Quaternion Mass Function

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Abstract

Mass function is used to handle uncertainty. Quaternion number is the extent of imaginary number. In this paper, the classical mass function is extended by quaternion number, named as Quaternion Mass Function (QMF). The proposed QMF has the advantage to deal with uncertain information. When the quaternion number degenerates into the complex number, then the QMF degenerates into the complex mass function. In addition, if the complex mass function is degenerated as real number, the QMF is the same as mass function in classical evidence theory. In the case when the quaternion number degenerates into the real number and the QMF focus on the frame of discernment with single subsets, the QMF is the same as the probability distribution in probability theory. The combination rule is also presented to combine two QMFs, which is the generalization of Dempster rule. In the case when the quaternion mass function degenerates into the real number and assigns only to single subsets, the proposed combination rule is degenerated as Beyesian updation in probability theory. Numerical examples are applied to prove the efficiency of the proposed model. The experimental results show that the proposed model can apply the quaternion theory to mass function effectively and successfully.

Keywords: Quaternion, Mass function, Quaternion mass function, Complex mass function, Probability, Beyesian updating

1. Introduction

There are many uncertainties in the real world. In order to deal with the uncertain issues, many mathematical models and theories have been proposed,

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such as quaternion [1], evidence theory [2, 3], belief entropy [4], belief function [5], quaternion bayesian updating [6]. Among those models and theories, the mass function is very efficient in representing uncertainty, which is based on the frame of discernment. There are many models to combine uncertainties. Relying on the advantages on representing uncertainty, the mass function have been widely studied by scholars at home and abroad.

Quaternion theory, which can extend the classical probability theory to the four-dimensional space, which has great promise for discovery. However, how to apply quaternion probability to evidence theory is still an open issue.

This paper proposes the QMF, which doesn't conform to the communitative property of multiplication. The QMF uses addition, subtraction, multiplication, and division operations. The quaternions mass function is based on the quaternion theory, which is can extend the mass fucntion to the four-dimensional space. In order to fusing the different QMFs, the quaternion dempster's rule of combination has been proposed, which is an extent of dempster's rule of combination under the quaternion theory.

The remain of this paper is structured as follows. Section 2 introduces the preliminary. Section 3 presents the QMF. Section 4 illustrates the flexibility of QMF. Section 5 summarizes the whole paper.

2. Preliminaries

In this section, mass function, quaternion are briefly introduced.

2.1. Quaternion

Quaternions are a four dimensional hypercomplex numbers system. The definition of quaternions as follow:

Definition 2.1. *(Quaternion) [1]*

$$q = a + bi + cj + dk \tag{1}$$

Given two quaternions, $q_1 = a + bi + cj + dk$ and $q_2 = e + fi + gj + hk$, the definition of quaternions multiplication as follow:

Definition 2.2. *(Quaternion Multiplication) [1]*

$$\begin{aligned} q_1 q_2 =& (ae - (bf + cg + dh)) + \\ & (be + af + ch - dg)i + \\ & (ce + ag + df - bh)j + \\ & (de + ah + bg - cf)k \end{aligned} \tag{2}$$

2.2. Frame of Discernment

Given a frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, the power set of frame of discernment is defined as follows:

Definition 2.3. *(Power Set of Frame of Discernment) [2]*

$$2^{\Omega} = \{\emptyset, \{x_1\}, \{x_2\}, \ldots, \{x_n\}, \{x_1, x_2\}, \ldots, \{x_1, x_2, \ldots, x_i\}, \ldots, \Omega\} \quad (3)$$

2.3. Mass Function

Given a frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, the mass function, m, on 2^{Ω} is defined as follows:

Definition 2.4. *(Mass Function) [2]*

$$m : 2^{\Omega} \to [0, 1] \quad (4)$$

Where, $m(\emptyset) = 0$ *and* $\Sigma_{B \in 2^{\Omega}} m(B) = 0$ *with a focal element,* B*, of* 2^{Ω}.

2.4. Dempster's Rule of Combination

Given two mass functions, m_1 and m_2, in Ω. The dempster'rule of combination, $m = m_1 \oplus m_2$, between m_1 and m_2 is defined as follows:

Definition 2.5. *(Dempster's Rule of Combination) [2]*

$$m(D) = \begin{cases} \frac{1}{1-K} \sum_{E \bigcap F = D} m(E)m(F) & D \neq \emptyset \\ 0 & D = \emptyset \end{cases} \quad (5)$$

Where, $K = \sum_{E \bigcap F = \emptyset} m(E)m(F)$.

3. The proposed method

3.1. Quaternions Mass Function

In order to extend the mass function to four-dimensional space, this paper proposes the QMF, which includes several operations.

Given a frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, the QMF, QM, on Ω is defined as follows:

Definition 3.1. *(Quaternion Mass Function)*

$$QM(\emptyset) = 0 \quad (6)$$

$$\Sigma_{B \in 2^{\Omega}} QM(B) = 1 \quad (7)$$

$$QM(B) = m(B)e^{u\Theta}, B \in 2^{\Omega} \quad (8)$$

Where, $u = u_x i + u_y j + u_z k$ *with* $i = j = k = \sqrt{-1}$. *Assume* $m(B)$ *is the magnitude of the QMF* $QM(B)$. $\Theta \in [0, 2\pi]$.

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets $A, B \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the addition operator of QM is defined as follows:

Definition 3.2. *(The Addition Operator of Quaternion Mass Function)*

$$QM(A) + QM(B) = (a + e) + (b + f)i + (c + g)j + (d + h)k \tag{9}$$

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any subset $A \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$, the Modular operator of QM is defined as follows:

Definition 3.3. *(The Modular Operator of Quaternion Mass Function)*

$$|QM(A)| = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{10}$$

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets $A, B \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the subtraction operator of QM is defined as follows:

Definition 3.4. *(The Subtraction Operator of Quaternion Mass Function)*

$$QM(A) - QM(B) = (a - e) + (b - f)i + (c - g)j + (d - h)k \tag{11}$$

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets $A, B \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the multiplication operator of QM is defined as follows:

Definition 3.5. *(The Multiplication Operator of Quaternion Mass Function)*

$$\begin{aligned} QM(A)QM(B) =& m(A)e^{u_A \Theta_A} m(B)e^{u_B \Theta_B} \\ &= (ae - (bf + cg + dh)) \\ &+ (be + af + ch - dg)i \\ &+ (ce + ag + df - bh)j \\ &+ (de + ah + bg - cf)k \end{aligned} \tag{12}$$

Theorem 3.1. *QMF does not satisfy the commutative law of multiplication.*

Proof 3.1. *Assume there is a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets $A, B \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$,* *Relying on Eq.(3.1), one has the following equation:*

$$
\begin{aligned}
QM(A)QM(B) = &(ae-(bf+cg+dh))\\
&+(be+af+ch-dg)i\\
&+(ce+ag+df-bh)j\\
&+(de+ah+bg-cf)k
\end{aligned}
$$

Then, relying on Eq.(3.1), we can obtain the equation of $QM(B)QM(A)$ as follow:

$$
\begin{aligned}
QM(B)QM(A) = &(ea-(fb+gc+hd))\\
&+(fa+eb+gd-hc)i\\
&+(ga+ec+hb-fd)j\\
&+(ha+ed+fc-gb)k
\end{aligned}
$$

When $QM(A)QM(B) = QM(B)QM(A)$, it means the following equation holds:

$$
\begin{aligned}
QM(A)QM(B) = &(ae-(bf+cg+dh))\\
&+(be+af+ch-dg)i\\
&+(ce+ag+df-bh)j\\
&+(de+ah+bg-cf)k =\\
QM(B)QM(A) = &(ea-(fb+gc+hd))\\
&+(fa+eb+gd-hc)i\\
&+(ga+ec+hb-fd)j\\
&+(ha+ed+fc-gb)k
\end{aligned}
$$

Then, we can obtain that:

$$
\begin{aligned}
&(ch-dg)i+(df-bh)j+(bg-cf)k =\\
-&(ch-dg)i-(df-bh)j-(bg-cf)k
\end{aligned}
$$

The above equation is only true if $ch-dg$, $(df-bh)$ and $(bg-cf)$. Otherwise, $QM(A)QM(B) \neq QM(B)QM(A)$.

Hence, we can conclude that QMF does not satisfy the commutative law of multiplication. □

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets$A, B \in 2^{\Omega}$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the quadronic multiplication operator of QM is defined as follows:

Definition 3.6. *(The Quadronic Multiplication Operator of Quaternion Mass Function)*

$$QM(A) \cdot QM(B) = ae + bf + cg + dh \tag{13}$$

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets$A, B \in 2^\Omega$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the transpose operator of QM is defined as follows:

Definition 3.7. *(The Transpose Operator of Quaternion Mass Function)*

$$QM(A)^{-1} = \frac{QM(A)^*}{QM(A) \cdot QM(A)} \tag{14}$$

Where,

$$QM(A)^* = a - bi - cj - dk \tag{15}$$

Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any two subsets$A, B \in 2^\Omega$ such that $QM(A) = m(A)e^{u_A \Theta_A} = a + bi + cj + dk$ and $QM(B) = m(B)e^{u_B \Theta_B} = e + fi + gj + hk$, the division operator of QM is defined as follows:

Definition 3.8. *(The Division Operator of Quaternion Mass Function)*

$$QM(A)/QM(B) = QM(B)^{-1}QM(A) = \frac{QM(B)^* QM(A)}{QM(B) \cdot QM(B)} \tag{16}$$

Example 3.1. *Given a QMF QM on frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$, then for any $A, B \in 2^\Omega$ such that $QM(A) = 0.1 + 0.1i + 0.1j + 0.1k$ and $QM(B) = 0.2 + 0.2i + 0.2j + 0.2k$ and $QM(A, B) = 0.7 - 0.3i - 0.3j - 0.3k$, then*

$$QM(A)^* = 0.1 - 0.1i - 0.1j - 0.1k$$
$$QM(B)^* = 0.2 - 0.2i - 0.2j - 0.2k$$
$$QM(A, B)^* = 0.7 + 0.3i + 0.3j + 0.3k$$
$$|QM(A)| = \sqrt{0.1^2 + 0.1^2 + 0.1^2 + 0.1^2} = 0.2$$
$$|QM(B)| = \sqrt{0.2^2 + 0.2^2 + 0.2^2 + 0.2^2} = 0.4$$
$$|QM(A, B)| = \sqrt{0.7^2 + 0.3^2 + 0.3^2 + 0.3^2} = \sqrt{0.76}$$
$$QM(A)^{-1} = \frac{QM(A)^*}{QM(A) \cdot QM(A)} = 2.5 - 2.5i - 2.5j - 2.5k$$
$$QM(B)^{-1} = \frac{QM(B)^*}{QM(B) \cdot QM(B)} = 1.25 - 1.25i - 1.25j - 1.25k$$
$$QM(A, B)^{-1} = \frac{QM(A, B)^*}{QM(A, B) \cdot QM(A, B)} = \frac{70}{76} + \frac{3}{76}i + \frac{3}{76}j + \frac{3}{76}k$$

$$QM(A) + QM(B) + QM(A,B) = 1$$

$$QM(B) - QM(A) = 0.1 + 0.1i + 0.1j + 0.1k$$

$$QM(A)QM(B) = -0.04 + 0.04(i + j + k)$$

$$QM(A)/QM(B) = QM(B)^{-1}QM(A) = (1.25{-}1.25i{-}1.25j{-}1.25k)(0.1{+}0.1i{+}0.1j{+}0.1k) = 0.5$$

3.2. Combination rule of Quaternions Mass Function

Given two mass functions, QM_1 and QM_2, in Ω. The Combination rule of Quaternions Mass Function, $QM = QM_1 \oplus QM_2$, between QM_1 and QM_2 is defined as follows:

Definition 3.9. *(Combination rule of Quaternions Mass Function)*

$$QM(D) = \begin{cases} \frac{1}{1-QK} \sum_{E \bigcap F = D} QM_1(E)QM_2(F) & D \neq \emptyset \\ 0 & D = \emptyset \end{cases} \tag{17}$$

Where, $QK = \sum_{E \bigcap F = \emptyset} QM(E)QM(F)$. $E, F \in 2^{\Omega}$ *and* QK *is the conflict coefficient between* QM_1 *and* QM_2.

3.3. Discussion

Mass function has high efficiency in handling uncertainty. Quaternion number is the extent of complex number. This paper proposes the QMF, which is combined with mass function and quaternion number. When the quaternion number degenerates into the complex number, then the QMF degenerates into the complex mass function. In addition, if the complex mass function is degenerated as real number, the QMF is the same as mass function in classical evidence theory. In the case when the quaternion number degenerates into the real number and the QMF focus on the frame of discernment with single subsets, the QMF is the same as the probability distribution in probability theory. The combination rule is also presented to combine two QMFs, which is the generalization of Dempster rule. When the quaternion number is degenerated as the complex number, then the combination rule of quaternions mass function degenerates into the generalized dempster's rule of combination. In addition, when the complex mass function degenerates into real number, the combination rule of quaternions mass function is the same as dempster's rule of combination in classical evidence theory. In the case when the quaternion mass function degenerates into the real number and assigns only to single subsets, the proposed combination rule is degenerated as Beyesian updation in probability theory. The relationship of quaternion evidence theory(QET), complex evidence theory(CET), evidence theory(ET) and probability theory can be shown in Fig. 1.

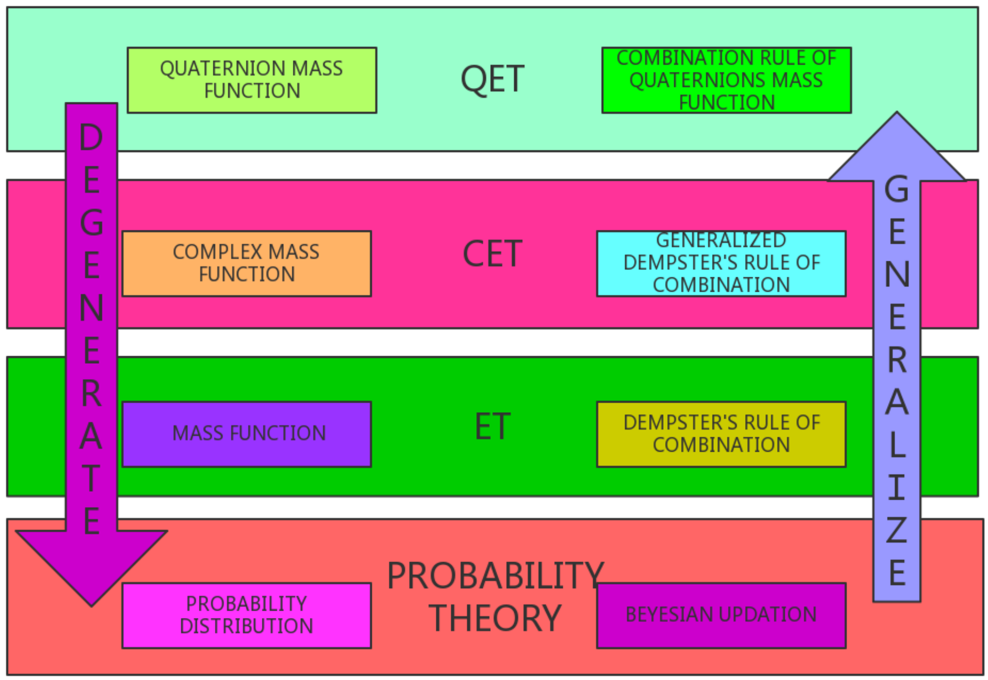

Figure 1: The relationship of four theories

4. Numerical examples

Example 4.1. *Supposing that there are two QMFs QM_1 and QM_2 in the frame of discernment $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$. Here, $A = \{x_1, x_2\}$ and $B = \{x_3, x_4, x_5\}$.*

$$
\begin{aligned}
QM_1 : & QM_1(A) = \sqrt{0.5^2 + 0.5^2} e^{u_1 \Theta_1} \\
& QM_1(B) = \sqrt{(1-0.5)^2 + (-0.5)^2} e^{u_2 \Theta_2}
\end{aligned}
$$

$$
\begin{aligned}
QM_2 : & QM_2(A) = \sqrt{1^2 + 0^2} e^{u_3 \Theta_3} = 1 \\
& QM_2(B) = \sqrt{(1-1)^2 + (-0)^2} e^{u_4 \Theta_4} = 0
\end{aligned}
$$

Where, $\Theta_1 = \frac{\pi}{4}$ or $\Theta_1 = \frac{7\pi}{4}$. $\Theta_2 = \frac{3\pi}{4}$ or $\Theta_2 = \frac{5\pi}{4}$. $\Theta_3 = 0$ or $\Theta_3 = 2\pi$. Since A is proportional to B, then $u_1 = u_2 = u_3 = u_4$.

According to the QM_1 and QM_2 above, the QK as follows:

$$
\begin{aligned}
QK &= QM_1(A) \times QM_2(B) + QM_1(B) \times QM_2(A) \\
&= QM_1(A) \times 0 + QM_1(B) \times 1 \\
&= QM_1(B) \\
&= \frac{1}{\sqrt{2}} e^{u_2 \Theta_2}
\end{aligned}
$$

The magnitude of conflict coefficient $|QK|$ between QM_1 and QM_2 as follow:

$$|QK| = 0.7071$$

In this case, from the value of $|QK|$, we can obtain the conclusion that there is a degree of conflict between QM_1 and QM_2.

When QM_1 in the frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$ as follow:

$$
\begin{aligned}
QM_1: QM_1(A) &= \sqrt{1^2 + 0^2} e^{u_1 \Theta_1} \\
QM_1(B) &= \sqrt{(1-1)^2 + (-0)^2} e^{u_2 \Theta_2}
\end{aligned}
$$

According to the QM_1 and QM_2 above, the QK as follows:

$$
\begin{aligned}
QK &= QM_1(A) \times QM_2(B) + QM_1(B) \times QM_2(A) \\
&= 1 \times 0 + 0 \times 1 \\
&= 0
\end{aligned}
$$

The magnitude of conflict coefficient $|QK|$ between QM_1 and QM_2 as follow:

$$|QK| = 0$$

In this case, from the value of $|QK|$, we can obtain the conclusion that there is no conflict between QM_1 and QM_2. It is rational, since $QM_1 = QM_2$ in this way. This is intuitively true.

When QM_1 in the frame of discernment $\Omega = \{x_1, x_2, \ldots, x_n\}$ as follow:

$$
\begin{aligned}
QM_1: QM_1(A) &= \sqrt{(1-1)^2 + (-0)^2} e^{u_1 \Theta_1} = 0 \\
QM_1(B) &= \sqrt{1^2 + 0^2} e^{u_2 \Theta_2} = 1
\end{aligned}
$$

According to the QM_1 and QM_2 above, the QK as follows:

$$
\begin{aligned}
QK &= QM_1(A) \times QM_2(B) + QM_1(B) \times QM_2(A) \\
&= 0 \times 0 + 1 \times 1 \\
&= 1
\end{aligned}
$$

The magnitude of conflict coefficient $|QK|$ between QM_1 and QM_2 as follow:

$$|QK| = 1$$

In this case, from the value of $|QK|$, we can obtain the conclusion that QM_1 and QM_2 are full conflict. Intuitively, QM_1 and QM_2 are completely opposite QMF.

Example 4.2. *Supposing that there are two QMFs QM_1 and QM_2 in the frame of discernment $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$. Here, $A = \{x_1, x_2\}$, $B = \{x_3, x_4, x_5\}$ and $C = \{x_1, x_2, x_3, x_4, x_5\}$.*

$$\begin{aligned} QM_1 : & QM_1(A) = 0.2 + 0.1i + 0.1j + 0.2k \\ & QM_1(B) = 0.3 + 0.2i + 0.2j + 0.3k \\ & QM_1(A,B) = 0.5 - 0.3i - 0.3j - 0.5k \end{aligned}$$

$$\begin{aligned} QM_2 : & QM_2(A) = 0.3 + 0.2i + 0.3j + 0.2k \\ & QM_2(B) = 0.2 + 0.2i + 0.2j + 0.2k \\ & QM_2(A,B) = 0.5 - 0.4i - 0.5j - 0.4k \end{aligned}$$

Relying on the Eq.(17), we can obtain the equation of combination of $QM(A)$ and $QM(B)$ as follow:

$$\begin{aligned} QK_{1 \cdot 2} &= QM_1(A)QM_2(B) + QM_1(B)QM_2(A) \\ &= (-0.04 + 0.04i + 0.08j + 0.08k) + (-0.07 + 0.07i + 0.17j + 0.17k) \\ &= -0.11 + 0.11i + 0.25j + 0.25k \end{aligned}$$

Then, we can obtain that:

$$1 - QK_{1 \cdot 2} = 1 - (-0.11 + 0.11i + 0.25j + 0.25k) = 1.11 - 0.11i - 0.25j - 0.25k$$

$$
\begin{aligned}
QM_{1\cdot 2}(A) &= \frac{QM_1(A)QM_2(A) + QM_1(A)QM_2(A,B) + QM_1(A,B)QM_2(A)}{1 - QK_{1\cdot 2}} \\
&= \frac{(-0.03 + 0.03i + 0.11j + 0.11k) + (0.27 + 0.03i - 0.09j + 0.01k) + (0.4 + 0.1i + 0.02j - 0.08k)}{1 - QK_{1\cdot 2}} \\
&= \frac{0.64 + 0.16i + 0.04j + 0.04k}{1 - QK_{1\cdot 2}} \\
&= (1 - QK_{1\cdot 2})^{-1}(0.64 + 0.16i + 0.04j + 0.04k) \\
&= \frac{(1 - QK_{1\cdot 2})^*(0.64 + 0.16i + 0.04j + 0.04k)}{(1 - QK_{1\cdot 2}) \cdot (1 - QK_{1\cdot 2})} \\
&= \frac{(1.11 - 0.11i - 0.25j - 0.25k)^*(0.64 + 0.16i + 0.04j + 0.04k)}{(1.11 - 0.11i - 0.25j - 0.25k) \cdot (1.11 - 0.11i - 0.25j - 0.25k)} \\
&= \frac{(1.11 + 0.11i + 0.25j + 0.25k)(0.64 + 0.16i + 0.04j + 0.04k)}{(1.11 - 0.11i - 0.25j - 0.25k) \cdot (1.11 - 0.11i - 0.25j - 0.25k)} \\
&= \frac{0.6728 + 0.248i + 0.24j + 0.1688k}{1.3692} \\
&= 0.4914 + 0.1811i + 0.1753j + 0.1233k
\end{aligned}
$$

$$
\begin{aligned}
QM_{1\cdot 2}(B) &= \frac{QM_1(B)QM_2(B) + QM_1(B)QM_2(A,B) + QM_1(A,B)QM_2(B)}{1 - QK_{1\cdot 2}} \\
&= \frac{(-0.08 + 0.08i + 0.12j + 0.12k) + (0.45 + 0.05i - 0.09j + 0.01k) + (0.32 + 0.08i)}{1 - QK_{1\cdot 2}} \\
&= \frac{0.69 + 0.21i + 0.03j + 0.13k}{1 - QK_{1\cdot 2}} \\
&= (1 - QK_{1\cdot 2})^{-1}(0.69 + 0.21i + 0.03j + 0.13k) \\
&= \frac{(1 - QK_{1\cdot 2})^*(0.69 + 0.21i + 0.03j + 0.13k)}{(1 - QK_{1\cdot 2}) \cdot (1 - QK_{1\cdot 2})} \\
&= \frac{(1.11 - 0.11i - 0.25j - 0.25k)^*(0.69 + 0.21i + 0.03j + 0.13k)}{(1.11 - 0.11i - 0.25j - 0.25k) \cdot (1.11 - 0.11i - 0.25j - 0.25k)} \\
&= \frac{(1.11 + 0.11i + 0.25j + 0.25k)(0.69 + 0.21i + 0.03j + 0.13k)}{(1.11 - 0.11i - 0.25j - 0.25k) \cdot (1.11 - 0.11i - 0.25j - 0.25k)} \\
&= \frac{0.7028 + 0.334i + 0.244j + 0.2676k}{1.3692} \\
&= 0.5133 + 0.2439i + 0.1782j + 0.1954k
\end{aligned}
$$

$$
\begin{aligned}
QM_{1\cdot 2}(A,B) &= \frac{QM_1(A,B)QM_2(A,B)}{1-QK_{1\cdot 2}} \\
&= \frac{(0.5-0.3i-0.3j-0.5k)(0.5-0.4i-0.5j-0.4k)}{1-QK_{1\cdot 2}} \\
&= \frac{-0.22-0.48i-0.32j-0.42k}{1-QK_{1\cdot 2}} \\
&= (1-QK_{1\cdot 2})^{-1}(-0.22-0.48i-0.32j-0.42k) \\
&= \frac{(1-QK_{1\cdot 2})^{*}(-0.22-0.48i-0.32j-0.42k)}{(1-QK_{1\cdot 2})\cdot(1-QK_{1\cdot 2})} \\
&= \frac{(1.11-0.11i-0.25j-0.25k)^{*}(-0.22-0.48i-0.32j-0.42k)}{(1.11-0.11i-0.25j-0.25k)\cdot(1.11-0.11i-0.25j-0.25k)} \\
&= \frac{(1.11+0.11i+0.25j+0.25k)(-0.22-0.48i-0.32j-0.42k)}{(1.11-0.11i-0.25j-0.25k)\cdot(1.11-0.11i-0.25j-0.25k)} \\
&= \frac{-0.0064-0.582i-0.484j-0.4364k}{1.3692} \\
&= -0.0047-0.425i-0.3535j-0.3187k
\end{aligned}
$$

Here, $QM_{1\cdot 2}(A)+QM_{1\cdot 2}(B)+QM_{1\cdot 2}(A,B)=1$. So, the $QM_{1\cdot 2}$ is a QMF.

Then, since the theorem 3.1, we know the QMF doesn't conform to the commutative property of multiplication. Now, we change the multiplication order of QM_1 and QM_2.

$$
\begin{aligned}
QK_{2\cdot 1} &= QM_2(A)QM_1(B)+QM_2(B)QM_1(A) \\
&= (-0.07+0.17i+0.13j+0.13k)+(-0.04+0.08i+0.04j+0.08k) \\
&= -0.11+0.25i+0.17j+0.21k
\end{aligned}
$$

Then, we can obtain that:

$$
1-QK_{2\cdot 1} = 1-(-0.11+0.25i+0.17j+0.21k) = 1.11-0.25i-0.17j-0.21k
$$

$$
\begin{aligned}
QM_{2\cdot1}(A) &= \frac{QM_2(A)QM_1(A) + QM_2(A)QM_1(A,B) + QM_2(A,B)QM_1(A)}{1 - QK_{2\cdot1}} \\
&= \frac{(-0.03 + 0.11i + 0.07j + 0.09k) + (0.40 - 0.08i + 0.10j - 0.02k) + (0.27 - 0.09i - 0.01j + 0.03k)}{1 - QK_{2\cdot1}} \\
&= \frac{0.64 - 0.06i + 0.16j + 0.1k}{1 - QK_{2\cdot1}} \\
&= (1 - QK_{2\cdot1})^{-1}(0.64 - 0.06i + 0.16j + 0.1k) \\
&= \frac{(1 - QK_{2\cdot1})^{*}(0.64 - 0.06i + 0.16j + 0.1k)}{(1 - QK_{2\cdot1}) \cdot (1 - QK_{2\cdot1})} \\
&= \frac{(1.11 - 0.25i - 0.17j - 0.21k)^{*}(0.64 - 0.06i + 0.16j + 0.1k)}{(1.11 - 0.25i - 0.17j - 0.21k) \cdot (1.11 - 0.25i - 0.17j - 0.21k)} \\
&= \frac{(1.11 + 0.25i + 0.17j + 0.21k)(0.64 - 0.06i + 0.16j + 0.1k)}{(1.11 - 0.25i - 0.17j - 0.21k) \cdot (1.11 - 0.25i - 0.17j - 0.21k)} \\
&= \frac{0.6772 + 0.0768i + 0.2488j + 0.2956k}{1.3676} \\
&= 0.4952 + 0.0561i + 0.1817j + 0.2159k
\end{aligned}
$$

$$
\begin{aligned}
QM_{2\cdot1}(B) &= \frac{QM_2(B)QM_1(B) + QM_2(B)QM_1(A,B) + QM_2(A,B)QM_1(B)}{1 - QK_{2\cdot1}} \\
&= \frac{(-0.08 + 0.12i + 0.08j + 0.12k) + (0.32 + 0.08j) + (0.45 - 0.09i - 0.01j + 0.05k)}{1 - QK_{2\cdot1}} \\
&= \frac{0.69 + 0.03i + 0.15j + 0.17k}{1 - QK_{2\cdot1}} \\
&= (1 - QK_{2\cdot1})^{-1}(0.69 + 0.03i + 0.15j + 0.17k) \\
&= \frac{(1 - QK_{2\cdot1})^{*}(0.69 + 0.03i + 0.15j + 0.17k)}{(1 - QK_{2\cdot1}) \cdot (1 - QK_{2\cdot1})} \\
&= \frac{(1.11 - 0.25i - 0.17j - 0.21k)^{*}(0.69 + 0.03i + 0.15j + 0.17k)}{(1.11 - 0.25i - 0.17j - 0.21k) \cdot (1.11 - 0.25i - 0.17j - 0.21k)} \\
&= \frac{(1.11 + 0.25i + 0.17j + 0.21k)(0.69 + 0.03i + 0.15j + 0.17k)}{(1.11 - 0.25i - 0.17j - 0.21k) \cdot (1.11 - 0.25i - 0.17j - 0.21k)} \\
&= \frac{0.6972 + 0.2032i + 0.2476j + 0.366k}{1.3676} \\
&= 0.5098 + 0.1484i + 0.1808j + 0.2673k
\end{aligned}
$$

$$
\begin{aligned}
QM_{2\cdot 1}(A,B) &= \frac{QM_2(A,B)QM_1(A,B)}{1-QK_{2\cdot 1}} \\
&= \frac{(0.5-0.4i-0.5j-0.4k)(0.5-0.3i-0.3j-0.5k)}{1-QK_{2\cdot 1}} \\
&= \frac{-0.22-0.22i-0.48j-0.48k}{1-QK_{2\cdot 1}} \\
&= (1-QK_{2\cdot 1})^{-1}(-0.22-0.22i-0.48j-0.48k) \\
&= \frac{(1-QK_{2\cdot 1})^{*}(-0.22-0.22i-0.48j-0.48k)}{(1-QK_{2\cdot 1})\cdot(1-QK_{2\cdot 1})} \\
&= \frac{(1.11-0.25i-0.17j-0.21k)^{*}(-0.22-0.22i-0.48j-0.48k)}{(1.11-0.25i-0.17j-0.21k)\cdot(1.11-0.25i-0.17j-0.21k)} \\
&= \frac{(1.11+0.25i+0.17j+0.21k)(-0.22-0.22i-0.48j-0.48k)}{(1.11-0.25i-0.17j-0.21k)\cdot(1.11-0.25i-0.17j-0.21k)} \\
&= \frac{-0.0068-0.28i-0.4964j-0.6616k}{1.3676} \\
&= -0.005-0.2045i-0.3625j-0.4832k
\end{aligned}
$$

Here, $QM_{2\cdot 1}(A)+QM_{2\cdot 1}(B)+QM_{2\cdot 1}(A,B)=1$. So, the $QM_{2\cdot 1}$ is a QMF.

However, we can find that $QM_{1\cdot 2}(A) \neq QM_{2\cdot 1}(A)$, $QM_{1\cdot 2}(B) \neq QM_{2\cdot 1}(B)$ and $QM_{1\cdot 2}(A,B) \neq QM_{2\cdot 1}(A,B)$, which is the result of that the QMF doesn't conform to the commutative property of multiplication. Now, we change the multiplication order of QM_1 and QM_2. .

Example 4.3. *Supposing that there are two QMFs QM_1 and QM_2 in the frame of discernment $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$. Here, $A = \{x_1, x_2\}$, $B = \{x_3, x_4, x_5\}$ and $C = \{x_1, x_2, x_3, x_4, x_5\}$.*

$$
\begin{aligned}
QM_1 :& QM_1(A) = 0.2+0.1i \\
& QM_1(B) = 0.3+0.2i \\
& QM_1(A,B) = 0.5-0.3i
\end{aligned}
$$

$$
\begin{aligned}
QM_2 :& QM_2(A) = 0.3+0.2i \\
& QM_2(B) = 0.2+0.2i \\
& QM_2(A,B) = 0.5-0.4i
\end{aligned}
$$

Now, the QMF degenerates into the complex mass function.

Relying on the Eq.(17), we can obtain the equation of combination of $QM(A)$ and $QM(B)$ as follow:

$$
\begin{aligned}
QK &= QM_1(A)QM_2(B)+QM_1(B)QM_2(A) \\
&= (0.2+0.1i)(0.2+0.2i)+(0.3+0.2i)(0.3+0.2i) \\
&= (0.04+0.06i-0.02)+(0.09+0.12i-0.04) \\
&= 0.07+0.18i
\end{aligned}
$$

Then, we can obtain that:

$$1 - QK = 1 - (0.07 + 0.18i) = 0.93 - 0.18i$$

$$\begin{aligned}
QM(A) &= \frac{QM_1(A)QM_2(A) + QM_1(A)QM_2(A,B) + QM_1(A,B)QM_2(A)}{1-QK} \\
&= \frac{(0.2+0.1i)(0.3+0.2i) + (0.2+0.1i)(0.5-0.4i) + (0.5-0.3i)(0.3+0.2i)}{1-QK} \\
&= \frac{(0.04+0.07i) + (0.14-0.03i) + (0.21+0.01i)}{1-QK} \\
&= \frac{0.39+0.05i}{1-QK} \\
&= \frac{0.39+0.05i}{0.93-0.18i} \\
&= \frac{0.3537+0.1167i}{0.8973} \\
&= \frac{0.3537+0.1167i}{0.8973}
\end{aligned}$$

$$\begin{aligned}
QM(B) &= \frac{QM_1(B)QM_2(B) + QM_1(B)QM_2(A,B) + QM_1(A,B)QM_2(B)}{1-QK} \\
&= \frac{(0.3+0.2i)(0.2+0.2i) + (0.3+0.2i)(0.5-0.4i) + (0.5-0.3i)(0.2+0.2i)}{1-QK} \\
&= \frac{(0.02+0.1i) + (0.23-0.02i) + (0.16+0.04i)}{1-QK} \\
&= \frac{0.41+0.12i}{1-QK} \\
&= \frac{0.41+0.12i}{0.93-0.18i} \\
&= \frac{0.3597+0.1854i}{0.8973}
\end{aligned}$$

$$\begin{aligned}
QM(A,B) &= \frac{QM_1(A,B)QM_2(A,B)}{1-QK} \\
&= \frac{(0.5-0.3i)(0.5-0.4i)}{1-QK} \\
&= \frac{0.13-0.35i}{1-QK} \\
&= \frac{0.13-0.35i}{0.93-0.18i} \\
&= \frac{0.1839-0.3021i}{0.8973}
\end{aligned}$$

Here, $QM(A)+QM(B)+QM(A,B)=1$*. So, the* $QM_{1\cdot 2}$ *is a complex mass function.*

5. Conclusion

This paper proposes QMF, which is the extent of complex mass function. The QMF consists of mass function and quaternion theory. The inputs of the proposed model are based on quaternion numbers. When the quaternion number degenerates into the complex number, then the QMF degenerates into the complex mass function. When the quaternion number degenerates into the real number, then the QMF degenerates into the classical mass function. When the quaternion number degenerates into the real number and the QMF focus on the frame of discernment with single subsets, then the QMF degenerates into the probability of probability theory. The QMFs can be combined by the aid of Combination rule of QMF, which has been proposed in this paper. When the quaternion number degenerates into the real number, the QMF will degenerate into the classical mass function. Numerical examples are applied to verify the validity of the QMF and Combination rule of QMF. The experimental results demonstrate that the proposed models can address the conflict combination issues of mass function with the aid of quaternion theory effectively.

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