# ULAM NUMBERS HAVE ZERO DENSITY

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Abstract. In this paper we show that the natural density $\mathcal{D}[(U_m)]$ of Ulam numbers $(U_m)$ satisfies $\mathcal{D}[(U_m)] = 0$. That is, we show that for $(U_m) \subset [1, k]$ then

$$\lim_{k \longrightarrow \infty} \frac{|(U_m) \cap [1, k]|}{k} = 0.$$

## 1. Introduction

The notion of Ulam numbers was first introduced by the Polish mathematician Stanislaw Ulam, in 1964 [2]. Let us denote, as is standard, the sequence of Ulam numbers by $(U_n)$, then each term in the sequence of Ulam numbers has the unique representation as the sum of two prior distinct Ulam numbers, and it is the smallest such number. More precisely, Ulam numbers is a sequence of **distinct** numbers of the form $1, 2, 3, 4, 6, \ldots, U_i, U_{i+1}, \ldots$, where each term in the sequence is distinct and has the unique representation $U_i = U_j + U_k$ for $i - 1 \geq j > k$ and $U_i$ is the smallest such number. The main problem of the sequence of Ulam numbers very much concerns their natural density. This problem is now known as the Ulam density problem, which can be stated as

*Question* 1.1. Do the Ulam numbers have positive density?

Ulam is said to have conjectured that the density of these numbers is zero. In this paper we answer this question in the negative by showing that

**Theorem 1.2.** *Let $(U_m)$ be the infinite sequence of Ulam numbers and denote by $D[(U_m)]$ their natural density. Then we have the relation*

$$\mathcal{D}[(U_m)] = 0.$$

## 2. Overview and structure of the paper

In this section we provide a summary sketch with some of the ingredients employed in establishing the main results of the paper. We lay them down in a chronologically in the sequel.

- First we recall the notion of an addition chain producing a given number and their corresponding regulators and determiners.

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- Next we recall an inequality of the length of an addition chain upper and lower bounded by an expression involving the least and the worst regulators of the chain.

- We recall the notion of the Ulam numbers and prove the infinitude of those numbers. That is, we show that those numbers increases without bound using a certain well-known construction. Additionally we prove that the gap between any consecutive Ulam numbers can be made arbitrarily large.

- Next we show that we can embed any finite sequence of Ulam numbers into a certain addition chain.

- Applying the a piori inequality we can now get control on the cardinality of the covered finite Ulam numbers by the length of the chain, which in turn can be control above by the gain of the contest between the unit left translate over the worst Ulam number in the sequence of the least scale of the regulators and below the same gain of the unit left translate of the worst number in the sequence over the worst regulator of the chain.

- The previous step allows us to write the length of this addition chain producing the largest Ulam number as the gain over the contest betwen the unit left translate of the largest Ulam number in the finite sequence over a certain function depending on the index of the worst Ulam number in the chain.

- We now produce the localized natural density function of the Ulam numbers considered and take limits on both sides of the resulting inequality. We are left with understanding the behaviour of the function majorizing the density function. The result of the previous steps allows us to take this function arbitrarily small thereby squeezing the density of the Ulam numbers

## 3. **The notion of an addition chains**

In this section we recall the notion of an addition chain and the notion of the regulators and associated determiners and prove an inequality introduced earlier on by the author.

**Definition 3.1.** Let $n \geq 3$, then by the addition chain of length $k-1$ producing $n$, we mean the finite sequence

$$1, 2, \ldots, s_{k-1}, s_k = n$$

where each term $s_j$ $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ for $2 \leq i \leq k$, where $a_i = s_{i-1}$. We recall the partition the $i$th generator of the chain. We call the sequence $r_i$ and $a_i$ the regulator and the determiner of the $i$ th generator of the chain. We call the sequence $(a_i)$ and $(r_i)$ the determiners and the regulators of the addition chain for $2 \leq i \leq k$.

*Remark* 3.2. Next we recall and reprove an important inequality in our inquiry. It puts a a threshold-upper and lower-on the length of any addition chain. We first prove an identity of the partial sums of the regulators of an addition chain with a given argument $n$.

[1]

**Theorem 3.3.** *Let* $1, 2, \ldots, s_{k-1}, s_k = n$ *be any addition chain producing* $n$ *with* $n \geq 3$*. Then the identity holds*

$$\sum_{j=2}^{k} r_j = n - 1.$$

*Proof.* First we observe that $r_k = n - a_k$. It follows that

$$\begin{aligned} r_k + r_{k-1} &= n - a_k + r_{k-1} \\ &= n - (a_{k-1} + r_{k-1}) + r_{k-1} \\ &= n - a_{k-1}. \end{aligned}$$

Again we obtain the relation

$$\begin{aligned} r_k + r_{k-1} + r_{k-2} &= n - a_{k-1} + r_{k-2} \\ &= n - (a_{k-2} + r_{k-2}) + r_{k-2} \\ &= n - a_{k-2}. \end{aligned}$$

By iterating downwards in this manner and noting that $a_2 = 1$ establishes the identity. □

*Remark* 3.4. Next we write down an expression for the length of any addition chain incorporating the arguments and a certain implicit function locally bounded by the worst and the least scale of the regulators of the chain. It is a consequence of the following inequality.

**Proposition 3.5.** Let $1, 2, \ldots s_{k-1}, s_k = n$ be any addition chain producing $n \geq 3$ with associated generators

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

If the length of the chain is $\delta(n)$, then there exist some $\mathrm{Inf}(r_i)_{i=2}^{\delta(n)+1} \leq \mathcal{C} := \mathcal{C}(n) \leq \sup(r_i)_{i=2}^{\delta(n)+1}$ such that

$$\delta(n) = \frac{n-1}{\mathcal{C}}.$$

*Proof.* By denoting the length of the addition producing $n$ by $\delta(n)$ and using the identity in Theorem 3.3, we obtain the inequality

$$\frac{n-1}{\sup(r_i)_{i=2}^{\delta(n)+1}} \leq \delta(n) \leq \frac{n-1}{\mathrm{Inf}(r_i)_{i=2}^{\delta(n)+1}}$$

by noting that the regulators in the chain with multiplicity counts as the length of the chain producing $n$. The result follows immediately from the above inequality. □

[1]Visionary Ulam conjectured absolutely right; Ulam numbers are very special but can be covered.

**Lemma 3.6.** *Let $\iota(n)$ denote the shortest addition chain producing $n$. Then we have the inequality*

$$\log_2(n)+\log_2(\nu(n))-2.13\leq \iota(n)\leq \log_2(n)+\frac{(1+o(1))\log_2(n)}{\log_2(\log_2(n))}$$

*where $\nu(n)$ is the hamming weight - the number of ones of the binary expansion of $n$.*

*Proof.* For a proof see [3]. □

*Remark* 3.7. Albeit determining an exact expression for the implicit constant in Proposition 3.5 is by no means an easy tussle, we can however obtain a lower bound for the purposes of our work.

**Proposition 3.8.** Let $1,2,\ldots s_{k-1},s_k=n$ be any addition chain producing $n\geq 3$ with associated generators

$$2=1+1,\ldots,s_{k-1}=a_{k-1}+r_{k-1},s_k=a_k+r_k=n.$$

If the length of the addition chain producing $n$ is $\delta(n)$, then there exist some $\mathrm{Inf}(r_i)_{i=2}^{\delta(n)+1}\leq \mathcal{C}:=\mathcal{C}(n)\leq \sup(r_i)_{i=2}^{\delta(n)+1}$ such that

$$\delta(n)=\frac{n-1}{\mathcal{C}}$$

where

$$\mathcal{C}\gg \frac{n}{\log_2(n)}.$$

*Proof.* The first part of the result has already been proven in Proposition 3.5. In particular, we can write

$$\mathcal{C}=\frac{n-1}{\delta(n)}$$

where $\delta(n)$ runs over all addition chains producing $n$. It follows that

$$\mathcal{C}\geq \inf(\mathcal{C})\gg \frac{n}{\log_2(n)}$$

by appealing to the upper bound in Lemma 3.6. □

## 4. The notion of Ulam numbers

In this section we recall the concept of Ulam numbers and review some of its properties. We recall the well-known construction that confirms the infinitude of these numbers. First we recall the following definitions.

**Definition 4.1.** By Ulam numbers we mean sequence of **distinct** numbers of the form $1,2,3,4,6,\ldots,U_i,U_{i+1},\ldots$, where each term in the sequence is distinct and has the unique representation $U_i=U_j+U_k$ for $i-1\geq j>k$ and $U_i$ is the smallest such number.

Next we ascertain the infinitude of the sequence of Ulam numbers. The following construction is well-known and standard, yet we do not feel hesitant to reproduce it here [1].

**Lemma 4.2.** *There are infinitely many Ulam numbers $(U_m)_{m\geq 1}$.*

*Proof.* Suppose the first $n$ Ulam numbers have already been determined, namely $1, 2, 3, 4, \ldots, U_{n-1}, U_n$. Then the representation $U_n + U_{n-1}$ is unique and the number so represented in this form could be the next Ulam number. If not then this number is not the smallest such number and since there are other numbers with such unique representations, we choose the smallest from among them bigger than $U_n$ and assigns to $U_{n+1}$ as the next Ulam number. This construction can then be repeated indefinitely thereby generating an infinite sequence of Ulam numbers. This completes the proof. □

**Lemma 4.3.** *No Ulam number $U_m$ for $m > 3$ can be the sum of it's prior consecutive Ulam numbers.*

*Proof.* Suppose on the contrary that $U_{n-1} + U_n = U_{n+1}$. Then necessarily the representation $U_n + U_{n-2}$ must be unique. Suppose it is not unique, then there exist some $U_i < U_{n-2}$ and $U_j > U_n$ such that

$$\begin{aligned} U_n + U_{n-2} &= U_i + U_j \\ &> U_{n+1} \\ &= U_n + U_{n-1} \end{aligned}$$

and it follows that $U_{n-2} > U_{n-1}$, which is absurd. Now we observe that

$$U_n \leq U_n + U_{n-2} < U_{n+1}$$

contradicting the fact that $U_{n+1}$ is the next Ulam number. □

*Remark* 4.4. Next we show that we can squeeze any finite sequence of Ulam numbers $(U_n)$ into a certain addition chain by carefully choosing the regulators of the chain.

**Proposition 4.5.** Let $(U_m)_{m=1}^{n}$ be a finite sequence of Ulam numbers. Then there exist an addition chain $(s_k)$ producing $U_n$ such that

$$(U_m)_{m=1}^{n} \subseteq (s_k).$$

*Proof.* Let $1, 2, 3, 4, \ldots, U_n$ be a finite sequence of Ulam numbers. Then for each term $U_m$ for $m \geq 1$, we choose the regulator $r_j \geq 1$ such that $U_m + r_j \leq U_{m+1}$. If it is the case that $U_m + r_j = U_{m+1}$ then the consecutive sequence $U_m, U_{m+1}$ is also a consecutive sequence in the sought-after addition chain. If not then we continue this process by choosing the regulator $r_i \geq 1$ such that $U_m + r_j + r_i = U_{m+1}$. Then in such a case the consecutive Ulam numbers $U_m, U_{m+1}$ are not consecutive numbers in the corresponding addition chain. This construction can be carried out to generate an addition chain producing $U_n$ and yet covering the finite sequence of Ulam numbers. This completes the proof of the proposition. □

## 5. **Density of Ulam numbers**

In this section we show that Ulam numbers indeed have natural density zero. By denoting the natural density of Ulam numbers $(U_m)$ of the form $\mathcal{D}[(U_m)]$, we obtain the following result.

**Theorem 5.1.** *Let $(U_m)$ be the infinite sequence of Ulam numbers and denote by $D[(U_m)]$ their natural density. Then we have the relation*

$$\mathcal{D}[(U_m)] = 0.$$

*Proof.* First let us construct the first $n$ sequence of Ulam number $1, 2, 3, \ldots, U_{n-2}, U_{n-1}, U_n$. Then by Proposition 4.5 there exists at least one addition chain $(s_k)$ producing $U_n$ that covers the original enumerated sequence of Ulam numbers. Then we obtain the following relation

$$n \leq \delta(U_n) + 1$$
$$= \frac{U_n - 1}{\mathcal{C}(n)} + 1$$

by virtue of Proposition 3.5. For any $l \geq U_n > n$, we have

$$\frac{n}{l} \leq \frac{U_n - 1}{l\mathcal{C}(n)} + \frac{1}{l}$$
$$\leq \frac{1}{\mathcal{C}(n)} - \frac{1}{l\mathcal{C}(n)} + \frac{1}{l}$$
$$\leq \frac{1}{\mathcal{C}(n)} + \frac{1}{l}.$$

Taking limits $n \longrightarrow \infty$ on both sides we have

$$\mathcal{D}[(U_m)_{m=1}^{\infty}] \leq \lim_{n \longrightarrow \infty} \frac{1}{\mathcal{C}(n)}$$
$$\ll \lim_{n \longrightarrow \infty} \frac{\log_2(n)}{n}$$
$$= 0$$

by appealing to Proposition 3.8. This completes the proof of the theorem. □

## References


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