 0$ is zero.

We will note its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x \to a} \frac{F(x)}{x-a} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a). \tag{1.2}$$

For many applications, see the papers cited in the reference.

The identity (1.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x\to a} \frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x=a$ which is defined independently of $f(a)$ in (1.2). This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n=0$ for the general formula; that is,

$$\frac{f(x)}{(x-a)^0}|_{x=a} = \frac{f^{(0)}(a)}{0!} = f(a).$$

In the expression (1.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{1.3}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

When we apply the relation (1.1) to the elementary formulas for differentiable functions, we can imagine some deep results for the division by zero calculus. For example, in the simple formula

$$(u+v)' = u' + v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition. This is a result from the property of derivatives.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u\prime}{u} + \frac{v\prime}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u}\right).$$

For the second order differentials, we have the familiar formulas:

$$(uv)'' = u''v + 2u'v' + uv'',$$

$$\frac{d^2 f(g(t))}{dt^2} = f''(g(t))g'(t) + f'(g(t))g''(t),$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3}$$

and

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}.$$

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta - a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. See [30].

2 Differential quotients and division by zero

We state an interesting interpretation of the relation between differential quotients and division by zero. For the differential quotient

$$\frac{dy}{dx},$$

if it is zero in some interval, then, of course, we have that $y = C$ in the interval with a constant C. This will mean that if $dy = 0$, then $y = C$ in some interval with a constant C and $y' = 0$.

Meanwhile, if $dx = 0$, then, by the division by zero, we have

$$\frac{dy}{dx} = 0$$

and so, we have that $y' = 0$. Then, however, $x = D$ with a constant D in some y interval. This interpretation shows that the gradient of the y axis is zero, that is

$$\tan \frac{\pi}{2} = 0.$$

We see that this result is very fundamental and appeared in many situations, as we can see from the cited references.

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