Approximation of Harmonic Series

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Abstract

Background : Harmonic Series is the sum of Harmonic Progression. There have been multiple formulas to approximate the harmonic series, from Euler's formula to even a few in the 21st Century. Mathematicians have concluded that the sum cannot be calculated, however any approximation better than the previous others is always needed.

In this paper we will discuss the flaws in Euler's formula for approximation of harmonic series and provide a better formula. We will also use the infinite harmonic series to determine the approximations of finite harmonic series using the Euler-Mascheroni constant. We will also look at the Leibniz series for Pi and determine the correction factor that Leibniz discussed in his paper which he found using Euler numbers.

Each subsequent approximation we find in this paper is better than all previous ones. Different approximations for different types of harmonic series are calculated, best fit for the given type of harmonic series. The correction factor for Leibniz series might not provide any applied results but it is a great way to ponder some other infinite harmonic series.

Keywords: Harmonic series, Euler-Mascheroni constant, Correction factor for Leibniz series, Infinite harmonic series

1. Introduction

1.1. Introduction to Harmonic Series

Harmonic series (HS) is the sum of harmonic progression (HP) which is the progression formed using reciprocals of consecutive terms of an Arithmetic Progression (AP)

$AP \ = \ a, \ a+d, \ a+2d, \ a+3d, ..., \ a+(n-1)d$

In standard notations, 'a' is the first term, 'd' is the common difference, 'n' is the number of terms, The last term is often noted as 'L'.

$HP \ = \ \frac{1}{(a)}, \ \frac{1}{(a+d)}, \ \frac{1}{(a+2d)}, \ .... \ , \ \frac{1}{(L)}$

$HS \ = \ \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \ .... \ + \ \frac{1}{(L)}$

The most trivial case of the HS is when 'a' and 'd' is equal to one. This case is the most studied and used for approximations.

$= \ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

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1.2. Euler's Formula for Harmonic series [1]

Euler derived an equation for the sum of infinite harmonic series

$$S(\infty) \;=\; \ln{(\infty)} \;+\; \gamma \tag{1}$$

$\gamma \;=\; 0.5772156649$ ....
where gamma is the Euler-Mascheroni constant.

1.3. Formula in a previous paper [2]

I used the approximation theory to integrate a function similar to the harmonic series in order to find its approximation.

$$S^a(L)^d \;=\; \frac{\ln\left(\dfrac{L}{a}\right)}{d} + \frac{1}{2a} + \frac{1}{2L} + \;\triangle E^a(L)^d \tag{2}$$

Here $\triangle E^a(L)^d \;=\; \textit{Error in the approximation}$

$$S^a(L)^d \;\approx\; \frac{\ln\left(\dfrac{L}{x}\right)}{d} + \frac{1}{2x} + \frac{1}{2L} + \;S^a(x-d)^d \tag{3}$$

Here 'x' is an arbitrarily chosen term of the AP.

1.4. Leibniz series for Pi [3]

Leibniz derived a series to determine the value of Pi.
$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ .... $= \frac{\pi}{4}$
While calculating this manually gives a pretty close approximation, Leibniz introduced the correction factor for the series that accounts for the terms not calculated manually, whose position can be determined using the Euler Numbers, however value is still debated.

2. Results

2.1. Correction in Euler's formula

By Equation (1), mathematicians determined a formula to approximate finite harmonic series.
$S(n) \;\cong\; \ln{(n)} \;+\; \gamma$
But with Equation (2) as a reference point we determine an even better approximation.

$$S(n) \;\cong\; \ln{(n)} + \frac{1}{2n} + \gamma \tag{4}$$

2.2. Errors in Formula in Equation (2)

Using both Equation (1) and Equation (2) we determine the error in the infinite series calculated using Equation (2) .

$\triangle E^1(\infty)^1 \;=\; \gamma \;-\; \frac{1}{2}$

It was found experimentally

$$\triangle E^1 \left(L\right)^1 \;\cong\; \gamma(k+2) \;-\; \frac{1}{2} \tag{5}$$

$\gamma(p) \;=\; \gamma \; to \; 'p' \; decimal \; places$

$eg. \; \gamma(3) \;=\; 0.577$

$L \;\leq 5 \times 4^k$

Subsequently,

$$\triangle E^a \left(\infty\right)^1 \;\cong\; \gamma \;-\; \gamma(k+2) \tag{6}$$

Here, $a \;\leq\; 5 \times 4^k$

Combining Both,

$$\triangle E^a \left(L\right)^1 \;\cong\; \gamma(k_2+2) \;-\; \gamma(k_1+2) \tag{7}$$

Here, $a \leq\; 5 \times 4^{k_1}$
$L \leq\; 5 \times 4^{k_2}$

Using both Equation (1) and Equation (2)

$$\triangle E^1(\infty)^d \;\cong\; \frac{\ln(d!)}{d^2} - \frac{\ln(d)}{2d} - \frac{1}{4d^2} + \frac{\gamma}{d} - \frac{\gamma(k+2)}{2d} \tag{8}$$

Subsequently,

$$\triangle E^a(L)^d \;\cong\; \frac{\ln\left(\frac{(a+d-1)!}{a^d \times (a-1)!}\right)}{d^2} - \frac{\ln\left(\frac{a+d-1}{a}\right)}{2d} + \frac{d-1}{4ad(a+d-1)} + \frac{\gamma}{d} - \frac{\gamma(k_1+2)}{2d} - \frac{\gamma(k_2+2)}{2d} \tag{9}$$

Here $a \leq 5 \times 4^{k_1}$
$a+d-1 \;\leq\; 5 \times 4^{k_2}$

2.3. Correction factor for Leibniz series

$$CF \;=\; \frac{\ln\left(\frac{x+2}{x}\right)}{4} + \frac{1}{x(x+2)} \tag{10}$$

Here, $CF \;=\; correction \; factor$
$x \;=\; First \; term \; that \; isnt \; calculated \; manually$

3. Discussion

3.1. Proof for Equation (4)

$S(\infty) \; = \; \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \; \ldots\ldots\ldots\ldots$

By formula in Equation (2)

$S^1(\infty)^1 \; = \; \frac{\ln\left(\dfrac{\infty}{1}\right)}{1} + \frac{1}{2} + \frac{1}{2\infty} + \triangle E^1(\infty)^1$

By Euler's equation Equation (1)

$\ln(\infty) \; + \; \gamma \; = \; \ln(\infty) \; + \frac{1}{2} + \triangle E^1(\infty)^1$

Therefore, $\triangle E^1(\infty)^1 \; = \; \gamma \; - \frac{1}{2}$

It is a well-established result that $\triangle E^1(n)^1 \; \leq \; \triangle E^1(\infty)^1$ *for all positive integral values of n*

If these two errors are equated

$S(n) \; \approx \; \ln(n) \; + \; \frac{1}{2} + \frac{1}{2n} + \triangle E(\infty)$

$= \; \ln(n) + \frac{1}{2} + \frac{1}{2n} + \gamma - \frac{1}{2}$

Hence,

$$S(n) \; \approx \; \ln(n) \; + \; \frac{1}{2n} + \gamma$$

3.2. Proof for derived Errors in Formula in Equation (2)

3.2.1. Proof for Equation (5)

It is a well-established result that $\triangle E^1(n)^1 \; \leq \; \triangle E^1(\infty)^1$ *for all positive integral values of n*

It was found experimentally that the Error increases by one decimal place of the Euler-Mascheroni constant after multiplying the Last term by 4.

$\triangle E^1(5)^1 = \; 0.07\;389542$

Eg. $\triangle E^1(20)^1 = \; 0.077\;007383$

$\triangle E^1(80)^1 = \; 0.0772\;02126$

We used these to conclude that

$$\triangle E^1\,(L)^1 \; \cong \; \gamma(k+2) \; - \; \frac{1}{2}$$

$\gamma(p) \; = \; \gamma \; to \; 'p' \; decimal \; places$

$eg.\; \gamma(3) \; = \; 0.577$

$L \; \leq 5 \times 4^k$

3.2.2. Proof for Equation (6)

$S^1(a)^1 + \; S^a(\infty)^1 \; = \; S^1(\infty)^1 \; + \; \frac{1}{a}$

By using Equation (1) and Equation (2)

$\ln(a) \; + \; \frac{1}{2} \; + \; \frac{1}{2a} \; + \; \triangle E^1(a)^1 \; + \; \ln\left(\frac{\infty}{a}\right) \; + \frac{1}{2a} + \frac{1}{2\infty} \; + \; \triangle E^a(\infty)^1 \; = \; \ln(\infty) + \gamma \; + \frac{1}{a}$

We conclude that

$\triangle E^1(a)^1 \; + \; \triangle E^a(\infty)^1 \; = \; \gamma \; - \; \frac{1}{2}$

We know from Equation (5) that $\triangle E^1(a)^1 \; \cong \; \gamma(k+2) \; - \; \frac{1}{2}$

Putting this in the equation above we get,

$$\triangle E^{a}\left(\infty\right)^{1} \;\cong\; \gamma \;-\; \gamma(k+2)$$

Here, $a \;\leq\; 5 \times 4^{k}$

3.2.3. Proof for Equation (7)

$S^{1}(a)^{1} \;+\; S^{a}(L)^{1} \;+\; S^{L}(\infty)^{1} \;=\; S^{1}(\infty)^{1} \;+\; \frac{1}{a} + \frac{1}{L}$

By using Equation (1) and Equation (2)

$\ln(a) + \frac{1}{2} + \frac{1}{2a} + \triangle E^{1}(a)^{1} + \;\ln\left(\frac{L}{a}\right) + \frac{1}{2a} + \frac{1}{2L} + \triangle E^{a}(L)^{1} + \ln\left(\frac{\infty}{L}\right) + \frac{1}{2L} + \frac{1}{2\infty}$
$= \;\ln(\infty) \;+\; \gamma$

We can conclude that $\triangle E^{1}(a)^{1} + \triangle E^{a}(L)^{1} + \triangle E^{L}(\infty)^{1} \;=\; \gamma \;-\; \frac{1}{2}$

We know from Equation (5) and Equation (6) that $\begin{array}{ll} \triangle E^{1}(a)^{1} \;\cong\; \gamma(k_1+2) \;-\; \frac{1}{2} & a \leq 5 \times 4^{k_1} \\ \triangle E^{L}(\infty)^{1} \;\cong\; \gamma \;-\; \gamma(k_2+2) & L \leq 5 \times 4^{k_2} \end{array}$

Putting these in the equation above we get,

$$\triangle E^{a}\left(L\right)^{1} \;\cong\; \gamma(k_2+2) \;-\; \gamma(k_1+2)$$

Here, $\begin{array}{l} a \leq\; 5 \times 4^{k_1} \\ L \leq\; 5 \times 4^{k_2} \end{array}$

3.2.4. Proof for Equation (8)

$S^{1}(\infty)^{d} + S^{2}(\infty)^{d} + ... + S^{d}(\infty)^{d} \;=\; S^{1}(\infty)^{1}$

From Equation (2) and Equation (1)

$\frac{\ln\left(\dfrac{\infty}{1}\right)}{d} + ... + \frac{\ln\left(\dfrac{\infty}{d}\right)}{d} + \frac{1}{2} + ... + \frac{1}{2d} + \triangle E^{1}(\infty)^{d} + ... + \triangle E^{d}(\infty)^{d} = \frac{\ln\left(\infty^{d}\right)}{d} + \gamma$

We conclude that

$\frac{\ln\left(\dfrac{\infty^{d}}{d!}\right)}{d} + \frac{S^{1}(d)^{1}}{2} + \triangle E^{1}(\infty)^{d} + ... + \triangle E^{d}(\infty)^{d} \;=\; \frac{\ln\left(\infty^{d}\right)}{d} + \gamma$

And

$\triangle E^{1}(\infty)^{d} + ... + \triangle E^{d}(\infty)^{d} \;=\; \frac{\ln(d!)}{d} + \gamma \;-\; \frac{S^{1}(d)^{1}}{2}$

Because the number of errors is equal to 'd', we assume that each error is equal and divide both sides by 'd'

$\triangle E^{1}(\infty)^{d} \;\cong\; \frac{\ln(d!)}{d^2} \;+\; \frac{\gamma}{d} \;-\; \frac{S^{1}(d)^{1}}{2d}$

From Equation (2) and Equation (5)

$\triangle E^{1}(\infty)^{d} \;\cong\; \frac{\ln(d!)}{d^2} + \frac{\gamma}{d} \;-\; \frac{\ln(d) + \dfrac{1}{2d} + \gamma(k+2)}{2d}$

Therefore,

$$\triangle E^{1}(\infty)^{d} \;\cong\; \frac{\ln(d!)}{d^2} - \frac{\ln(d)}{2d} - \frac{1}{4d^2} + \frac{\gamma}{d} - \frac{\gamma(k+2)}{2d}$$

3.2.5. Proof for Equation (9)

$S^a(\infty)^d + S^{a+d}(\infty)^d$ + ... + $S^{a+d-1}(\infty)^d = S^a(\infty)^1$

Using Equation (2) and Equation (6)

$$\frac{\ln\left(\dfrac{\infty^d \times (a-1)!}{(a+d-1)!}\right)}{d} + \frac{S^a(a+d-1)^1}{2} + \triangle E = \ln\left(\frac{\infty}{a}\right) + \frac{1}{2a} + \gamma - \gamma(k_1+2)$$

$$\triangle E = \frac{\ln\left(\dfrac{\infty^d}{a^d}\right)}{d} - \frac{\ln\left(\dfrac{\infty^d \times (a-1)!}{(a+d-1)!}\right)}{d} + \frac{1}{2a} - \frac{S^a(a+d-1)^1}{2} + \gamma - \gamma(k_1+2)$$

Here $\triangle E = \triangle E^a(\infty)^d + \triangle E^{a+d}(\infty)^d +$ .... + $\triangle E^{a+d-1}(\infty)^d$

Using Equation (2)

$$\triangle E = \frac{\ln\left(\dfrac{(a+d-1)!}{a^d \times (a-1)!}\right)}{d} - \frac{\ln\left(\dfrac{a+d-1}{a}\right)}{2} + \frac{1}{4a} - \frac{1}{4(a+d-1)} + \gamma - \frac{\gamma(k_1+2)}{2} - \frac{\gamma(k_2+2)}{2}$$

Here $a+d-1 \leq 5 \times 4^{k_2}$

Because the number of errors is equal to 'd', we assume each error to be equal and divide both sides by 'd' to find the approximate value of one error. We also neglect the difference in error created by the last term being 'L' instead of infinity.

$$\triangle E^a(L)^d \cong \frac{\ln\left(\dfrac{(a+d-1)!}{a^d \times (a-1)!}\right)}{d^2} - \frac{\ln\left(\dfrac{a+d-1}{a}\right)}{2d} + \frac{d-1}{4ad(a+d-1)} + \frac{\gamma}{d} - \frac{\gamma(k_1+2)}{2d} - \frac{\gamma(k_2+2)}{2d}$$

Here $a \leq 5 \times 4^{k_1}$
$a+d-1 \leq 5 \times 4^{k_2}$

3.3. Proof for correction factor of Leibniz series

$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ ....

$= (\frac{1}{1} + \frac{1}{5} + \frac{1}{9} +$ ....$) - (\frac{1}{3} + \frac{1}{7} + \frac{1}{11} +$ ....$)$

If we assume that 'x' number of terms have been calculated manually, we cas use the formula in Equation (3) to find the rest of the sum.

By Equation (3)

$$\frac{\pi}{4} \cong \left[\frac{\ln\left(\dfrac{\infty}{x}\right)}{4} + \frac{1}{2x} + S^1(x-4)^4\right] - \left[\frac{\ln\left(\dfrac{\infty}{x+2}\right)}{4} + \frac{1}{2(x+2)} + S^3(x-2)^4\right]$$

Solving this we get

$$\frac{\pi}{4} \cong \frac{\ln\left(\dfrac{\mathrm{x}+2}{\mathrm{x}}\right)}{4} + \frac{1}{\mathrm{x}(\mathrm{x}+2)} + \mathrm{S}^1(\mathrm{x}-4)^4 - \mathrm{S}^3(\mathrm{x}-2)^4$$

Now if we calculate all terms of the Leibniz series up to and including (x-2), we will have found the value of $S^1(x-4)^4 - S^3(x-2)^4$

The remaining portion of the equation can be used as the correction factor

Therefore,

$$CF \;=\; \frac{\ln\left(\dfrac{x+2}{x}\right)}{4} + \frac{1}{x(x+2)}$$

Here, $CF \;=\; correction\ factor$
$x \;=\; First\ term\ that\ isnt\ calculated\ manually$

3.4. Verification

Some Important Notations :- $\triangle E \;=\; Percent\ Error$
$A \;=\; Percent\ Accuracy$

Let's consider some Harmonic Series

- $1 \;+\; \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{10}$
  $a = 1\ ;\ d \;=\; 1\ ;\ n = 10\ ;\ L \;=\; 10$

$S^1(10)^1 \;=\; 2.928968254$
$(calculated\ manually)$

By Formula derived from Euler's equation

$S(10) \;\cong\; \ln(10) + \gamma \;=\; 2.879800758$ $\triangle E \;=\; 01.67867\ \%$
$A \;=\; 98.32133\%$

By Equation (4)

$S(10) \;\cong\; \ln(10) \;+\; \frac{1}{20} \;+\gamma \;=\; 2.929800758$ $\triangle E \;=\; 00.02842\%$
$A \;=\; 100.02842\%$

- $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{20}$
  $a = 1\ ;\ d = 1\ ;\ n = 20\ ;\ L = 20$

$S(20) \;=\; 3.597739657$
$(calculated\ manually)$

By Equation (4)

$S(20) \;\cong\; \ln(20) + \frac{1}{40} + \gamma \;=\; 3.597947938$ $\triangle E \;=\; 0.00578\%$
$A \;=\; 99.99421\%$

By Equation (5)

$S(20) \;\cong\; \ln(20) \;+ \frac{1}{40} + \;\gamma(3) \;=\; 3.597732274$ $\triangle E \;=\; 00.00020\%$
$A \;=\; 99.99979\%$

- $\frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \;\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{50}$
  $a = 11\ ;\ d = 1\ ;\ n = 40\ ;\ L = 50$

$S^{11}(50) \;=\; 1.570237084$
$(calculated\ manually)$

By Equation (5)

$S^{11}(50) \cong \ln\left(\frac{50}{11}\right) + \frac{1}{22} + \frac{1}{100} + 0.0772 \;=\; 1.641782278$ $\triangle E \;=\; 4.55633$
$A \;=\; 104.55633$

By Equation (7)

$S^{11}(50) \;\cong\; \ln\left(\frac{50}{11}\right) + \frac{1}{22} + \frac{1}{100} + 0.0002 \;=\; 1.564782278$

$\triangle E \;=\; 0.34738$

$A \;=\; 99.65261$

- $\frac{1}{20} + \frac{1}{25} + \frac{1}{30} + \ldots\ldots\ldots\ldots + \frac{1}{100}$
  $a = 20 \;;\; d = 5 \;;\; n = 17 \;;\; L = 100$

$S^{20}(100)^5 \;=\; 0.352881264$

$(calculated\ manually)$

By Equation (7)

$S^{20}(100)^5 \;\cong\; \frac{\ln(5)}{5} + \frac{1}{40} + \frac{1}{200} + 0.00021 \;=\; 0.352097582$

$\triangle E \;=\; 0.22208$

$A \;=\; 99.77791$

By Equation (9)

$S^{20}(100)^5 \;\cong\; \frac{\ln(5)}{5} + \frac{1}{40} + \frac{1}{200} + 0.00085499 \;=\; 0.352742579$

$\triangle E \;=\; 0.03930$

$A \;=\; 99.96069$

4. Conclusions

1. Approximation of harmonic series by Euler's method has a distinct error that occurs due to the error of the series converging slowly. Using Equation (2) we developed a better formula that converts the error from slowly decreasing to the Euler-Mascheroni constant to rapidly increasing to it. This molds the error in such a way that increase in the value of 'n' decreases the error.
2. Although the equations derived from the infinite harmonic series are obtained using experimental data, it certainly holds true for practical purposes. The value of error increases in a certain pattern differing with the type of harmonic series. Different equations are a formula and solution for these different types of harmonic series.
3. Although the equation when the common difference has a value other than one, it proves useful in limiting the error and increases the accuracy in a significant way. We use the same formula for both when last term is a finite and when it is infinity.
4. The value of correction factor derived in this paper is very useful. If a thousand terms of the Leibniz series are calculated, then the rest of terms can be accounted for by the correction factor. Also, worth noting is the fact that the correction factor when added to the sum of the thousand terms will always result in a number lesser than Pi by 4. What this signifies is that the sum added to the correction factor will never exceed but slowly converge to the value of Pi by 4.

4.1. Future Research and Potential

1. As mentioned in the abstract, a newer and better approximation of harmonic series is always needed. While, this might not provide a completed research, it is a step in the right way.
2. Many other infinite harmonic series might also have correction factors that can be calculated using the same method, using Equation (3) .

5. List of abbreviations

HS = harmonic series
$S^a(L)^d \ = \ sum\ of\ hs\ where$
$a = first\ term\ ;\ d = common\ difference\ ;\ L \ = \ last\ term$
$of\ the\ corresponding\ arithmetic\ progression$

References

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