Formulation And Validation Of First Order Lagrangian

Eric Su
eric.su.mobile@gmail.com
https://sites.google.com/view/physics-news/home
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An unproven lagrangian generates erroneous theory. The unknown lagrangian can be validated with Hamiltonian. The invalid lagrangian can be formulated into a valid lagrangian with a three-phase process based on Euler-Lagrange equation and conservation law. The formulation process modifies the lagrangian with system invariant. The process is superior to the popular trial-and-error approach.

I. INTRODUCTION

An unproven lagrangian generates invalid Euler-Lagrange equation and incorrect theory. The new lagragian requires validation. The popular trial-and-error approach relies on the Euler-Lagrange equation to verify the unknown lagrangian. The trial-and-error approach fails if the Euler-Lagrange equation is also unknown.

However, a system invariant associated with the Euler-Lagrange equation is always valid. It is equivalent to Hamiltonian in classical mechanics. The invariant can be used to validate the unproven lagrangian.

The unproven lagrangian can also be formulated into a valid lagrangian. The formulation process requires conservation law and Euler-Lagrange equation. A first-order derivative term must be present in the unknown lagrangian to interact with the system invariant which is the single degree of freedom in the formulation process.

II. PROOF

A. Euler-Lagrange Equation

Let L be a multi-variable function.

$$L = L(t, x_1, x_2, .., x_N, u_1, u_2, .., u_N) \tag{1}$$

$$\frac{dx_i}{dt} = u_i \tag{2}$$

L is a valid lagrangian if L satisfies the Euler-Lagrange equation as

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}(\frac{\partial L}{\partial u_i}) \tag{3}$$

The total derivative of L from equation (1) is

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{i=1}^{N}(\frac{\partial L}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial L}{\partial u_i}\frac{du_i}{dt}) \tag{4}$$

From equations (2,3,4),

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{i=1}^{N}(u_i\frac{d}{dt}(\frac{\partial L}{\partial u_i}) + \frac{\partial L}{\partial u_i}\frac{du_i}{dt}) \tag{5}$$

$$= \frac{\partial L}{\partial t} + \sum_{i=1}^{N}\frac{d}{dt}(u_i\frac{\partial L}{\partial u_i}) \tag{6}$$

Seperate the total derivative from partial derivative.

$$\frac{d}{dt}(L - \sum_{i=1}^{N} u_i\frac{\partial L}{\partial u_i}) = \frac{\partial L}{\partial t} \tag{7}$$

Equation (7) is the requirement for L to be a valid lagrangian. Hence, L can be formulated as

$$L = \sum_{i=1}^{N} u_i\frac{\partial L}{\partial u_i} + \int \frac{\partial L}{\partial t}dt + const. \tag{8}$$

The validation prevents any unproven lagrangian from generating incorrect Euler-Lagrange equation.

Two lagrangians in the following sections serves as validation example. One lagrangian for electromagnetic field in classic mechanics. Another lagrangian for relativsitic harmonic oscillation.

B. Electromagnetic Lagrangian

The lagrangian for electromagnetic source j_μ can be represented as

$$L = \frac{1}{c}j_\mu A_\mu - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{9}$$

with

$$F_{\mu\nu} = u_{\nu\mu} - u_{\mu\nu} \tag{10}$$

$$u_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} \tag{11}$$

From equations (9,10,11),

$$u_{\mu\nu}\frac{\partial L}{\partial u_{\mu\nu}} = u_{\mu\nu}F_{\mu\nu} \tag{12}$$

From equations (7,9,12), the lagrangian L is valid only if

$$\frac{d}{dx_\nu}(\frac{1}{c}j_\mu A_\mu - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - u_{\mu\nu}F_{\mu\nu}) = 0 \tag{13}$$

C. Kinetic Lagrangian

The lagrangian for relativistic harmonic oscillation can be represented as

$$L = -\frac{1}{\gamma}mc^2 - \frac{1}{2}kx^2 \tag{14}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{15}$$

From equations (14,15),

$$u\frac{\partial L}{\partial u} = u(\gamma\frac{u}{c^2}mc^2) \tag{16}$$

From equations (7,14,16), the lagrangian L is valid only if

$$\frac{d}{dt}(-\frac{1}{\gamma}mc^2 - \frac{1}{2}kx^2 - \gamma\frac{u^2}{c^2}mc^2) = 0 \tag{17}$$

$$\frac{d}{dt}(-\gamma mc^2 - \frac{1}{2}kx^2) = 0 \tag{18}$$

D. Constant Lagrangian

The lagrangian is not constant in general. If L is a constant and

$$\frac{\partial L}{\partial t} = 0 \tag{19}$$

the validation of lagrangian can be simplified from equations (7,19) as

$$\frac{d}{dt}(\sum_{i=1}^{N} u_i\frac{\partial L}{\partial u_i}) = 0 \tag{20}$$

The simplifed validation is useful in certain area such as Riemannian geometry.

E. Line Element

The line element in Riemannian manifold is represented by

$$ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \tag{21}$$

The lagrangian for $ds \neq 0$ can be represented as

$$L = 1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = g_{\mu\nu}u_\mu u_\nu \tag{22}$$

$$u_\mu = \frac{dx^\mu}{ds} \tag{23}$$

From equations (22,23),

$$\frac{\partial L}{\partial u_\mu} = 2g_{\mu\nu}u_\nu \tag{24}$$

From equations (20,22,24), the lagrangian L is valid only if

$$\frac{d}{ds}(u_\mu 2g_{\mu\nu}u_\nu) = 0 \tag{25}$$

$$\frac{d}{ds}(g_{\mu\nu}u_\nu u_\mu) = 0 \tag{26}$$

$$\frac{d}{ds}(1) = 0 \tag{27}$$

Line element from any metric such as Minkowski metric and Schwarzschild metric is a valid lagrangian.

F. Schwarzschild Metric

Schwarzschild metric[1,2] was derived by Karl Schwarzschild in 1916. The metric describes an isotropic manifold for a point mass.

$$ds^2 = (1-\frac{\alpha}{R})dt^2 - \frac{dR^2}{1-\frac{\alpha}{R}} - R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{28}$$

Schwarzschild applied rotational symmetry to the line element by setting

$$\theta = \frac{\pi}{2} \tag{29}$$

From equations (28,29),

$$ds^2 = (1-\frac{\alpha}{R})dt^2 - \frac{dR^2}{1-\frac{\alpha}{R}} - R^2 d\phi^2 \tag{30}$$

The lagrangian can be represented as

$$L = 1 = f(\frac{dt}{ds})^2 - \frac{1}{f}(\frac{dR}{ds})^2 - R^2(\frac{d\phi}{ds})^2 \tag{31}$$

$$f = (1-\frac{\alpha}{R}) \tag{32}$$

L is a valid lagrangian because L is a constant and satisfies the requirement from equation (27) as

$$1 = g_{tt}(\frac{dt}{ds})^2 + g_{RR}(\frac{dR}{ds})^2 + g_{\phi\phi}(\frac{d\phi}{ds})^2 \tag{33}$$

G. Single Variable Lagrangian

For any lagrangian with single variable, the validation from equation (20) can be further simplified as

$$\frac{d}{dt}(u\frac{\partial L}{\partial u}) = 0 \tag{34}$$

$$L = ln(u)K + f(x) \tag{35}$$

with K as a constant and f(x) as an arbitrary function of x.

The $ln(u)$ term must be present in the valid lagrangian. A good example is the total energy of an isolated system.

H. Conservation Of Energy

Total energy of an isolated harmonic oscillator is a constant but can not be a valid lagrangian

$$L \neq \frac{m}{2}u^2 + \frac{k}{2}x^2 = E \tag{36}$$

because the $ln(u)$ term is not present as required by equation (35).

I. Lagrangian Formulation

The valid lagrangian can be formulated from an invalid lagrangian with the aid of conservation law. A very useful process when a new lagrangian is proposed for a new system or theory.

Given a possible lagrangian $f(x_i, u_i, t)$, the lagrangian can be formulated from equations (8) as

$$L = \sum_{i=1}^{N} u_i \frac{\partial f}{\partial u_i} + \int \frac{\partial f}{\partial t} dt + constant \tag{37}$$

Employ the conservation law to set the constant to nC with C as a known system invariant and n as an unknown variable.

$$L = \sum_{i=1}^{N} u_i \frac{\partial f}{\partial u_i} + \int \frac{\partial f}{\partial t} dt + nC. \tag{38}$$

The new lagrangian is validated with equation (7) to solve for n.

$$\frac{d}{dt}(L - \sum_{i=1}^{N} u_i \frac{\partial L}{\partial u_i}) = \frac{\partial L}{\partial t} \tag{39}$$

L is formulated if $n \neq 0$.

The choice of C depends on the system represented by $f(x_i, u_i, t)$. Any invariant related to the conservation law for the given system may serve the purpose. Two examples are provided in the following sections.

J. Harmonic Oscillation

A possible lagrangain for harmonic oscillation in classical mechanics is proposed as

$$f = \frac{m}{2}u^2 + t \tag{40}$$

From equations (37,40),

$$L = mu^2 + t + constant \tag{41}$$

The choice of constant depends on the system for f. Let the constant be proportional to the total energy of this isolated system.

$$L = mu^2 + t + n(\frac{m}{2}u^2 + \frac{k}{2}x^2) \tag{42}$$

From equations (39,42), L is valid only if

$$\frac{d}{dt}(mu^2 + t + n(\frac{m}{2}u^2 + \frac{k}{2}x^2) - (2+n)mu^2) = 1 \tag{43}$$

$$\frac{d}{dt}(mu^2 - (2+n)mu^2) = 0 \tag{44}$$

$$n = -1 \tag{45}$$

From equations (42,45), the valid lagrangian is

$$L = t + \frac{m}{2}u^2 - \frac{k}{2}x^2 \tag{46}$$

K. Harmonic Oscillation 2

Another lagrangain is proposed as

$$f = u^2 + g(t) \tag{47}$$

From equations (37,47),

$$L = 2u^2 + g + constant \tag{48}$$

Let the constant be proportional to the total energy.

$$L = 2u^2 + g + n(\frac{m}{2}u^2 + \frac{k}{2}x^2) \tag{49}$$

From equations (39,49), L is valid only if

$$\frac{d}{dt}(2u^2 + g + n(\frac{m}{2}u^2 + \frac{k}{2}x^2) - 4u^2 - nmu^2) = \frac{\partial g}{\partial t} \tag{50}$$

$$\frac{d}{dt}(2u^2 - 4u^2 - nmu^2) = 0 \tag{51}$$

$$n = -\frac{2}{m} \tag{52}$$

From equations (49,52), the valid lagrangian is

$$L = g + u^2 - \frac{k}{m}x^2 \tag{53}$$

III. CONCLUSION

The lagrangian can be validated by a system invariant equivalent to Hamiltonian of classical mechanics.

An unproven lagrangian is invalid if it fails to generate the invariant. The invalid lagrangian needs revision but is useful for the formulation of a valid lagrangian.

Given a new lagrangian, a valid lagrangian can be formulated by conservation law and the Euler-Lagrange equation. The formulation process is superior to the popular trial-and-error approach which fails if the Euler-Lagrange equation is also unknown.

The formulation process modifies the lagrangian with the system invariant, nC, which can be replaced with h(C), a random function of C, as along as h(C) is also invariant.

[1] K. Schwarzschild: ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. https://de.wikisource.org/w/index.php?title=das Gravitationsfeld eines

[2] Su, Eric: Tangent Velocity Of Schwarzschild Geodesics. viXra: Relativity and Cosmology/2006.0137 (2020). http://vixra.org/abs/2006.0137

[3] Eric Su: List of Publications, http://vixra.org/author/eric_su