Pythagoras theorem is an alternate form of Ptolemy's theorem

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Abstract

Generally, the proofs given to demonstrate Ptolemy's theorem prove Pythagoras theorem as a special case of Ptolemy's theorem when certain special conditions are imposed. We prove in this article, that Pythagoras theorem follows from Ptolemy's theorem in all cases. Therefore, we may say that Pythagoras rediscovered Ptolemy's theorem.

Key words

Ptolemy's theorem, geometry, Pythagoras theorem, cyclic quadrilateral, mean proportional

Ptolemy's theorem is a well-known theorem in geometry. Many different methods of proof are available in literature[1-4]. In a recent article[1] we gave a geometrical proof of Ptolemy's theorem. Therein we showed that the products of pairs of opposite sides of a cyclic quadrilateral can be geometrically represented by rectangles. Similarly the product of the diagonals of the same quadrilateral can be represented by another rectangle. Then we showed that the sum of the areas of the rectangles obtained from the products of sides is equal to the area of the rectangle obtained from the product of the diagonals.

In this article we show the areas products of pairs of opposite sides as well as the product of the diagonals of a cyclic quadrilateral can be geometrically represented by circles. Then we show that the sum of the areas of the circles obtained from the products of sides is equal to the area of the circle obtained from the product of the diagonals. To obtain the radii of these circles we take the mean proportional of the pairs of opposite sides of the cyclic quadrilateral and the mean proportional of the pair of its diagonals.

Statement of Ptolemy's theorem[2]

Ptolemy's theorem states that, the sum of the products of pairs of opposite sides of a cyclic quadrilateral is equal to the product of the diagonals.

Proof

Draw any circle. Choose any four points A, B, C and D on it. Join A, B; A, C; A, D; and B, D. Now ABCD is a cyclic quadrilateral. AB, CD is a pair of opposite sides, AD, BC is the other pair of opposite sides and AC, BD are the diagonals (see Fig. 1).

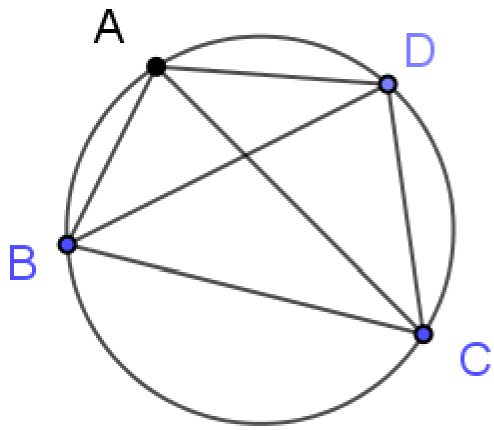

Fig. 1. ABCD is a cyclic quadrilateral. Ptolemy's theorem says, AB.DC + AD.BC = AC.BD.

We can write Ptolemy's theorem in the form of equation (1) below.

$$AB \times CD + AD \times BC = AC \times BD \quad (1)$$

Instead of taking the products of the line segments, we can take their mean proportionals. Let r_1 be the mean proportional of the pair of opposite sides AB, CD; r_2 be the mean proportional of the pair of opposite sides AD, BC and r_3 be the mean proportional of the diagonals AC, BD. Then we get,

$$AB \times CD = r_1^2 \quad (2)$$

$$AD \times BC = r_2^2 \quad (3)$$

$$AC \times BD = r_3^2 \quad (4)$$

We can now state Ptolemy's theorem in an alternate form as follows:

Alternate statement of Ptolemy's theorem

The sum of the squares of the mean proportionals of the pairs opposite sides of a cyclic quadrilateral is equal to the square of the mean proportional of the diagonals. We can write this alternate form of statement of Ptolemy's theorem in the form of Eq (5) below.

$$(r_1^2 + r_2^2) = r_3^2 \quad (5)$$

Equation (5) is a statement of Pythogorus' theorem.

Proof

We construct mean proportionals of the line segments as needed.

We move CD in Fig. 1 so that C coincides with A and, AB, CD form a straight line BAD' (see Fig. 2).

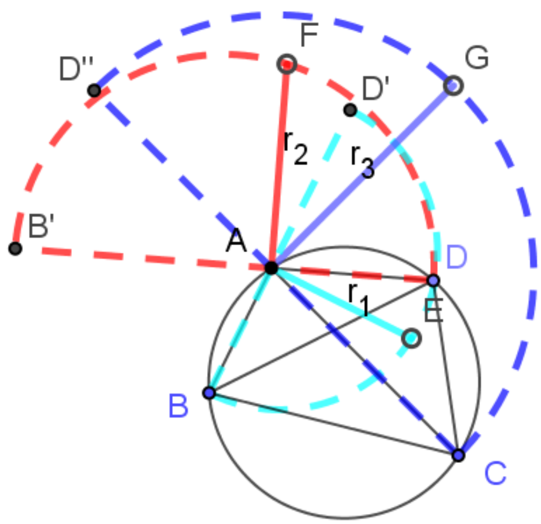

Fig. 2. Mean proportionals of AB, CD; AD, BC and AC, BD are constructed. The results are shown in green, red and blue colored solid lines respectively.

Draw a semicircle on BD' and draw a perpendicular to BD' at A (see Fig. 2). Let it cut the semicircle at E. AE is the meanproportional r_1 of AB and CD.

Similarly, we move BC so that C coincides with A and, AD, CB form a straight line B'AD. Draw a semicircle on B'D and draw a perpendicular to B'D at A (see Fig. 2). Let it cut the semicircle at F. AF is the meanproportional r_2 of AD and BC.

Finally, we move BD so that B coincides with A and, AC, BD form a straight line CAD". Draw a semicircle on CD" and draw a perpendicular to CD" at A (see Fig. 2). Let it cut the semicircle at G. AG is the meanproportional r_3 of AC and BD.

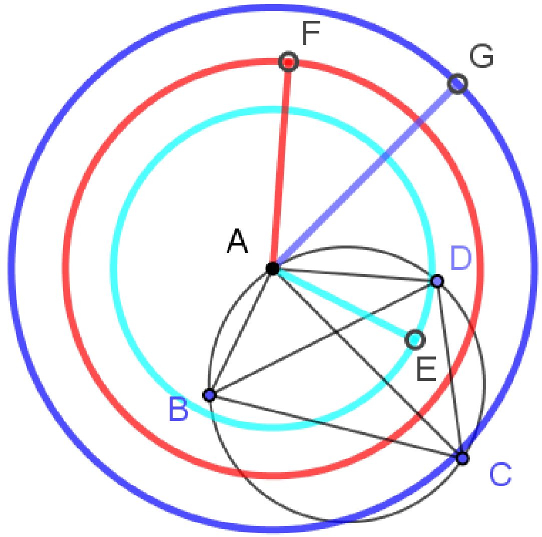

Fig. 3. Circles are drawn with A as center and the mean proportionals as radii.

We draw the three concentric circles with A as center and radii r_1 (green), r_2 (red), r_3 (blue). The areas of these circles are πr_1^2, πr_2^2, πr_3^2. From Eq. (5) we see that the sum of the areas of the two inner circles is equal to the area of the outer most circle. To demonstrate that it is so, we proceed as follows.

Draw a circle with AG as diameter (see Fig. 4). From the opposite ends A, G of AG draw circles with radii, r_1, r_2. These circles intersect at four points marked P_1, P_2, P_1', P_2', We join A, P_1; P_1, G; A, P_2'and G P_2'. We get the cyclic quadrilateral A P_1GP_2'. We apply Ptolemy's theorem to this quadrilateral and get Eq. (5).

Since the radii r_1, r_2 are mean proportionals of the pairs of opposite sides, and r_3, the mean proportional of the diagonals of the cyclic quadrilateral ABCD it follows that the sum of the squares of the mean proportionals of the pairs of opposite sides, is equal to the square of the mean proportional of the diagonals of a cyclic quadrilateral. Thus we find Pythogorus' theorem is an alternate form Ptolemy's theorem. Therefore, we may say Pythogorus rediscovered Ptolemy's theorem.

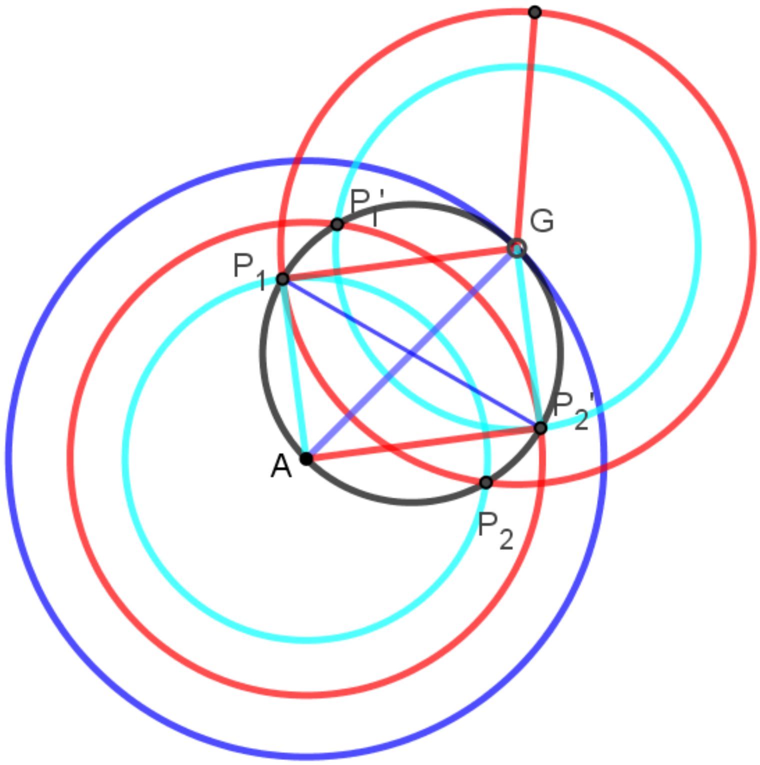

Fig. 4. Circles are drawn with AG as diameter. From the opposite ends A and G as centers, circles are drawn with radii r_1 and r_2. The points of intersection are marked as P_1, P_2, P_1', P_2'. Cyclic quadrilateral AP_1GP_2' is drawn.

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