A vector interpretation of Quaternion Mass Function

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Abstract

Mass function vector is used to handle uncertainty. Quaternion number is the extent of real number. The mass function vector can extend the mass function by combining the vector. In this paper, the mass function vector is extended by quaternion number, named as Quaternion Mass Function Vector(QMFV). The proposed QMFV has the advantage to deal with uncertain information. When the quaternion number degenerates into the real number, then the QMFV degenerates into the quaternion mass function. In addition, if the probability of multiple subsets of frame of discernment is not assigned to the single subsets, then the mass function vector will degenerate into mass function in classical evidence theory. When the quaternion number degenerates into the real number, then the combination rule of quaternion mass function vectors degenerates into the combination rule of mass function vectors. In the case when the probability of multiple subsets of frame of discernment is not assigned to the single subsets, the combination rule of mass function vectors degenerates into generalized dempster's rule of combination. Numerical examples are applied to prove the efficiency of the proposed model. The experimental results show that the proposed model can apply the quaternion theory to mass function vector effectively and successfully.

Keywords: Quaternion, Mass function, Quaternion mass function, Quaternion mass function vector, Vector

1. Introduction

There are many uncertainties in the real world. In order to deal with the uncertain issues, many mathematical models and theories have been proposed, such as quaternion [1], evidence theory [2, 3], belief entropy [4], belief function [5], quaternion beyesian updating [6], quaternion mass function [7]. Among those models and theories, the quaternion mass function is very efficient in representing uncertainty, which is based on the quaternion theory. The quaternion

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mass function doesn't conform to the multiplication commutative law. Relying on the advantages on representing uncertainty, the quaternion mass function have been widely studied by scholars at home and abroad.

Recently, Luo and Deng proposed the mass function vector [8], which has a meaningful model. The mass function vector can interpret the mass function as a vector clearly with the aid of vector. However, what is the QMFV is still an open issue.

This paper proposed the QMFV, which is an extent of mass function vector under quaternion environment. QMFV can interpret the quaternion mass function as a vector. Based on the QMFV, this paper proposed the combination rule of QMFVs, which can combine the QMFVs effectively. Meanwhile, the combination rule of QMFVs doesn't conform to the multiplicatin commutative law.

The remain of this paper is structured as follows. Section 2 introduces the preliminary. Section 3 presents the QMFV. Section 4 illustrates the flexibility of QMFV. Section 5 summarizes the whole paper.

2. Preliminaries

In this section, mass function, quaternion are briefly introduced.

2.1. Quaternion

Quaternion numbers are a four dimensional hypercomplex numbers system. The definition of quaternion numbers as follow:

Definition 2.1. *(Quaternion) [1]*

$$q = a + bi + cj + dk \tag{1}$$

Given two quaternion numbers, $q_1 = a+bi+cj+dk$ and $q_2 = e+fi+gj+hk$, the definition of quaternion numbers multiplication as follow:

Definition 2.2. *(Quaternion Multiplication) [1]*

$$\begin{aligned} q_1 q_2 =&(ae - (bf + cg + dh))+ \\ &(be + af + ch - dg)i+ \\ &(ce + ag + df - bh)j+ \\ &(de + ah + bg - cf)k \end{aligned} \tag{2}$$

2.2. Frame of Discernment

Given a frame of discernment $\Omega = \{x_1, x_2, \dots, x_n\}$, the power set of frame of discernment is defined as follows:

Definition 2.3. *(Power Set of Frame of Discernment) [2]*

$$2^\Omega = \{\emptyset, \{x_1\}, \{x_2\}, \dots, \{x_n\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_i\}, \dots, \Omega\} \tag{3}$$

2.3. Mass Function

Given a frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, the mass function, m, on 2^Ω is defined as follows:

Definition 2.4. *(Mass Function) [2]*

$$m : 2^\Omega \rightarrow [0, 1] \tag{4}$$

Where, $m(\emptyset) = 0$ *and* $\Sigma_{B \in 2^\Omega} m(B) = 0$ *with a focal element,* B*, of* 2^Ω.

2.4. Dempster's Rule of Combination

Given two mass functions, m_1 and m_2, in Ω. The dempster'rule of combination, $m = m_1 \oplus m_2$, between m_1 and m_2 is defined as follows:

Definition 2.5. *(Dempster's Rule of Combination) [2]*

$$m^{(12)}(A) = \begin{cases} \frac{1}{1-K} \sum_{E \bigcap F = A} m^{(1)}(E) m^{(2)}(F) & A \neq \emptyset \\ 0 & A = \emptyset \end{cases} \tag{5}$$

Where, $K = \sum_{E \bigcap F = \emptyset} m^{(1)}(E) m^{(2)}(F)$.

If we ignore the normalization factor, the above formula can be simplified as [8]

$$m^{(12)}(A) = \sum_{E \bigcap F = A} m^{(1)}(E) m^{(2)}(F) \tag{6}$$

2.5. Quaternions Mass Function

In order to extend the mass function to four-dimensional space, this paper proposes the quaternion mass function, which includes several operations.

Given a frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, the quaternion mass function, QM, on Ω is defined as follows:

Definition 2.6. *(Quaternion Mass Function) [7]*

$$QM(\emptyset) = 0 \tag{7}$$

$$\Sigma_{B \in 2^\Omega} QM(B) = 1 \tag{8}$$

$$QM(B) = m(B) e^{u\Theta}, B \in 2^\Omega \tag{9}$$

Where, $u = u_x i + u_y j + u_z k$ *with* $i = j = k = \sqrt{-1}$*. Assume* $m(B)$ *is the magnitude of the quaternion mass function* $QM(B)$*.* $\Theta \in [0, 2\pi]$.

2.6. Combination rule of Quaternions Mass Function

Given two mass functions, QM_1 and QM_2, in Ω. The Combination rule of Quaternions Mass Function, $QM = QM_1 \oplus QM_2$, between QM_1 and QM_2 is defined as follows:

Definition 2.7. *(Combination rule of Quaternions Mass Function) [7]*

$$QM(D) = \begin{cases} \frac{1}{1-QK} \sum_{E \bigcap F = D} QM_1(E)QM_2(F) & D \neq \emptyset \\ 0 & D = \emptyset \end{cases} \tag{10}$$

Where, $QK = \sum_{E \bigcap F = \emptyset} QM(E)QM(F)$. $E, F \in 2^{\Omega}$ *and* QK *is the conflict coefficient between* QM_1 *and* QM_2.

2.7. Mass function vector

Given a frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, the mass function vector M under a mass function based on Θ is defined as follows:

Definition 2.8. *(mass function vector) [8]*

$$M = (M_1, M_2, \ldots, M_N)^T \tag{11}$$

Where

$$M_j = \sum_{A_i \subseteq \Theta} m(A_i)\kappa_{(H_J|A_i)} \ (j = 1, 2, \ldots, N) \tag{12}$$

The variable parameter $\kappa_{(H_J|A_i)}(j = 1, 2, \ldots, N; i = 1, 2, \ldots, 2^N)$ *satisfies the following conditions:*

Remark 1. 1) $\kappa_{(H_j|A_i)} = 0$, *if* $H_j \notin A_i$.
2) $\kappa_{(H_j|A_i)} \in (0, 1]$, *if* $H_j \in A_i$.
3) $\sum\limits_{H_j \in A_i} \kappa_{(H_j|A_i)} = 1$, *for a fixed* $A_i \subseteq \Theta$.

2.8. Combination rule of mass function vectors

Given two mass function vectors on a given frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $M^{(1)} = (M_1^{(1)}, M_2^{(1)}, \ldots, M_N^{(1)})$ and $M^{(2)} = (M_1^{(2)}, M_2^{(2)}, \ldots, M_N^{(2)})$. The combination of $M^{(1)}$ and $M^{(2)}$, $M^{(12)}$ can be defined as follows:

Definition 2.9. *(Combination rule of mass function vectors) [8]*

$$M^{(12)} = M^{(1)} \otimes M^{(2)} = (M_1^{(1)} \cdot M_1^{(2)}, M_2^{(1)} \cdot M_2^{(2)}, \ldots, M_N^{(1)} \cdot M_N^{(2)}) \tag{13}$$

Where

$$M_k^{(1)} \cdot M_k^{(2)} = \sum_{i=1}^{2^N} \sum_{j=1}^{2^N} m^{(1)}(A_i)\kappa_{(H_k|A_i)} \cdot m^{(2)}(A_j)\kappa_{(H_k|A_j)} \tag{14}$$

2.9. The distance between mass function vectors

Given two mass function vectors on a given frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $M^{(1)} = (M_1^{(1)}, M_2^{(1)}, \ldots, M_N^{(1)})$ and $M^{(2)} = (M_1^{(2)}, M_2^{(2)}, \ldots, M_N^{(2)})$. The distance of $M^{(1)}$ and $M^{(2)}$ is defined as follows:

Definition 2.10. *(The distance between mass function vectors) [8]*

$$D^2(M^{(1)}, M^{(2)}) = \sum_{i=1}^{N} (M_i^{(1)} - M_i^{(2)})^2 \tag{15}$$

3. The proposed method

3.1. Quaternion mass function vector

Given a frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, the QMFV, $QMFV$, under a quaternion mass function QMF based on Θ is defined as follows:

Definition 3.1. *(Quaternion mass function vector)*

$$QMFV = (QMFV_1, QMFV_2, \ldots, QMFV_N)^T \tag{16}$$

Where

$$QMFV_j = \sum_{A_i \subseteq \Theta} QMF(A_i)\kappa_{(H_J|A_i)} \ (j = 1, 2, \ldots, N) \tag{17}$$

$$\sum_{j=1}^{N} QMFV_j = 1 \tag{18}$$

The variable parameter $\kappa_{(H_j|A_i)}(j = 1, 2, \ldots, N; i = 1, 2, \ldots, 2^N)$ satisfies the following conditions:

Remark 2. 1) $\kappa_{(H_j|A_i)} = 0$, *if* $H_j \notin A_i$.
2) $\kappa_{(H_j|A_i)} \in (0, 1]$, *if* $H_j \in A_i$.
3) $\sum_{H_j \in A_i} \kappa_{(H_j|A_i)} = 1$, *for a fixed* $A_i \subseteq \Theta$.

Theorem 3.1. *Given a frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $QMFV$ is a QMFV under a quaternion mass function QMF based on Θ. If the QMF degenerates into the classical mass function m, then the $QMFV$ will be degenerated as mass function vector.*

Proof 3.1. *Since QMF is degenerated as a classical mass function m, relying on the Eq.(17), then we can obtain the following equation:*

$$\begin{aligned} QMFV_j &= \sum_{A_i \subseteq \Theta} QMF(A_i)\kappa_{(H_j|A_i)} \ (j = 1, 2, \ldots, N) \\ &= \sum_{A_i \subseteq \Theta} m(A_i)\kappa_{(H_j|A_i)} \ (j = 1, 2, \ldots, N) \end{aligned}$$

Then, we can see that the above formula and Eq.(12) are equivalent. Hence, we can find that Eq.(11) and Eq.(16) are equivalent, which means that in this way the $QMFV$ degenerates into mass function vector.

□

3.2. Combination rule of quaternion mass function vectors

Given two QMFVs on a given frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $QMFV^{(1)} = (QMFV_1^{(1)}, QMFV_2^{(1)}, \ldots, QMFV_N^{(1)})$ and $QMFV^{(2)} = (QMFV_1^{(2)}, QMFV_2^{(2)}, \ldots, QMFV_N^{(2)})$. The combination of $QMFV^{(1)}$ and $QMFV^{(2)}$, $QMFV^{(12)}$ can be defined as follows:

Definition 3.2. *(Combination rule of quaternion mass function vectors)*

$$
\begin{aligned}
&QMFV^{(12)} \\
&= QMFV^{(1)} \otimes QMFV^{(2)} \\
&= (QMFV_1^{(1)} \cdot QMFV_1^{(2)}, \ldots, QMFV_N^{(1)} \cdot QMFV_N^{(2)})
\end{aligned}
\tag{19}
$$

Where

$$
\begin{aligned}
&QMFV_k^{(1)} \cdot QMFV_k^{(2)} \\
&= \sum_{i=1}^{2^N} \sum_{j=1}^{2^N} QMF^{(1)}(A_i)\kappa_{(H_k|A_i)} \cdot QMF^{(2)}(A_j)\kappa_{(H_k|A_j)} \\
&= \sum_{i=1}^{2^N} \sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)[\kappa_{(H_k|A_i)} \cdot \kappa_{(H_k|A_j)}] \\
&= \sum_{i=1}^{2^N} \sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)\kappa_{(H_k|A_i \cap A_j)}
\end{aligned}
\tag{20}
$$

When the quaternion number degenerates into the real number, the combination rule of QMFVs will degenerate into the Dempster's rule of combination.

Theorem 3.2. *Given a frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $QMFV^{(1)}$ and $QMFV^{(2)}$ are two QMFVs under a quaternion mass function QMF based on Θ. Then, $QMFV^{(12)} \neq QMFV^{(21)}$.*

Proof 3.2. *According to the Eq.(17), the following equations can be obtained:*

$$
\begin{aligned}
QMFV_k^{(1)} \cdot QMFV_k^{(2)} &= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i)\kappa_{(H_k|A_i)} \cdot QMF^{(2)}(A_j)\kappa_{(H_k|A_j)} \\
&= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)[\kappa_{(H_k|A_i)} \cdot \kappa_{(H_k|A_j)}] \\
&= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)\kappa_{(H_k|A_i \cap A_j)}
\end{aligned}
$$

$$
\begin{aligned}
QMFV_k^{(2)} \cdot QMFV_k^{(1)} &= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(2)}(A_i)\kappa_{(H_k|A_i)} \cdot QMF^{(1)}(A_j)\kappa_{(H_k|A_j)} \\
&= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(2)}(A_i) \cdot QMF^{(1)}(A_j)[\kappa_{(H_k|A_i)} \cdot \kappa_{(H_k|A_j)}] \\
&= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(2)}(A_i) \cdot QMF^{(1)}(A_j)\kappa_{(H_k|A_i \cap A_j)}
\end{aligned}
$$

Since the quaternion mass functions doesn't conform to the commutative property of multiplication, which means that $QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j) \neq QMF^{(2)}(A_i) \cdot QMF^{(1)}(A_j)$.

Then, we can obtain the equation as follows:

$$
\sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)\kappa_{(H_k|A_i \cap A_j)} \neq \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(2)}(A_i) \cdot QMF^{(1)}(A_j)\kappa_{(H_k|A_i \cap A_j)}
$$

Hence, we can get that $QMFV_k^{(1)} \cdot QMFV_k^{(2)} = QMFV_k^{(2)} \cdot QMFV_k^{(1)}$.

3.3. The distance between quaternion mass function vectors

Given two QMFVs on a given frame of discernment $\Theta = \{H_1, H_2, \ldots, H_N\}$, $QMFV^{(1)} = (QMFV_1^{(1)}, QMFV_2^{(1)}, \ldots, QMFV_N^{(1)})$ and $QMFV^{(2)} = (QMFV_1^{(2)}, QMFV_2^{(2)}, \ldots, QMFV_N^{(2)})$. The distance of $QMFV^{(1)}$ and $QMFV^{(2)}$ is defined as follows:

Definition 3.3. *(The distance between quaternion mass function vectors)*

$$
D^2(QMFV^{(1)}, QMFV^{(2)}) = \sum_{i=1}^{N} |QMFV_i^{(1)} - QMFV_i^{(2)}|^2 \tag{21}
$$

Theorem 3.3. *Given a frame of discernment* $\Theta = \{H_1, H_2, \dots, H_N\}$, $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *are two QMFV under a quaternion mass function* QMF *based on* Θ. *Then, when* $QMFV^{(1)} = QMFV^{(2)}$, *the distance between QMFVs* $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *reach the minimum value 0.*

Proof 3.3. *According to the Eq.(22), the following equations can be obtained:*

$$D^2(QMFV^{(1)}, QMFV^{(2)}) = \sum_{i=1}^{N} |QMFV_i^{(1)} - QMFV_i^{(2)}|^2 \qquad (22)$$

Since the $|QMFV_i^{(1)} - QMFV_i^{(2)}|^2 \geq 0$, *then* $\sum_{i=1}^{N} |QMFV_i^{(1)} - QMFV_i^{(2)}|^2 geq 0$.

So, the distance between QMFVs $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *reach the minimum value 0.*

When $QMFV^{(1)} = QMFV^{(2)}$, *then* $|QMFV_i^{(1)} - QMFV_i^{(2)}|^2 = 0$. *Hence, when* $QMFV^{(1)} = QMFV^{(2)}$, *the distance between QMFVs* $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *reach the minimum value 0.*

3.4. The belief function and plausibility function of quaternion mass function vector

Given a QMFV on a given frame of discernment $\Theta = \{H_1, H_2, \dots, H_N\}$, $QMFV = (QMFV_1, QMFV_2, \dots, QMFV_N)$. The belief function and plausibility function of $QMFV$ is defined as follows:

Definition 3.4. *(The belief function and plausibility function of quaternion mass function vector)*

$$Bel(H_i) = min\{QMFV_i\} \ (i = 1, 2, \dots, N) \qquad (23)$$

Where $min\{QMFV_i\}$ *means that* $|QMFV_i|$ *get the minimum value.*

$$Pl(H_i) = max\{QMFV_i\} \ (i = 1, 2, \dots, N) \qquad (24)$$

Where $max\{QMFV_i\}$ *means that* $|QMFV_i|$ *get the maximum value.*

3.5. Discussion

Mass function has high effective in dealing with uncertainties. Luo and Deng [8] proposed the mass function vector and combination rule of mass function vectors, which has promising aspects. This paper proposed the QMFV, which is an extent of mass function vector with the aid of quaternion number. When the quaternion number degenerates into the real number, then the QMFV degenerates into the quaternion mass function. In addition, if the probability of multiple subsets of frame of discernment is not assigned to the single subsets, then the mass function vector will degenerate into mass function in classical evidence theory. When the quaternion number degenerates into the real number, then the combination rule of quaternion mass function vectors degenerates into the combination rule of mass function vectors. In the case when the probability of multiple subsets of frame of discernment is not assigned to the single subsets, the combination rule of mass function vectors degenerates into generalized dempster's rule of combination.

4. Numerical examples

Example 4.1. *Supposing that there are two quaternion mass functions QM in the frame of discernment $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$. Here, $A_1 = \{x_1, x_2\}$, $A_2 = \{x_3, x_4, x_5\}$.*

$$\begin{aligned} QM :& QM(A_1) = 0.2 + 0.1i + 0.1j + 0.2k \\ & QM(A_2) = 0.3 + 0.2i + 0.2j + 0.3k \\ & QM(A_1, A_2) = 0.5 - 0.3i - 0.3j - 0.5k \end{aligned}$$

Assume the variable parameter satisfies that $\kappa_{(A_2|A_1,A_2)} = \kappa_{(A_2|A_1,A_2)} = 1/2$. Relying on the Eq.(17), the QMFV under the QM can be obtained as follows:

$$\begin{aligned} QMFV_1 &= QM(A_1) + QM(A_1, A_2)\kappa_{(A_1|A_1,A_2)} \\ &= 0.2 + 0.1i + 0.1j + 0.2k + (0.5 - 0.3i - 0.3j - 0.5k)\kappa_{(A_1|A_1,A_2)} \\ &= 0.45 - 0.05i - 0.05j - 0.05k \\ QMFV_2 &= QM(A_2) + QM(A_1, A_2)\kappa_{(A_2|A_1,A_2)} \\ &= 0.3 + 0.2i + 0.2j + 0.3k + (0.5 - 0.3i - 0.3j - 0.5k)\kappa_{(A_2|A_1,A_2)} \\ &= 0.8 + 0.05i + 0.05j + 0.05k \end{aligned}$$

Since $QMFV_1 + QMFV_2 = 1$, then the $QMFV$ is a QMFV.

Example 4.2. *Assume there are two QMFVs as follows:*

$$QMFV^{(1)} = (0.1 + 0.2i + 0.3j + 0.3k, 0.9 - 0.2i - 0.3j - 0.3k)$$

$$QMFV^{(2)} = (0.6 + 0.1i + 0.2j + 0.1k, 0.4 - 0.1i - 0.2j - 0.1k)$$

Relying on the Eq.(22), the distance between these QMFVs can be calculated as follows:

$$\begin{aligned} D^2(QMFV^{(1)}, QMFV^{(2)}) &= |QMFV_1^{(1)} - QMFV_1^{(2)}|^2 + |QMFV_2^{(1)} - QMFV_2^{(2)}|^2 \\ &= (0.1 - 0.6)^2 + (0.2 - 0.1)^2 + (0.3 - 0.2)^2 + (0.3 - 0.1)^2 \\ &+ (0.9 - 0.4)^2 + (-0.2 - (-0.1))^2 + (-0.3 - (-0.2))^2 + (-0.3 - (-0.1))^2 \\ &= 0.62 \end{aligned}$$

Example 4.3. *Supposing that there are two quaternion mass functions $QMF^{(1)}$ and $QMF^{(2)}$ in the frame of discernment $\Theta = \{H_1, H_2\}$. The quaternion mass functions are known as follows:*

$$QMF^{(1)}(H_1) = 0.1 + 0.4j + 0.2k$$
$$QMF^{(1)}(H_2) = 0.3 + 0.3i + 0.2j + 0.1k$$
$$QMF^{(1)}(H_1, H_2) = 0.6 - 0.3i - 0.6j - 0.3k$$

$$QMF^{(2)}(H_1) = 0.7 + 0.6i + 0.6j + 0.6k$$
$$QMF^{(2)}(H_1, H_2) = 0.3 - 0.6i - 0.6j - 0.6k$$

Assume the variable parameter satisfies that $\kappa_{(H_1|H_1,H_2)} = 2/3$ and $\kappa_{(H_2|H_1,H_2)} = 1/3$.

Relying on the Eq.(17), the QMFVs under the these quaternion mass functions can be obtained as follows:

$$\begin{aligned} QMFV_1^{(1)} &= QMF^{(1)}(H_1) + QMF^{(1)}(H_1, H_2)\kappa_{(H_1|H_1,H_2)} \\ &= 0.1 + 0.4j + 0.2k + (0.6 - 0.3i - 0.6j - 0.3k)(2/3) \\ &= 0.5 - 0.2i \\ QMFV_2^{(1)} &= QMF^{(1)}(H_2) + QMF^{(1)}(H_1, H_2)\kappa_{(H_2|H_1,H_2)} \\ &= 0.3 + 0.3i + 0.2j + 0.1k + (0.6 - 0.3i - 0.6j - 0.3k)(1/3) \\ &= 0.5 + 0.2i \end{aligned}$$

$$\begin{aligned} QMFV_1^{(2)} &= QMF^{(2)}(H_1) + QMF^{(2)}(H_1, H_2)\kappa_{(H_1|H_1,H_2)} \\ &= 0.7 + 0.6i + 0.6j + 0.6k + (0.3 - 0.6i - 0.6j - 0.6k)(2/3) \\ &= 0.9 + 0.2i + 0.2j + 0.2k \\ QMFV_2^{(2)} &= QMF^{(2)}(H_2) + QMF^{(2)}(H_1, H_2)\kappa_{(H_2|H_1,H_2)} \\ &= 0 + (0.3 - 0.6i - 0.6j - 0.6k)(1/3) \\ &= 0.1 - 0.2i - 0.2j - 0.2k \end{aligned}$$

Then, we can obtain the vector form of $QMDV^{(1)}$ and $QMDV^{(2)}$ as follows:

$$QMFV^{(1)} = (0.5 - 0.2i, 0.5 + 0.2i)$$

$$QMFV^{(2)} = (0.9 + 0.2i + 0.2j + 0.2k, 0.1 - 0.2i - 0.2j - 0.2k)$$

Relying on the Eq.(19), the combination of $QMFV^{(1)}$ *and* $QMFV^{(2)}$, $QMFV^{(12)}$ *as follows:*

$$
\begin{aligned}
QMFV_1^{(1)} \cdot QMFV_1^{(2)} &= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i) \cdot QMF^{(2)}(A_j)\kappa_{(H_1|A_i \cap A_j)} \\
&= QMF^{(1)}(H_1) \cdot QMF^{(2)}(H_1)\kappa_{(H_1|H_1 \cap H_1)} \\
&+ QMF^{(1)}(H_1) \cdot QMF^{(2)}(H_2)\kappa_{(H_1|H_1 \cap H_2)} \\
&+ QMF^{(1)}(H_1) \cdot QMF^{(2)}(H_1, H_2)\kappa_{(H_1|H_1 \cap \{H_1, H_2\})} \\
&+ QMF^{(1)}(H_2) \cdot QMF^{(2)}(H_1)\kappa_{(H_1|H_2 \cap H_1)} \\
&+ QMF^{(1)}(H_2) \cdot QMF^{(2)}(H_2)\kappa_{(H_1|H_2 \cap H_2)} \\
&+ QMF^{(1)}(H_2) \cdot QMF^{(2)}(H_1, H_2)\kappa_{(H_1|H_2 \cap \{H_1, H_2\})} \\
&+ QMF^{(1)}(H_1, H_2) \cdot QMF^{(2)}(H_1)\kappa_{(H_1|\{H_1, H_2\} \cap H_1)} \\
&+ QMF^{(1)}(H_1, H_2) \cdot QMF^{(2)}(H_2)\kappa_{(H_1|\{H_1, H_2\} \cap H_2)} \\
&+ QMF^{(1)}(H_1, H_2) \cdot QMF^{(2)}(H_1, H_2)\kappa_{(H_1|\{H_1, H_2\} \cap \{H_1, H_2\})} \\
&= QMF^{(1)}(H_1) \cdot QMF^{(2)}(H_1) \\
&+ QMF^{(1)}(H_1) \cdot QMF^{(2)}(H_1, H_2) \\
&+ QMF^{(1)}(H_1, H_2) \cdot QMF^{(2)}(H_1) \\
&+ QMF^{(1)}(H_1, H_2) \cdot QMF^{(2)}(H_1, H_2)\kappa_{(H_1|\{H_1, H_2\})} \\
&= (0.1 + 0.4j + 0.2k) \cdot (0.7 + 0.6i + 0.6j + 0.6k) \\
&+ (0.1 + 0.4j + 0.2k) \cdot (0.3 - 0.6i - 0.6j - 0.6k) \\
&+ (0.6 - 0.3i - 0.6j - 0.3k) \cdot (0.7 + 0.6i + 0.6j + 0.6k) \\
&+ (0.6 - 0.3i - 0.6j - 0.3k) \cdot (0.3 - 0.6i - 0.6j - 0.6k)(2/3) \\
&= (0.1 + 0.4j + 0.2k) \cdot (0.7 + 0.6i + 0.6j + 0.6k) \\
&+ (0.1 + 0.4j + 0.2k) \cdot (0.3 - 0.6i - 0.6j - 0.6k) \\
&+ (0.6 - 0.3i - 0.6j - 0.3k) \cdot (0.7 + 0.6i + 0.6j + 0.6k) \\
&+ (0.6 - 0.3i - 0.6j - 0.3k) \cdot (0.3 - 0.6i - 0.6j - 0.6k)(2/3) \\
&= (-0.29 + 0.18i + 0.46j - 0.04k) \\
&+ (0.39 - 0.18i - 0.06j + 0.24k) \\
&+ (1.14 - 0.03i - 0.06j + 0.33k) \\
&+ (-0.36 - 0.18i - 0.36j - 0.42k) \\
&= 0.88 - 0.21i - 0.02j + 0.11k
\end{aligned}
$$

$$
\begin{aligned}
QMFV_2^{(1)} \cdot QMFV_2^{(2)} &= \sum_{i=1}^{2^N}\sum_{j=1}^{2^N} QMF^{(1)}(A_i)\cdot QMF^{(2)}(A_j)\kappa_{(H_2|A_i\cap A_j)} \\
&= QMF^{(1)}(H_1)\cdot QMF^{(2)}(H_1)\kappa_{(H_2|H_1\cap H_1)} \\
&+ QMF^{(1)}(H_1)\cdot QMF^{(2)}(H_2)\kappa_{(H_2|H_1\cap H_2)} \\
&+ QMF^{(1)}(H_1)\cdot QMF^{(2)}(H_1,H_2)\kappa_{(H_2|H_1\cap \{H_1,H_2\})} \\
&+ QMF^{(1)}(H_2)\cdot QMF^{(2)}(H_1)\kappa_{(H_2|H_2\cap H_1)} \\
&+ QMF^{(1)}(H_2)\cdot QMF^{(2)}(H_2)\kappa_{(H_2|H_2\cap H_2)} \\
&+ QMF^{(1)}(H_2)\cdot QMF^{(2)}(H_1,H_2)\kappa_{(H_2|H_2\cap \{H_1,H_2\})} \\
&+ QMF^{(1)}(H_1,H_2)\cdot QMF^{(2)}(H_1)\kappa_{(H_2|\{H_1,H_2\}\cap H_1)} \\
&+ QMF^{(1)}(H_1,H_2)\cdot QMF^{(2)}(H_2)\kappa_{(H_2|\{H_1,H_2\}\cap H_2)} \\
&+ QMF^{(1)}(H_1,H_2)\cdot QMF^{(2)}(H_1,H_2)\kappa_{(H_2|\{H_1,H_2\}\cap \{H_1,H_2\})} \\
&= QMF^{(1)}(H_2)\cdot QMF^{(2)}(H_2)\kappa_{(H_2|H_2)} \\
&+ QMF^{(1)}(H_2)\cdot QMF^{(2)}(H_1,H_2)\kappa_{(H_2|H_2)} \\
&+ QMF^{(1)}(H_1,H_2)\cdot QMF^{(2)}(H_2)\kappa_{(H_2|H_2)} \\
&+ QMF^{(1)}(H_1,H_2)\cdot QMF^{(2)}(H_1,H_2)\kappa_{(H_2|\{H_1,H_2\})} \\
&= (0.3+0.3i+0.2j+0.1k))\cdot(0) \\
&+ (0.3+0.3i+0.2j+0.1k)\cdot(0.3-0.6i-0.6j-0.6k) \\
&+ (0.6-0.3i-0.6j-0.3k)\cdot(0) \\
&+ (0.6-0.3i-0.6j-0.3k)\cdot(0.3-0.6i-0.6j-0.6k)(1/3) \\
&= (0.45-0.15i+0j-0.21k) \\
&+ (-0.18-0.09i-0.18j-0.21k) \\
&= 0.27-0.24i-0.18j-0.42k
\end{aligned}
$$

Then, the combination of $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *is as follows:*

$$QMFV^{(12)} = (0.648+0.11i+0.02312j+0.2602k, 0.352-0.11i-0.02312j-0.2602k)$$

Since the QMFVs doesn't conform to the commutative property of multiplication, then $QMFV^{(12)} \neq QMFV^{(21)}$.

Relying on the Eq.(19), the combination of $QMFV^{(1)}$ *and* $QMFV^{(2)}$, $QMFV^{(21)}$ *as follows:*

$$
\begin{aligned}
QMFV_1^{(2)} \cdot QMFV_1^{(1)} &= \sum_{j=1}^{2^N}\sum_{i=1}^{2^N} QMF^{(2)}(A_i)\cdot QMF^{(1)}(A_j)\kappa_{(H_1|A_i\cap A_j)} \\
&= QMF^{(2)}(H_1)\cdot QMF^{(1)}(H_1)\kappa_{(H_1|H_1\cap H_1)} \\
&+ QMF^{(2)}(H_1)\cdot QMF^{(1)}(H_2)\kappa_{(H_1|H_1\cap H_2)} \\
&+ QMF^{(2)}(H_1)\cdot QMF^{(1)}(H_1,H_2)\kappa_{(H_1|H_1\cap \{H_1,H_2\})} \\
&+ QMF^{(2)}(H_2)\cdot QMF^{(1)}(H_1)\kappa_{(H_1|H_2\cap H_1)} \\
&+ QMF^{(2)}(H_2)\cdot QMF^{(1)}(H_2)\kappa_{(H_1|H_2\cap H_2)} \\
&+ QMF^{(2)}(H_2)\cdot QMF^{(1)}(H_1,H_2)\kappa_{(H_1|H_2\cap \{H_1,H_2\})} \\
&+ QMF^{(2)}(H_1,H_2)\cdot QMF^{(1)}(H_1)\kappa_{(H_1|\{H_1,H_2\}\cap H_1)} \\
&+ QMF^{(2)}(H_1,H_2)\cdot QMF^{(1)}(H_2)\kappa_{(H_1|\{H_1,H_2\}\cap H_2)} \\
&+ QMF^{(2)}(H_1,H_2)\cdot QMF^{(1)}(H_1,H_2)\kappa_{(H_1|\{H_1,H_2\}\cap \{H_1,H_2\})} \\
&= QMF^{(2)}(H_1)\cdot QMF^{(1)}(H_1) \\
&+ QMF^{(2)}(H_1)\cdot QMF^{(1)}(H_1,H_2) \\
&+ QMF^{(2)}(H_1,H_2)\cdot QMF^{(1)}(H_1) \\
&+ QMF^{(2)}(H_1,H_2)\cdot QMF^{(1)}(H_1,H_2)\kappa_{(H_1|\{H_1,H_2\})} \\
&= (0.7+0.6i+0.6j+0.6k)\cdot(0.1+0.4j+0.2k) \\
&+ (0.7+0.6i+0.6j+0.6k)\cdot(0.6-0.3i-0.6j-0.3k) \\
&+ (0.3-0.6i-0.6j-0.6k)\cdot(0.1+0.4j+0.2k) \\
&+ (0.3-0.6i-0.6j-0.6k)\cdot(0.6-0.3i-0.6j-0.3k)(2/3) \\
&= (-0.29-0.06i+0.22j+0.44k) \\
&+ (1.14+0.33i-0.06j-0.03k) \\
&+ (0.39+0.06i+0.18j-0.24k) \\
&+ (-0.36-0.42i-0.36j-0.18k) \\
&= 0.88-0.09i-0.02j-0.01k
\end{aligned}
$$

$$
\begin{aligned}
QMFV_2^{(2)} \cdot QMFV_2^{(1)} &= \sum_{j=1}^{2^N}\sum_{i=1}^{2^N} QMF^{(2)}(A_i) \cdot QMF^{(1)}(A_j)\kappa_{(H_2|A_i \cap A_j)} \\
&= QMF^{(2)}(H_1) \cdot QMF^{(1)}(H_1)\kappa_{(H_2|H_1 \cap H_1)} \\
&+ QMF^{(2)}(H_1) \cdot QMF^{(1)}(H_2)\kappa_{(H_2|H_1 \cap H_2)} \\
&+ QMF^{(2)}(H_1) \cdot QMF^{(1)}(H_1, H_2)\kappa_{(H_2|H_1 \cap \{H_1,H_2\})} \\
&+ QMF^{(2)}(H_2) \cdot QMF^{(1)}(H_1)\kappa_{(H_2|H_2 \cap H_1)} \\
&+ QMF^{(2)}(H_2) \cdot QMF^{(1)}(H_2)\kappa_{(H_2|H_2 \cap H_2)} \\
&+ QMF^{(2)}(H_2) \cdot QMF^{(1)}(H_1, H_2)\kappa_{(H_2|H_2 \cap \{H_1,H_2\})} \\
&+ QMF^{(2)}(H_1, H_2) \cdot QMF^{(1)}(H_1)\kappa_{(H_2|\{H_1,H_2\} \cap H_1)} \\
&+ QMF^{(2)}(H_1, H_2) \cdot QMF^{(1)}(H_2)\kappa_{(H_2|\{H_1,H_2\} \cap H_2)} \\
&+ QMF^{(2)}(H_1, H_2) \cdot QMF^{(1)}(H_1, H_2)\kappa_{(H_2|\{H_1,H_2\} \cap \{H_1,H_2\})} \\
&= QMF^{(2)}(H_2) \cdot QMF^{(1)}(H_2) \\
&+ QMF^{(2)}(H_2) \cdot QMF^{(1)}(H_1, H_2) \\
&+ QMF^{(2)}(H_1, H_2) \cdot QMF^{(1)}(H_2) \\
&+ QMF^{(2)}(H_1, H_2) \cdot QMF^{(1)}(H_1, H_2)\kappa_{(H_2|\{H_1,H_2\})} \\
&= (0) \cdot (0.3 + 0.3i + 0.2j + 0.1k) \\
&+ (0) \cdot (0.6 - 0.3i - 0.6j - 0.3k) \\
&+ (0.3 - 0.6i - 0.6j - 0.6k) \cdot (0.3 + 0.3i + 0.2j + 0.1k) \\
&+ (0.3 - 0.6i - 0.6j - 0.6k) \cdot (0.6 - 0.3i - 0.6j - 0.3k)(1/3) \\
&= (0.45 - 0.03i - 0.24j - 0.09k) \\
&+ (-0.18 - 0.21i - 0.18j - 0.09k) \\
&= 0.22 - 0.24i - 0.42j - 0.18k
\end{aligned}
$$

Then, the QMFV of combination of $QMFV^{(1)}$ *and* $QMFV^{(2)}$ *is as follows:*

$$
QMFV^{(21)} = (0.6512 + 0.1232i + 0.2269j + 0.1222k, 0.3488 - 0.1232i - 0.2269j - 0.1222k)
$$

We can obtain that $QMFV_1^{(12)} \neq QMFV_1^{(21)}$ *and* $QMFV_2^{(12)} \neq QMFV_2^{(21)}$.

5. Conclusion

This paper proposes QMFV, which is the extent of mass function vector. The QMFV consists of mass function vector and quaternion theory. The inputs of the proposed model are based on quaternion numbers. Mass function vector is used to handle uncertainty. Quaternion number is the extent of real number. The mass function vector can extend the mass function by combining the vector. The proposed QMFV has the advantage to deal with uncertain information. When the quaternion number degenerates into the real number, then the QMFV degenerates into the quaternion mass function. In addition, if the probability of multiple subsets of frame of discernment is not assigned to the single subsets, then the mass function vector will degenerate into mass function in classical evidence theory. When the quaternion number degenerates into the real number, then the combination rule of quaternion mass function vectors degenerates into the combination rule of mass function vectors. In the case when the probability of multiple subsets of frame of discernment is not assigned to the single subsets, the combination rule of mass function vectors degenerates into generalized dempster's rule of combination. Numerical examples are applied to prove the efficiency of the proposed model. The experimental results show that the proposed model can apply the quaternion theory to mass function vector effectively and successfully.

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