MAJORIZATION IN THE FRAMEWORK OF 2–CONVEX SYSTEMS

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Abstract. We define a 2-convex system by the restrictions $x_1+x_2+\ldots+x_n=ns$, $e(x_1)+e(x_2)+\ldots+e(x_n)=nk$, $x_1\ge x_2\ge\ldots\ge x_n$ where $e:I\to\mathbb{R}$ is a strictly convex function. We study the variation intervals for x_k and give a more general version of the Boyd-Hawkins inequalities. Next we define a majorization relation on A_S by $x\preccurlyeq_p y \Leftrightarrow T_k(x)\le T_k(y)\ \ \forall 1\le k\le p-1$ and $B_k(x)\le B_k(y)\ \ \forall p+2\le k\le n$ (for fixed $1\le p\le n-1$) where $T_k(x)=x_1+\ldots+x_k$, $B_k(x)=x_k+\ldots+x_n$. The following Karamata type theorem is given: if $x,y\in A_S$ and $x\preccurlyeq_p y$ then $f(x_1)+f(x_2)+\ldots+f(x_n)\le f(y_1)+f(y_2)+\ldots+f(y_n)$ $\forall f:I\to\mathbb{R}$ 3-convex with respect to e. As a consequence, we get an extended version of the equal variable method of V. Cîrtoaje

1. Introduction. The main results, definitions and notations

DEFINITION 1. Let $I\subset\mathbb{R}$ an interval. A continuous, strictly convex function $e:I\to\mathbb{R}$ is called *acceptable* if it cannot be further extended by continuity on $\bar{I}$.

Let $m=\inf(I)\in\overline{\mathbb{R}}$, $M=\sup(I)\in\overline{\mathbb{R}}$. If $m\notin I$ we infer from the above definition that either $m=-\infty$, or m is finite but $\lim_{x\to m}e(x)=+\infty$ (and similarly for M).

We will study systems of the form $(S):\begin{cases}x_1+x_2+\ldots+x_n=ns\\ e(x_1)+e(x_2)+\ldots+e(x_n)=nk\\ x_1\ge x_2\ge\ldots\ge x_n\end{cases}$ where $n\ge 3$, $e:I\to\mathbb{R}$ is a continuous, strictly convex, *acceptable* function and s,k are real constants with $s\in\mathring{I}$. We call such a system *2-convex* or *(S)–sistem* and use the notation $S(e,s,k,n)$. We denote the solutions set by A_S. A necessary condition for A_S to be nonempty is that $e(s)\le k$ (by the convexity of e). A nonempty (S)–system it's called trivial if A_S has only one element. Because e is strictly convex we see that $e(s)=k$ $\Leftrightarrow A_S=\{(s,s\ldots,s)\}$, so (S) it's trivial in this case. We will prove in the next sections that A_S is a compact and connected set.

REMARK 1. We can also consider 2-concave systems $S(e,s,k,n)$ (for which the function e is strictly concave) and their theory is completely similar. In practice, we can associate to each concave system $S(e,s,k,n)$ the convex system $S'(-e,s,-k,n)$ for which $A'_S=A_S$ etc.

An important role in the study of the (S)–systems will be played by the so-called *p–invariants*.

DEFINITION 2. Let $S(e,s,k,n)$ be an (S)–system and $1\le p\le n-1$. We say that (S) admits invariants of order p if the following system

$$\begin{cases}pa+(n-p)b=ns\\ pe(a)+(n-p)e(b)=nk\\ a\ge b\end{cases}$$

is nonempty.

As we shall see, any such solution (a_p, b_p) is unique and we denote by $(a_p|b_p)_S$ the n-tuple $(a_p, \dots a_p, b_p, \dots b_p) \in A_S$. If (S) admits p–invariants $\forall 1 \le p \le n-1$ we say that (S) is *complete* and in this case we consider the intervals

$$I_p := \begin{cases} [a_{n-1},\ a_1] & \text{if } p = 1, \\ [b_{p-1},\ a_p] & \text{if } 1 < p < n, \\ [b_{n-1},\ b_1] & \text{if } p = n \end{cases}$$

We will show that every system $S(e,s,k,n)$ for which I_S is an open interval is complete and I_p is precisely the set of all possible values of component x_p $(x \in A_S)$. This extends the known inequalities of Boyd-Hawkins (see [4], pg. 155).

It is particularly important to consider the "poles" of the (S). It is shown that there is a single n-tuple ω (the lower pole) for which the minimum of x_1 is achieved, respectively a single n-tuple Ω (the upper pole) for which the maximum of x_n is achieved. Specifically, $\Omega = (a_1|b_1)_S$ (if S has 1–invariants) respectively $\omega = (a_{n-1}|b_{n-1})_S$ if S has $(n-1)$–invariants but, in general, ω and Ω have the form:

$$\begin{cases} \Omega = (\overbrace{M, \dots M}^{r \ge 0}, a, b \dots b) \\ \omega = (a, \dots a, b, \underbrace{m \dots m}_{r \ge 0}) \end{cases}$$

where $m = \inf(I_S)$, $M = \sup(I_S)$ with the observation that if $m \notin I_S$ (or $M \notin I_S$) then $r = 0$.

For $x \in \mathbb{R}^n$ and $1 \le k \le n$ we consider the "top" sums $T_k(x) = x_1 + \dots + x_k$ and also the "bottom" sums $B_k(x) = x_k + \dots + x_n$ (by convention $T_0(x) = 0$, $B_{n+1}(x) = 0$).

Given $x, y \in \mathbb{R}^n$ such that $x_1 \ge x_2 \ge \dots \ge x_n$ and $y_1 \ge y_2 \ge \dots \ge y_n$ then $x \preccurlyeq y$ (in the classical sense of the majorization theory) if:

$$\begin{cases} x_1 \le y_1 \\ x_1 + x_2 \le y_1 + y_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_1 + \dots + x_{n-1} \le y_1 + \dots + y_{n-1} \\ x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n \end{cases}$$

that is, more concisely, if $T_n(x) = T_n(y)$ and $T_k(x) \le T_k(y)$ $\forall 1 \le k \le n-1$.

We state here the classical result of Hardy-Littlewood-Polya (also known as Karamata's theorem):

THEOREM 1. *Let $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$ strictly convex and $x, y \in I^n$. If $x \preccurlyeq y$ then*

$$f(x_1) + f(x_2) + \dots + f(x_n) \le f(y_1) + f(y_2) + \dots + f(y_n)$$

Moreover, equality occurs if and only if $x = y$.

REMARK 2. The above condition $T_k(x) \le T_k(y)$ $\forall 1 \le k \le n-1$ can be replaced with:

$$\exists 1 \le p \le n \text{ such that} \begin{cases} T_k(x) \le T_k(y) & \forall 1 \le k \le p-1 \\ B_k(x) \ge B_k(y) & \forall p+1 \le k \le n \end{cases}$$

because $B_k(x) \ge B_k(y) \Leftrightarrow T_n(x) - T_{k-1}(x) \ge T_n(y) - T_{k-1}(y) \Leftrightarrow T_{k-1}(x) \le T_{k-1}(y)$ $\forall p+1 \le k \le n$ so $T_k(x) \le T_k(y)$ $\forall p \le k \le n-1$ and these inequalities, together with $T_k(x) \le T_k(y)$ $\forall 1 \le k \le p-1$ give us $T_k(x) \le T_k(y)$ $\forall 1 \le k \le n-1$.

Starting from this reformulation we will define in a very similar manner a majorization relation on A_S:

DEFINITION 3. Let $x, y \in A_S$ and $1 \le p \le n-1$ a fixed index. We say that $x \preccurlyeq_p y$ if

$$\begin{cases} T_k(x) \le T_k(y) & \forall 1 \le k \le p-1 \\ B_k(x) \le B_k(y) & \forall p+2 \le k \le n \end{cases}$$

In order to state the main result of the article we need the following definition:

DEFINITION 4. Let $f, e : I \subset \mathbb{R} \to \mathbb{R}$ continuous on I, differentiable on $\mathring{I}$. We say that f is (strictly) 3-convex with respect to e if $\exists g : J \to \mathbb{R}$ (strictly) convex with $e'(\mathring{I}) \subset J$ and such that $f' = g \circ e'$.

REMARK 3. In the particular case $e(x) = x^2$ this is equivalent with the standard definition of 3-convex functions (see for example [3]).

Now the main result:

THEOREM 2. (Karamata for 2-convex systems) *Let $S(e,s,k,n)$ a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$, $f : I_S \to \mathbb{R}$ strictly 3-convex with respect to e. Then $\forall x, y \in A_S$ with $x \preccurlyeq_p y$ we have:*

$$f(x_1) + f(x_2) + \ldots + f(x_n) \le f(y_1) + f(y_2) + \ldots + f(y_n)$$

Moreover, equality occurs if and only if $x = y$.

We will show that for any $x \in A_S$ $\exists p, q$ so that $\omega \preccurlyeq_p x \preccurlyeq_q \Omega$ and this allows us to obtain the following corollary (a generalization for the equal variable theorem of Vasile Cîrtoaje, see [1] and [2].

COROLLARY 1. (extension of the equal variable theorem) *Let $S(e,s,k,n)$ a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$, $f : I_S \to \mathbb{R}$ strictly 3-convex with respect to e. Then $\forall x \in A_S$ we have*

$$E_f(\omega) \le E_f(x) \le E_f(\Omega)$$

where $E_f(x) = f(x_1) + f(x_2) + \ldots + f(x_n)$ and ω, Ω are the poles of the (S). Moreover, equality occurs if and only if $x = \omega$ or $x = \Omega$.

2. The study of the invariants of an $S(e,s,k,n)$ system

We start here the study of the invariants of an $S(e,s,k,n)$ system (Definition 2).

LEMMA 1. *If $S(e,s,k,n)$ admits a pair (a_p,b_p) of invariants of order p for a certain $1 \leq p \leq n-1$ then this pair is unique.*

Proof. Suppose that (S) has a second pair of p–invariants $(a'_p,b'_p) \neq (a_p,b_p)$. We have, for example, $a_p < a'_p$ and then, using the relation $pa_p+(n-p)b_p = pa'_p+(n-p)b'_p = ns$ we infer $b_p > b'_p$.

Thus $(a'_p,\dots a'_p,b'_p,\dots b'_p) \succ (a_p,\dots a_p,b_p,\dots b_p)$ (strictly) and applying Karamata to the strictly convex function e we obtain $kn > kn$, a contradiction. □

LEMMA 2. *If $S(e,s,k,n)$ has $e(s) < k$ and $\exists(a_p|b_p)_S$ then $a_p > s > b_p$.*

Proof. From the definition of invariants, $pa_p+(n-p)b_p = ns$ and $a_p \geq b_p$.

Thus $p(a_p-s)+(n-p)(b_p-s) = 0 \quad (*)$ and we have the following cases :

Case 1. $a_p > s$ Then from $(*)$ it follows that $b_p < s$ and we get $a_p > s > b_p$

Case 2. $a_p = s$ Then from $(*)$ it follows that $b_p = s$. On the other hand $pe(a_p)+(n-p)e(b_p) = nk \Rightarrow e(s) = k$, contradiction.

Case 3. $a_p < s$ Then from $(*)$ it follows that $b_p > s$ which contradicts the fact that $a_p \geq b_p$. □

2.1. The extremal properties of invariants

THEOREM 3. *Let $S(e,s,k,n)$ be a nonempty system and $x \in A_S$.*

(a) Let $1 \leq p \leq n-1$. If $\exists(a_p|b_p)_S$ then $x_p \leq a_p$ with equality if and only if $x = (a_p|b_p)_S$.

(b) Let $2 \leq p \leq n-1$. If $\exists(a_{p-1}|b_{p-1})_S$ then $x_p \geq b_{p-1}$ with equality if and only if $x = (a_{p-1}|b_{p-1})_S$.

(c) If $\exists(a_1|b_1)_S$ then $x_n \leq b_1$ with equality if and only if $x = (a_1|b_1)_S$.

(d) If $\exists(a_{n-1}|b_{n-1})_S$ then $x_1 \geq a_{n-1}$ with equality if and only if $x = (a_{n-1}|b_{n-1})_S$.

Proof. (a) Suppose that $x_p > a_p$. We will show that $(x_1,\dots x_n) \succ (\underbrace{a_p,\dots a_p}_{p},\underbrace{b_p,\dots b_p}_{n-p})$.

Because $x_1 \geq \dots \geq x_p > a_p$ we get

$$x_1 > a_p,\ x_1+x_2 > 2a_p,\ \dots ,x_1+\dots +x_p > pa_p \qquad (*)$$

On the other hand, $(x_1+\dots+x_p)+(x_{p+1}+\dots+x_n) = pa_p+(n-p)b_p = ns$, but $x_1+\dots+x_p > pa_p$ and thus $x_{p+1}+\dots+x_n < (n-p)b_p$, so $\frac{x_{p+1}+\dots+x_n}{n-p} < b_p$.

But $x_{p+1} \geq x_{p+2} \geq \dots \geq x_n \Rightarrow x_n \leq \frac{x_n+x_{n-1}}{2} \leq \frac{x_n+x_{n-1}+x_{n-2}}{3} \leq \dots \leq \frac{x_n+\dots+x_{p+1}}{n-p} < b_p$ and so we get $x_n < b_p$, $x_n+x_{n-1} < 2b_p$, $\dots$, $(x_n+\dots+x_p) < (n-p)b_p \ (**)$

From $(*)$ and $(**)$ it follows that $x \succ (a_p|b_p)_S$ and applying Karamata to the strictly convex function e we get the contradiction $kn > kn$.

Therefore $x_p \le a_p$. If equality $x_p = a_p$ holds, then $x_p \ge a_p$ and, following exactly the above steps (from the $x_p > a_p$ case), we get the (not necessarily strictly) majorization $x \succcurlyeq (a_p|b_p)_S$. In fact, we must have $x = (a_p|b_p)_S$ otherwise $x \succ (a_p|b_p)_S$ and applying Karamata to e we get again $kn > kn$, contradiction. Thus $x_p = a_p$ imply $x = (a_p|b_p)_S$.

(b) Suppose that $x_p < b_{p-1}$. We will show that $x \succ (a_{p-1}|b_{p-1})_S$.
Using $b_{p-1} > x_p \ge x_{p+1} \ge \ldots \ge x_n$ we get

$$x_n < b_{p-1},\ (x_n + x_{n-1}) < 2b_{p-1}, \ldots, (x_n + \ldots + x_p) < (n-p+1)b_{p-1} \quad (*)$$

On the other hand, $(x_1 + \ldots + x_{p-1}) + (x_p + \ldots + x_n) = (p-1)a_{p-1} + (n-p+1)b_p = ns$, but $(x_p + \ldots + x_n) < (n-p+1)b_{p-1}$ and thus $x_1 + \ldots + x_{p-1} > (p-1)a_{p-1}$, so $\frac{x_1+\ldots+x_{p-1}}{p-1} > a_{p-1}$.
But $x_1 \ge x_2 \ge \ldots \ge x_{p-1} \Rightarrow x_1 \ge \frac{x_1+x_2}{2} \ge \frac{x_1+x_2+x_3}{3} \ge \ldots \ge \frac{x_1+\ldots+x_{p-1}}{p-1} > a_{p-1}$
and so we get $x_1 > a_{p-1},\ x_1 + x_2 > 2a_{p-1},\ \ldots,\ (x_1 + \ldots + x_{p-1}) > (p-1)a_{p-1} (**)$

From $(*)$ and $(**)$ it follows that $x \succ (a_p|b_p)_S$ and applying Karamata to the strictly convex function e we get $kn > kn$, contradiction.

Therefore $x_p \ge b_{p-1}$. If equality $x_p = b_{p-1}$ holds then $x_p \le b_{p-1}$ and, following exactly the above steps (from the $x_p < b_{p-1}$ case) we get the (not necessarily strictly) majorization $x \succcurlyeq (a_{p-1}|b_{p-1})_S$. We must have $x = (a_{p-1}|b_{p-1})_S$ otherwise $x \succ (a_{p-1}|b_{p-1})_S$ and applying Karamata to e we get again $kn > kn$, contradiction. Thus $x_p = b_{p-1}$ imply $x = (a_{p-1}|b_{p-1})_S$.

For (c), (d) the proofs use similar arguments. □

COROLLARY 2. *If* (S) *has* $e(s) < k$ *and admits* $(a_p|b_p)_S$, $(a_q|b_q)_S$ $(p < q)$ *then* $a_p > a_q$ *and* $b_p > b_q$.

Proof. Let $u = (a_p|b_p)_S$ and $v = (a_q|b_q)_S$. Notice that $v_p = a_q$ (because $p < q$) and applying theorem 3a we infer that $v_p \le a_p$ that is, $a_p \ge a_q$. But the equality case $a_p = a_q$ is not possible because, by the same theorem 3a, this would imply that $u = v$ and, using lemma 2 we get $s > b_p = u_{p+1} = v_{p+1} = a_{q+1} > s$, contradiction.

Thus $a_p > a_q$ and by theorem 3b we get similarly that $b_p > b_q$. □

EXAMPLE 1. Let $S(e,s,k,n)$ a 2-convex system where $k, s \in \mathbb{R}$, $k \ge s^2$ and $e : \mathbb{R} \to \mathbb{R}$ is given by $e(x) = x^2$. A straightforward computation shows that $\forall\ 1 \le p \le n-1$ the system 2 has the solution $(a_p, b_p) = \left(s + \sqrt{\frac{n-p}{p}\Delta},\ s - \sqrt{\frac{p}{n-p}\Delta}\right)$ where $\Delta = k - s^2 \ge 0$. Thus S is a complete system and $\forall x = (x_1, x_2 \ldots x_n) \in A_S$ we have $x_p \in I_p$ where

$$
I_p = \begin{cases} \left[s+\sqrt{\frac{\Delta}{n-1}},\ \ s+\sqrt{(n-1)\Delta}\ \right] & \text{if } p=1, \\ \left[s-\sqrt{\frac{p-1}{n-p+1}\Delta},\ \ s+\sqrt{\frac{n-p}{p}\Delta}\ \right] & \text{if } 1<p<n, \\ \left[s-\sqrt{(n-1)\Delta},\ \ s-\sqrt{\frac{\Delta}{n-1}}\ \right] & \text{if } p=n \end{cases}
$$

We obtain in this way the well-known Boyd-Hawkins's inequalities (see [4], pg. 155). and we can get many examples of this type by simply choosing another complete (S)–system, for example $S(e,s,k,n)$ with $s,k>0,\ ks\geq 1$ and $e:(0,\infty)\to\mathbb{R}$ given by $e(x)=\frac{1}{x}$ etc.

2.2. Existence conditions for invariants

Let $S(e,s,k,n)$ be un (S)–system and $1\leq p\leq n-1,\ I=I_S,\ m=\inf(I)\in\overline{\mathbb{R}}$, $M=\sup(I)\in\overline{\mathbb{R}}$.

Let $g_p:J_p\to\mathbb{R},\quad g_p(x)=pe(x)+(n-p)e(\frac{ns-px}{n-p})-kn$ where $J_p\subset I\cap[s,\infty)$ is the largest interval with the property that $\frac{ns-px}{n-p}\in I\cap(-\infty,s]$.

REMARK 4. J_p can be specified more precisely as follows: we consider the linear decreasing function $u:[s,\infty)\to(-\infty,s]$ given by $u(x)=\frac{ns-px}{n-p}$ and we see that $J_p=J\cap I$ where $J=u^{-1}(I\cap(-\infty,s])=\begin{cases}[s,\ u^{-1}(m)] & \text{if } m\in I\\ [s,\ u^{-1}(m)) & \text{if } m\notin I\end{cases}=\begin{cases}[s,\gamma_p] & \text{if } m\in I\\ [s,\gamma_p) & \text{if } m\notin I\end{cases}$

and $\gamma_p\overset{def}{=}\frac{ns-(n-p)m}{p}\in[s,\infty]$ and finally we get for J_p the expression

$$
\begin{cases} \text{If } M<\gamma_p \text{ then } J_p=\begin{cases}[s,M] & \text{if } M\in I\\ [s,M) & \text{if } M\notin I\end{cases} \\ \text{If } M>\gamma_p \text{ then } J_p=\begin{cases}[s,\gamma_p] & \text{if } m\in I\\ [s,\gamma_p) & \text{if } m\notin I\end{cases} \\ \text{If } M=\gamma_p \text{ then } J_p=\begin{cases}[s,M] & \text{if } m\in I \text{ and } M\in I\\ [s,M) & \text{if } m\notin I \text{ or } M\notin I\end{cases} \end{cases}
$$

□

LEMMA 3. *g_p is strictly increasing on J_p*

Proof. Let $c,d\in\mathring{J}_p$ with $c<d$. Then

$$
g_p(c)-g_p(d)=p[e(c)-e(d)]+(n-p)\left[e\left(\frac{ns-pc}{n-p}\right)-e\left(\frac{ns-pd}{n-p}\right)\right]
$$

which can be written as

$$
\frac{g_p(c)-g_p(d)}{c-d}=p\left[\frac{e(c)-e(d)}{c-d}-\frac{e\left(\frac{ns-pc}{n-p}\right)-e\left(\frac{ns-pd}{n-p}\right)}{\frac{ns-pc}{n-p}-\frac{ns-pd}{n-p}}\right] \tag{1}
$$

We observe that $d > \frac{ns-pd}{n-p} \Leftrightarrow d > s$ (true) and using the convexity of e we infer that

$$\frac{e(c)-e(d)}{c-d} > \frac{e(c)-e\left(\frac{ns-pd}{n-p}\right)}{c-\frac{ns-pd}{n-p}} \tag{2}$$

Similarly, $c > \frac{ns-pc}{n-p} \Leftrightarrow c > s$ (true) and from here we also get

$$\frac{e\left(\frac{ns-pd}{n-p}\right)-e(c)}{\frac{ns-pd}{n-p}-c} > \frac{e\left(\frac{ns-pd}{n-p}\right)-e\left(\frac{ns-pc}{n-p}\right)}{\frac{ns-pd}{n-p}-\frac{ns-pc}{n-p}} \tag{3}$$

From (2) and (3) we deduce that the right side of the relation (1) is positive $\Rightarrow$ $\frac{g_p(c)-g_p(d)}{c-d} > 0 \Rightarrow g_p(c)-g_p(d) < 0$, ie g_p is strictly increasing on $\mathring{J}_p$, so also on J_p because g_p is continuous. ☐

From this lemma we infer the existence of the limit

$$L_p \overset{def}{=} \lim_{x\to\sup J_p} g_p(x) \in \overline{\mathbb{R}}$$

THEOREM 4. *Let $S(e,s,k,n)$ be an (S)–system with $f(s) < k$, $1 \le p \le n-1$ and L_p the limit defined above. Then (S) has invariants of order p if and only if*

$$\begin{cases} L_p \ge 0 & \text{if } J_p \text{ is compact} \\ L_p > 0 & \text{if } J_p \text{ is not compact} \end{cases}$$

Proof. We see that $g_p(s) = n(e(s)-k) < 0$ and the theorem follows considering that g_p is strictly increasing (according to the previous lemma). ☐

COROLLARY 3. *Let $S_1(e,s,k_1,n)$ and $S_2(e,s,k_2,n)$ be two non-empty (S)–systems with $k_1 \le k_2$. If S_2 has p–invariants for a certain $1 \le p \le n-1$ then S_1 has also p–invariants.*

Proof. Let $g_p^1, g_p^2 : J_p \to \mathbb{R}$, $g_p^1(t) = pe(t) + (n-p)e(\frac{ns-pt}{n-p}) - k_1 n$ and $g_p^2(t) = pe(t) + (n-p)e(\frac{ns-pt}{n-p}) - k_2 n$ defined as above. Notice that $g_p^1(t) + k_1 n = g_p^2(t) + k_2 n$ $\forall t \in J_p$ and so

$$\lim_{t\to\sup J_p} g_p^1(t) = \lim_{t\to\sup J_p} g_p^2(t) + (k_2 - k_1)n \ge 0$$

☐

THEOREM 5. *If $S(e,s,k,n)$ has $e(s) \le k$ and I_S is an open interval then (S) is non-empty and complete.*

Proof. If $e(s) = k$ then $A_S = \{(s,s\ldots s)\}$ and the theorem is trivially true. We can therefore assume from now on that $e(s) < k$.

Let $1 \le p \le n-1$ and $g_p : J_p \to \mathbb{R}$, $g_p(x) = pe(x) + (n-p)e\left(\frac{ns-px}{n-p}\right) - kn$. According to remark 4 we have $J_p = \begin{cases} [s,M) & \text{if } M \le \gamma_p \\ [s,\gamma_p) & \text{if } M > \gamma_p \end{cases}$ and noting $\lambda = \sup J_p$ we have to show that $L_p = \lim_{x\to\lambda} g_p(x) > 0$.

Case 1. $M = \gamma_p = +\infty \Rightarrow J_p = [s,+\infty)$

Observe that for $x \in J_p$, $x > s$ we can write

$$g_p(x) = p(x-s)\left[\frac{e(x)-e(s)}{x-s} - \frac{e\left(\frac{ns-px}{n-p}\right)-e(s)}{\frac{ns-px}{n-p}-s}\right] + n(e(s)-k) \tag{4}$$

Let $r_1 < r_2$ arbitrarily fixed in (s,∞). For any $x > r_2 \Rightarrow \frac{ns-px}{n-p} < s < r_1 < r_2 < x$ and using the strict convexity of e we infer:

$$\underbrace{\frac{e\left(\frac{ns-px}{n-p}\right)-e(s)}{\frac{ns-px}{n-p}-s}}_{E_1} < \underbrace{\frac{e(r_1)-e(s)}{r_1-s}}_{E_2} < \underbrace{\frac{e(r_2)-e(s)}{r_2-s}}_{E_3} < \underbrace{\frac{e(x)-e(s)}{x-s}}_{E_4}$$

We see that $E_4 - E_1 > E_3 - E_2 \overset{def}{=} \lambda_0 > 0$ and thus for any $x > r_2$ we have

$$g_p(x) = p(x-s)(E_4-E_1) + n(e(s)-k) > p\lambda_0(x-s) + n(e(s)-k)$$

therefore $L_p = \lim_{x\to\infty} g_p(x) = +\infty$ (so >0).

Case 2. $M < \gamma_p \Rightarrow J_p = [s,M)$, $\lambda = M$.

Now M is finite $\Rightarrow \lim_{x\to M} e(x) = +\infty$ (because e is an acceptable function). On the other hand, $M < \gamma_p = \frac{ns-(n-p)m}{p} \Rightarrow m < \frac{ns-pM}{n-p} < s$ and so $\frac{ns-pM}{n-p} \in I_S$. Therefore

$$\lim_{x\to\lambda} g_p(x) = \lim_{x\to M}\left[pe(x) + (n-p)e\left(\frac{ns-px}{n-p}\right) - kn\right] = +\infty$$

Case 3. $M > \gamma_p \Rightarrow J_p = [s,\gamma_p)$, $\lambda = \gamma_p$.

Now γ_p is finite so m is also finite and $\lim_{x\to m} e(x) = +\infty$. Notice that $\frac{ns-p\gamma_p}{n-p} = m$ and so $\lim_{x\to\gamma_p} e\left(\frac{ns-px}{n-p}\right) = +\infty$. Therefore

$$\lim_{x\to\lambda} g_p(x) = \lim_{x\to\gamma_p}\left[pe(x) + (n-p)e\left(\frac{ns-px}{n-p}\right) - kn\right] = +\infty$$

Case 4. $M = \gamma_p < +\infty \Rightarrow J_p = [s,M)$, $\lambda = M$.

M and m are both finite so $\lim_{x\to m} e(x) = +\infty$, $\lim_{x\to M} e(x) = +\infty$. Notice that $\frac{ns-pM}{n-p} = \frac{ns-p\gamma_p}{n-p} = m$ so $\lim_{x\to M} e\left(\frac{ns-px}{n-p}\right) = +\infty$. Therefore

$$\lim_{x\to\lambda} g_p(x) = \lim_{x\to M}\left[pe(x) + (n-p)e\left(\frac{ns-px}{n-p}\right) - kn\right] = +\infty$$

□

THEOREM 6. *Let $S(e,s,k,n)$ with $A_S \neq \emptyset$ and $m = \inf(I_S)$, $M = \sup(I_S)$. Then*

(a) If $M \notin I_S$ then (S) has the invariants of order 1

(b) If $m \notin I_S$ then (S) has the invariants of order $(n-1)$

Proof. Notice that $e(s) \geq k$ (because $A_S \neq \emptyset$) and let $c = (c_1 \dots c_n) \in A_S$.

(a) If we also have $m \notin I_S$ then I_S is an open interval and the conclusion follows from the theorem 5 and so we can further assume that $I_S = [m,M)$, M finite or not.

Let $g_1 : J_1 \to \mathbb{R}$, $\quad g_1(t) = e(t) + (n-1)e(\frac{ns-t}{n-1}) - kn$

According to remark 4, $J_1 = \begin{cases} [s,M) & \text{if } M \leq \gamma_1 \\ [s,\gamma_1] & \text{if } M > \gamma_1 \end{cases}$ where $\gamma_1 = ns - (n-1)m$

Case 1. $M > \gamma_1$ then $J_1 = [s,\gamma_1]$ and we have to show that $g_1(\gamma_1) \geq 0$.

Notice that $m = \frac{ns-\gamma_1}{n-1}$ so $g_1(\gamma_1) \geq 0 \Leftrightarrow e(\gamma_1) + (n-1)e(m) \geq kn \Leftrightarrow$

$$e(\gamma_1) + (n-1)e(m) \geq kn = e(c_1) + \dots e(c_n)$$

and this follows from Karamata because, obviously, $(\gamma_1, m, \dots, m) \succcurlyeq (c_1, c_2, \dots c_n)$.

Case 2. $M < \gamma_1$ (this case is only possible if M is finite)

Now $J_1 = [s,M)$ and we have to show that $\lim_{t \to M} g_1(t) > 0$.

But $M < \gamma_1$, thus $s \leq \frac{ns-M}{n-1} < m$ and so $\frac{ns-M}{n-1} \in I_S$ and using also the fact that $\lim_{r \to M} e(r) = +\infty$ (e being an acceptable function) we infer that

$$\lim_{t \to M} g_1(t) = \lim_{t \to M} \left[e(t) + (n-1)e\left(\frac{ns-t}{n-1}\right) - kn \right] = +\infty$$

Case 3. $M = \gamma_1$ (this case is only possible if M is finite)

In this case we also have $J_1 = [s,M)$ and we have to show that $\lim_{t \to M} g_1(t) > 0$.

Notice that $M = \gamma_1 \Rightarrow \frac{ns-M}{n-1} = m$ and we see that $\lim_{r \to M} e(r) = \lim_{r \to m} e(r) = +\infty$ (because M, m are finite and e is an acceptable function). Therefore

$$\lim_{t \to M} g_1(t) = \lim_{t \to M} \left[e(t) + (n-1)e\left(\frac{ns-t}{n-1}\right) - kn \right] = +\infty$$

(b) can be proved in a similar manner.

□

LEMMA 4. *Let $I = [m,M]$ a compact interval, $s \in \mathring{I}$ and $C = \{x \in I^n | x_1 + x_2 + \dots x_n = ns\}$. Then $\exists! u \in C$ of the form $u = (\underbrace{M, \dots M}_{l_0}, \theta, \underbrace{m, \dots m}_{n-l_0-1})$ where $0 \leq l_0 \leq n-1$ and $\theta \in [m,M)$.*

Proof. Let $\lambda = \frac{s-m}{M-m} \in (0,1)$ and $l_0 = [n\lambda] \in \{0, \dots n-1\}$

Next we define $\theta = ns - l_0 M - (n - l_0 - 1)m$ and a straightforward calculation give us $\theta = m + \{n\lambda\}(M-m) \in [m,M)$ and $u \stackrel{def}{=} (\underbrace{M, \dots M}_{l_0}, \theta, \underbrace{m, \dots m}_{n-l_0-1}) \in C$

For uniqueness, we notice that if $u' = (\underbrace{M,\dots M}_{l'_0}, \theta', \underbrace{m,\dots m}_{n-l'_0-1}) \in C$ with $0 \le l'_0 \le n-1$ and $\theta' \in [m,M)$ then $\theta' = ns - l'_0 M - (n-l'_0-1)m$ and from here we immediately get that $n\lambda - l'_0 = \frac{\theta'-m}{M-m} \in [0,1)$ so $l'_0 = [n\lambda] = l_0$ etc.

$\square$

THEOREM 7. *Let $S(e,s,k,n)$ with $A_S \neq \emptyset$ and $m = \inf I_S$, $M = \sup I_S$. Then:*

(a) If $M \in I_S$ and (S) has no invariants of order 1 then there are solutions $x \in A_S$ of the form $x = (M, x_2 \dots x_n)$

(b) If $m \in I_S$ and (S) has no invariants of order $n-1$ then there are solutions $x \in A_S$ of the form $x = (x_1 \dots x_{n-1}, m)$

Proof. (a) Let $\omega \in A_S$. We consider two cases.

Case 1 I_S is compact, so $I_S = [m,M]$.

According to lemma 4, ns has an unique representation of the form $ns = l_0 M + \theta + (n-l_0-1)m$ with $\theta \in [m,M)$ and $0 \le l_0 \le n-1$. First we shall show that $l_0 \ge 1$. If $l_0 = 0$ then we consider $\tilde{u} \stackrel{def}{=} (\theta, m \dots m)$, $\tilde{k} \stackrel{def}{=} \frac{e(\theta)+(n-1)e(m)}{n}$ and, after noticing that $(\omega_1, \omega_2 \dots \omega_n) \preccurlyeq (\theta, m \dots m)$, we infer from Karamata that $k \le \tilde{k}$. But, obviously, $\tilde{S}(e,s,\tilde{k},n)$ has invariants of order 1 (because $\tilde{u} \in A_{\tilde{S}}$) and using the corollary 3 we conclude that (S) also has invariants of order 1, contradiction. Therefore $l_0 \ge 1$.

Next, we prove that $M \le \gamma_1 \stackrel{def}{=} ns - (n-1)m$. If not, $M > \gamma_1$ and from $\gamma_1 \ge m$ we get $\gamma_1 \in [m,M)$, so $ns = \gamma_1 + (n-1)m \Rightarrow l_0 = 0$, contradiction. Therefore $M \le \gamma_1$ and from here we also infer that $\delta \stackrel{def}{=} \frac{ns-M}{n-1} \in [m,M]$.

Let $g_1 : J_1 \to \mathbb{R}$, $g_1(t) = e(t) + (n-1)e\left(\frac{ns-t}{n-1}\right) - kn$ where $J_1 = \begin{cases} [s,M] & \text{if } M < \gamma_1, \\ [s,\gamma_1] & \text{if } M \ge \gamma_1, \end{cases}$ but, according to the above observation, $M \le \gamma_1$ so $J_1 = [s,M]$.

But (S) has no invariants of order 1 and by theorem 4, we infer that $g_1(M) < 0$ so $e(M) + (n-1)e(\delta) < kn$.

Next we define $C = \{(x_2, \dots x_n) \in I^{n-1} | M \ge x_2 \ge \dots \ge x_n,\ M + x_2 + \dots + x_n = ns\}$ and we see that C is a convex set (so it is also connected). Let $u \stackrel{def}{=} (\underbrace{M,\dots M}_{l_0 \ge 1}, \theta, \underbrace{m,\dots m}_{n-l_0-1})$ respectively $v \stackrel{def}{=} (M, \delta \dots \delta)$ and it's clear that $u, v \in C$.

Let $E : C \to \mathbb{R}$, $E(x_2, \dots x_n) = e(x_2) + \dots e(x_n)$. We see that $E(v) < kn$, because $g_1(M) < 0$. On the other hand, we notice that $\omega \preccurlyeq u$ and using Karamata we get $E(\omega) \le E(u)$, therefore $E(u) \ge kn$. But E is a continuous function and C is a connected set and therefore we deduce that $\exists x \in C$ with $E(x) = kn$ which means that (S) has the solution $(M, x_2, \dots x_n)$.

Case 2. I is a non compact interval. This case can be reduced to the previous (compact) case. Indeed, we will first choose an $m < m_1 < M$ such that $m_1 < \omega_n$ and let $I_1 = [m_1, M]$, $e_1 = e\big|I_1$. It's clear that $S_1(e_1, s, k, n)$ is non-empty and has no invariants of order 1 (because they would be valid for (S) as well) and so, according to the compact case, we will find a solution $(M, x_2 \dots x_n) \in A_{S_1}$ but, obviously, this is also a solution for S.

$\square$

2.3. A_S is a compact set

THEOREM 8. *For any $S(e,s,k,n)$ the set A_S is compact.*

Proof. We can assume that $A_S \neq \emptyset$. Let $m = \inf(I) \in \overline{\mathbb{R}}$, $M = \sup(I) \in \overline{\mathbb{R}}$. We will first show that there is a compact interval $J \subset I_S$ with $A_S \subset J^n$.

Let x be an arbitrary point in A_S. According to theorem 6, if $M \notin I_S$ then $\exists (a_1|b_1)_S$ and, using theorem 3, we infer that $x_1 \leq a_1$. Similarly, if $m \notin I_S$ then $\exists (a_{n-1}|b_{n-1})_S$ and $x_n \geq b_{n-1}$. Thus, if we define $m_0 = \begin{cases} m & \text{if } m \in I_S \\ b_{n-1} & \text{if } m \notin I_S \end{cases}$, $M_0 = \begin{cases} M & \text{if } M \in I_S \\ a_1 & \text{if } M \notin I_S \end{cases}$ and $J = [m_0, M_0]$ it follows that $x \in J^n$ and therefore $A_S \subset J^n$.

Next, we see that we can write $A_S = A_1 \cap A_2 \cap E_1 \dots \cap E_{n-1}$ where

$$E_p = \{x \in \mathbb{R}^n | x_{p+1} - x_p \leq 0\} \quad \forall 1 \leq p \leq n-1$$

$$A_1 = \{x \in \mathbb{R}^n | x_1 + x_2 + \dots x_n = ns\}$$

$$A_2 = \{x \in J^n | e(x_1) + e(x_2) + \dots e(x_n) = nk\}$$

and, because these sets are all closed sets we conclude that A_S is a compact set. $\square$

3. Functional dependence. The T_ε transforms

3.1. The $n = 3$ case

LEMMA 5. *Let $S(e,s,k,3)$ be an (S)–system and let $x, y \in A_S$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ with $x_1 \leq y_1$. Then $y_1 \geq x_1 \geq x_2 \geq y_2 \geq y_3 \geq x_3$*

Proof. We have to show that $x_2 \geq y_2$ and also that $y_3 \geq x_3$, the other inequalities being obvious. If $x_3 > y_3$ then, using the fact that $x_1 \leq y_1$, we deduce that $x \prec y$ (strictly majorization) and from Karamata we get $e(x_1) + e(x_2) + e(x_3) < e(y_1) + e(y_2) + e(y_3)$ so $3k < 3k$, a contradiction. Thus $x_3 \leq y_3$. Next, if $x_2 < y_2$ then using $x_1 \leq y_1$ we infer that $x_1 + x_2 < y_1 + y_2$ so $x_3 > y_3$ and further we get a contradiction exactly as above. So we also have $x_2 \geq y_2$.

$\square$

LEMMA 6. *Let $S(e,s,k,3)$ be an (S)–system and let $x, y \in A_S$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. If $x_1 = y_1$ (respectively $x_2 = y_2$ or $x_3 = y_3$) then $x = y$.*

Proof. Let $x_1 = y_1$. Suppose that $x_3 \neq y_3$. Then, for example, $x_3 > y_3$ and from this we get immediately that $x \prec y$ (strict) and applying Karamata to the function e we get $3k < 3k$, a contradiction. So $x_3 = y_3$ and from here we also get $x_2 = 3s - (x_1 + x_3) = 3s - (y_1 + y_3) = y_2$, therefore $x = y$.

$\square$

Because A_S is a compact set we infer that $P_k \overset{def}{=} \text{Pr}_k(A_S)$ $(k = 1,2,3)$ are also compact sets and let $m_k = \min(P_k)$, $M_k = \max(P_k)$ $(k = 1,2,3)$. Thus $P_k \subseteq I_k \overset{def}{=} [m_k, M_k]$ $(k = 1,2,3)$. From now on, we denote by ω the point (unique, according to the lemma 6) for which $\omega_1 = m_1$, respectively by Ω the unique point for which $\Omega_3 = M_3$.

LEMMA 7. *Let $I_k = [m_k, M_k]$ and ω, Ω as above. Then:*

(a) $\omega = (m_1, M_2, m_3)$ *and* $\Omega = (M_1, m_2, M_3)$

(b) $M_1 \geq m_1 \geq M_2 \geq m_2 \geq M_3 \geq m_3$

Proof. 1) Let $\omega = (\omega_1, \omega_2, \omega_3)$ so $\omega_1 = m_1$ and let $x = (x_1, x_2, x_3) \in A_S$ be an arbitrary point. Then $x_3 \geq \omega_3$ because otherwise, using the fact that $x_1 \geq m_1 = \omega_1$, we infer that $\omega \prec x$ (strictly) and applying Karamata to the function e we arrive at the contradiction $3k < 3k$. Because $x \in A_S$ is arbitrary we deduce that $\omega_3 = m_3$. At the same time $x_2 = 3s - (x_1 + x_3) \leq 3s - (\omega_1 + \omega_3) = \omega_2$ but x is an arbitrary point so $\omega_2 = M_2$. Therefore $\omega = (m_1, M_2, m_3)$ and we get similarly that $\Omega = (M_1, m_2, M_3)$.

2) According to (a), $(m_1, M_2, m_3) \in A_S$, $(M_1, m_2, M_3) \in A_S$ but, obviously, $m_1 \leq M_1$ so, using lemma 5, we get $M_1 \geq m_1 \geq M_2 \geq m_2 \geq M_3 \geq m_3$. □

LEMMA 8. *Let $I_S = [m, M]$ and ω, Ω as above. Then:*

(a) Ω *is of the form* $\begin{cases} (a_1, b_1, b_1) = (a_1|b_1)_S & \textit{if } S \textit{ has 1–invariants} \\ (M, a, b) & \textit{if } S \textit{ doesn't have 1–invariants} \end{cases}$

(b) ω *is of the form* $\begin{cases} (a_2, a_2, b_2) = (a_2|b_2)_S & \textit{if } S \textit{ has 2–invariants} \\ (a, b, m) & \textit{if } S \textit{ doesn't have 2–invariants} \end{cases}$

Proof. (a) If $\exists (a_1|b_1)_S$ then, using the extremal properties of invariants, we deduce that $\forall x \in A_S$ $x_3 \leq b_1$ and so we must have $b_1 = M_3 = \Omega_3 \Rightarrow (a_1|b_1)_S = \Omega$.

If $\nexists (a_1|b_1)_S$ then, according to theorem 7 we deduce (S) has solutions of the form (M, a, b). This means that $M_1 = M$ but, according to the lemma 7, $\Omega = (M_1, m_2, M_3) = (M, m_2, M_3)$ and we infer (using lemma 6) that $\Omega = (M, a, b)$.

(b) can be proved in a similar manner. □

LEMMA 9. *A non-empty system $S(e, s, k, 3)$ is trivial if and only if $\omega = \Omega$.*

Proof. If (S) is trivial it's clear that $\omega = \Omega$.

If $\omega = \Omega \Rightarrow (m_1, M_2, m_3) = (M_1, m_2, M_3)$ so $m_k = M_k$ $(k = 1,2,3)$ and clearly $|A_S| = 1$ so (S) is trivial. □

REMARK 5. Thus, if $S(e, s, k, 3)$ is non-trivial, then $\omega \neq \Omega$ and it's clear that $m_k \neq M_k$, so $\mathring{I}_k \neq \emptyset$ $(k = 1,2,3)$. We also infer that $\forall x \in A_S$ with $x_1 \in \mathring{I}_1$ we have $x_2 \in \mathring{I}_2$ and $x_3 \in \mathring{I}_3$ (because if, for example, $x_2 = m_2$ then $x = \Omega$ etc.) and also that $\forall x \in A_S$ with $x_1 \in \mathring{I}_1 \Rightarrow x_1 > x_2 > x_3$.

LEMMA 10. *Let $S(e,s,k,3)$ be a non-empty (S)–system and $I_k = [m_k, M_k]$ as above. Then:*

(a) For any $x_1 \in I_1$ $\exists!(x_2,x_3) \in I_2 \times I_3$ with $(x_1,x_2,x_3) \in A_S$

(b) For any $x_3 \in I_3$ $\exists!(x_1,x_2) \in I_1 \times I_2$ with $(x_1,x_2,x_3) \in A_S$

Proof. (a) Fix $x_1^0 \in I_1$. If $x_1^0 = m_1$ or $x_1^0 = M_1$ then the conclusion follows (because $\omega, \Omega \in A_S$) so we can assume $x_1^0 \in (m_1, M_1)$. Let $f_0 = f\big|[m, x_1^0]$.
Because $\omega = (m_1, M_2, m_3) \in A_S$ and $x_1^0 > m_1 \geq M_2 \geq m_3 \geq m$ it follows that $s \in (m, x_1^0)$ so we have a well-defined (S)–system $S(f_0, s, k, 3)$ for which $\omega \in A_{S_0}$ and so $A_{S_0} \neq \emptyset$.

Observe that $A_{S_0} \subset A_S$ and also that, if S_0 has the 1–invariants (a_1^0, b_1^0) then they are valid for S as well.

We now show that S_0 doesn't have $1-$invariants.
Case 1. (S) doesn't have $1-$invariants. According to the previous observation, neither (S_0) doesn't have $1-$invariants.
Case 2. (S) has $1-$invariants (a_1, b_1) so $M_1 = a_1$. Suppose (S_0) has also $1-$invariants (a_1^0, b_1^0) and then, according to the previous observation, (a_1^0, b_1^0) are valid $1-$invariants for (S) as well and so $(a_1, b_1) = (a_1^0, b_1^0) \Rightarrow a_1^0 = a_1 = M_1$. But $M_1 > x_1^0 \geq a_1^0$ and so we get a contradiction.

Therefore (S_0) is non-empty and without $1-$invariants. According to Theorem 7a, (S_0) has a solution of the form $(x_1^0, x_2^0, x_3^0) \in A_{S_0} \subset A_S$ and this is unique (according to Lemma 6).

(b) Fix $x_3^0 \in I_3$. If $x_3^0 = m_3$ or $x_3^0 = M_3$ then the conclusion follows (because $\omega, \Omega \in A_S$) so we can assume $x_3^0 \in (m_3, M_3)$. Let $f_0 = f\big|[x_3^0, M]$.
Because $\Omega = (M_1, m_2, M_3) \in A_S$ and $M \geq M_1 \geq m_2 \geq M_3 > x_3^0$ it follows that $s \in (x_3^0, M)$ so we have a well-defined (S)–system $S(f_0, s, k, 3)$ for which $\Omega \in A_{S_0}$ and so $A_{S_0} \neq \emptyset$.

Observe that $A_{S_0} \subset A_S$ and also that, if S_0 has the 2–invariants (a_2^0, b_2^0) then they are valid for S as well.

We now show that S_0 doesn't have $2-$invariants.
Case 1. (S) doesn't have $2-$invariants. According to the previous observation, neither (S_0) doesn't have $2-$invariants.
Case 2. (S) has $2-$invariants (a_2, b_2) so $m_3 = b_2$. Suppose (S_0) has also $2-$invariants (a_2^0, b_2^0) and then, according to the previous observation, (a_2^0, b_2^0) would be valid $2-$invariants for (S) as well and so $(a_2, b_2) = (a_2^0, b_2^0) \Rightarrow b_2^0 = b_2 = m_3$. But $m_3 < x_3^0 \leq b_2^0$ and so we get a contradiction.

Therefore (S_0) is non-empty and without $2-$invariants. According to Theorem 7b

(S_0) has a solution of the form $(x_1^0, x_2^0, x_3^0) \in A_{S_0} \subset A_S$ and this is unique (according to Lemma 6). $\square$

THEOREM 9. (the functional dependence) *Let $S(e,s,k,3)$ be a non-empty system and $I_k = [m_k, M_k]$ as above. Then $\exists! u : I_1 \to I_2, v : I_1 \to I_3$ bijective, continuous, monotonic functions (u decreasing, v increasing) such that $A_S = \{(t, u(t), v(t)) | t \in I_1\}$.*

Proof. According to Lemma 10a, $\forall x_1 \in I_1 \;\; \exists!(x_2, x_3) \in I_2 \times I_3$ with $(x_1, x_2, x_3) \in A_S$ therefore $\exists!$ the functions $u : I_1 \to I_2, v : I_1 \to I_3$ with $A_S = \{(t, u(t), v(t)) | t \in I_1\}$. It remains to show that they are continuous, bijective and strictly monotone.

But Lemma 10b also give us the unique functions $\tilde{u} : I_1 \to I_2, \tilde{v} : I_1 \to I_3$ with the property $A_S = \{(\tilde{v}(t), \tilde{u}(t), t) | t \in I_3\}$ and so, for any fixed $(x_1^0, x_2^0, x_3^0) \Rightarrow \begin{cases} x_1^0 = \tilde{v}(x_3^0) = \tilde{v}(v(x_1^0)) \\ x_3^0 = v(x_1^0) = v(\tilde{v}(x_3^0)) \end{cases}$
and this means that $v, \tilde{v}$ are inverse of each other, so they are bijective functions.
Now we show that v is an increasing function on I_1. If not, it follows that $\exists x_1 < x_1' \in I_1$ with $v(x_1) > v(x_1')$. This imply that $(x_1', u(x_1'), v(x_1')) \succ (x_1, u(x_1), v(x_1))$ (strictly) and, applying Karamata to the function e we get the contradiction $3k < 3k$. Therefore v is increasing, in fact strictly increasing (because of bijectivity) and from here we also infer the continuity, because, in general, a bijective and monotone function $f : I \to J$ (where I, J are intervals) is continuous.

In the $u : I_1 \to I_2$ case, we use the relation $u(x_1) = 3s - x_1 - v(x_1)$ and we immediately infer the continuity of u and also that u is strictly decreasing, hence also injective. It remains to show that u is surjective. But $\Omega = (M_1, m_2, M_3) \in A_S \;\Rightarrow m_2 = u(M_1) \Rightarrow m_2 \in Im(u)$ and, similarly, $M_2 \in Im(u)$ and from continuity of u we deduce that $Im(u) = [m_2, M_2]$ so u is also surjective. $\square$

THEOREM 10. *Let $S(e,s,k,3)$ be a nontrivial system and $u : I_1 \to I_2$, $v : I_1 \to I_3$ as above. If, in addition, e is differentiable on $\mathring{I}_S$ then $e \in C^1(\mathring{I}_S)$ and $u, v \in C^1(\mathring{I}_1)$.*

Proof. Because e is strictly convex $\Rightarrow e'$ is strictly increasing on $\mathring{I}_S$ and, using also the intermediate value property of e', we infer that e' is continuous, hence $e \in C^1(\mathring{I}_S)$.

Because (S) is nontrivial it follows (according to Remark 5) that $\mathring{I}_k \neq \emptyset \;\; (k = 1,2,3)$. Next let $F : \mathring{I}_1 \times \mathring{I}_2 \times \mathring{I}_3 \to \mathbb{R}^2, \quad F(x_1,x_2,x_3) = (F_1(x_1,x_2,x_3), F_2(x_1,x_2,x_3))$ where

$$\begin{cases} F_1(x_1,x_2,x_3) = x_1 + x_2 + x_3 - 3s \\ F_2(x_1,x_2,x_3) = e(x_1) + e(x_2) + e(x_3) - 3k \end{cases}$$

Fix $c_1 \in \mathring{I}_1$ and let $c_2 = u(c_1) \in \mathring{I}_2$, $c_3 = v(c_1) \in \mathring{I}_3$. Observe that $c_1 > c_2 > c_3$ (see Remark 5) and also that $F(c_1,c_2,c_3) = 0$. The determinant of the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x_2}(c) & \dfrac{\partial F_1}{\partial x_3}(c) \\ \\ \dfrac{\partial F_2}{\partial x_2}(c) & \dfrac{\partial F_2}{\partial x_3}(c) \end{pmatrix}$$

is $e'(c_2) - e'(c_3) \neq 0$ (because e' is strictly monotone and $c_2 > c_3$). Therefore, by implicit function theorem applied to the C^1 class function $F \Rightarrow \exists I_{c_1} \subset \mathring{I}_1,\ I_{c_2} \subset \mathring{I}_2,\ I_{c_3} \subset \mathring{I}_3$ open intervals centered in c_1, c_2 respectively c_3 and the C^1 class function $g : I_{c_1} \to I_{c_2} \times I_{c_3}$, $g(x_1) = (g_1(x_1), g_2(x_1))$ such that $\forall (x_1, x_2, x_3) \in I_{c_1} \times I_{c_2} \times I_{c_3}$ we have the equivalence:

$$F(x_1, x_2, x_3) = 0 \Leftrightarrow (x_2, x_3) = (g_1(x_1), g_2(x_1))$$

But $\forall (x_1, x_2, x_3) \in I_{c_1} \times I_{c_2} \times I_{c_3} \Rightarrow x_1 > x_2 > x_3$ so $F(x_1, x_2, x_3) = 0 \Leftrightarrow (x_1, x_2, x_3) \in A_S$. On the other hand, we know that $A_S = \{(t, u(t), v(t)) | t \in I_1\}$ so $g_1 \equiv u\big|I_{c_1}$, $g_2 \equiv v\big|I_{c_2}$. We conclude that $u, v \in C^1(\mathring{I}_1)$.

□

3.2. The T_ε transforms. Preliminaries

Let $S(e,s,k,n)$ be an (S)–system given by $\begin{cases} x_1 + x_2 + \ldots + x_n = ns & (1) \\ e(x_1) + e(x_2) + \ldots + e(x_n) = nk & (2) \\ x_1 \geq x_2 \geq \ldots \geq x_n & (3) \end{cases}$

Fix $c = (c_1, \ldots c_n) \in A_S$, $1 \leq i < j < k \leq n$ and let $S'(e, s', k', 3)$ be the (S)–system given by

$$\begin{cases} x'_1 + x'_2 + x'_3 = c_i + c_j + c_k = 3s' \\ e(x'_1) + e(x'_2) + e(x'_3) = e(c_i) + e(c_j) + e(c_k) = 3k' \\ x'_1 \geq x'_2 \geq x'_3 \end{cases}$$

Obviously, $A_{S'} \neq \emptyset$. As in the previous section, we consider the intervals $x'_k \in I'_k = [m'_k, M'_k] (k = 1,2,3)$ and, according to Theorem 9, $\exists!$ the functions $u : I'_1 \to I'_2$, $v : I'_1 \to I'_3$ continuous, bijective, strictly monotonic (u decreasing, v increasing) such that $A_{S'} = \{(t, u(t), v(t)) | t \in I'_1\}$.

For any $t \in I'_1 = [m'_1, M'_1]$ we consider the n-tuple $D(t)$ constructed from c by replacing (c_i, c_j, c_k) with $(t, u(t), v(t))$, thus defining a continuous function $D = D[c_i, c_j, c_k] : I'_1 \to \mathbb{R}^n$. Notice that for any $t \in I'_1$, the n-tuple $D(t)$ satisfies the equalities (1) and (2) of the initial (S)–system, but not necessarily the ordering condition (3).

DEFINITION 5. Let $1 \leq i < j < k \leq n$.

(a) We say that $x \in I_S^n$ satisfies the "ascending" condition $(A^+_{i,j,k})$ if

$$\begin{cases} x_i < \begin{cases} M & \text{if } i = 1 \\ x_{i-1} & \text{if } i > 1 \end{cases} \\ x_j > x_{j+1} \\ x_k < x_{k-1} \end{cases}$$

(b) We say that $x \in I_S^n$ satisfies the "descending" condition $(A^-_{i,j,k})$ if

$$\begin{cases} x_i > x_{i+1} \\ x_j < x_{j-1} \\ x_k > \begin{cases} m & \text{if } k = n \\ x_{k+1} & \text{if } k < n \end{cases} \end{cases}$$

LEMMA 11. *Let $S(e,r,k,n)$ be a non-empty (S)–system, $c \in A_S$, $1 \le i < j < k \le n$ and $D = D[c_i, c_j, c_k] : I'_1 = [m'_1, M'_1] \to \mathbb{R}^n$ as above.*

(a) If c satisfies the $(A^+_{i,j,k})$ condition, then $c_i < M'_1$ and there is a largest interval $J^+ = [c_i, c_i + \varepsilon^*_T] \subset I'_1$ $(\varepsilon^*_T > 0)$ with the property that $D(J^+) \subset A_S$ and $D(t)$ satisfies $(A^+_{i,j,k})$ $\forall t \in [c_i, c_i + \varepsilon^*_T)$.

(b) If c satisfies the $(A^-_{i,j,k})$ condition, then $c_i > m'_1$ and there is a largest interval $J^- = [c_i - \varepsilon^*_B, c_i] \subset I'_1$ $(\varepsilon^*_B > 0)$ with the property that $D(J^-) \subset A_S$ and $D(t)$ satisfies $(A^-_{i,j,k})$ $\forall t \in (c_i - \varepsilon^*_B, c_i]$.

Proof. (a) According to Lemma 8, $\Omega' = \begin{cases} (a'_1, b'_1, b'_1) & \text{if } S' \text{ has 1–invariants} \\ (M, a', b') & \text{if } S' \text{ doesn't have 1–invariants} \end{cases}$ and from this it follows that $(c_i, c_j, c_k) \ne \Omega'$ (otherwise we have either $c_j = c_k$, either $c_i = M$, impossible). On the other hand, according to Lemma 7, we know that $\Omega' = (M'_1, m'_2, M'_3)$ and because $(c_i, c_j, c_k) \ne \Omega'$ it follows that $c_i < M'_1$.

The point $D(c_i) = c$ satisfies the strict inequalities in $(A^+_{i,j,k})$ and using the continuity of D we deduce that $\exists \varepsilon > 0$ such that $\forall t \in [c_i, c_i - \varepsilon)$, the point $D(t)$ also satisfies the strict inequalities in $(A^+_{i,j,k})$.

It's clear that $D(t)$ also satisfies the ordering condition (3) hence $D(t) \in A_S$ $\forall t \in [c_i, c_i + \varepsilon)$. Next we define

$$\varepsilon^*_T = \sup\{\varepsilon > 0 | D(t) \text{ satisfies } (A^+_{i,j,k}) \quad \forall t \in [c_i, c_i + \varepsilon)\}$$

and let $J^+ = [c_i, c_i + \varepsilon^*_T]$. It's clear that $D(t) \in A_S$ $\forall t \in [c_i, c_i + \varepsilon^*_T)$ and, at the same time $D(c_i + \varepsilon^*_T) \in A_S$ because we can choose a sequence $(t_m)_{m \ge 1} \subset [c_i, c_i + \varepsilon^*_T)$ with $t_m \to c_i + \varepsilon^*_T$ and from continuity of D we infer that $D(t_m) \to D(c_i + \varepsilon^*_T)$, but $D(t_m) \in A_S$ and A_S is a compact set, hence $D(c_i + \varepsilon^*_T) \in A_S$. □

REMARK 6. Let $d^* = D(c_i + \varepsilon^*_T) \in A_S$. Because $d^*_l = c_l$ $\forall l \ne i,j,k$ we have

$$M \ge \ldots \ge c_{i-1} \ge d^*_i \ge \ldots \ge d^*_j \ge c_{j+1} \ge \ldots \ge c_{k-1} \ge \ldots \ge d^*_k$$

On the other hand, it's clear that d^* cannot satisfies the strict conditions in $A^+_{i,j,k}$ (otherwise, following exactly the above steps, we could extend the interval J^+ but this contradict the maximality of J^+) and from this we infer that d^* must satisfy at least one of the following equalities

$$\begin{cases} d_i^* = \begin{cases} M & \text{if } i = 1 \\ c_{i-1} & \text{if } i > 1 \end{cases} \\ d_j^* = d_k^* \quad \text{if } j+1 = k \\ d_j^* = c_{j+1} \quad \text{if } j+1 < k \\ d_k^* = c_{k-1} \quad \text{if } j+1 < k \end{cases}$$

LEMMA 12. *Let $c \in A_S$ satisfying the $A^+_{i,j,k}$ condition and let J^+ be the interval given by Lemma 11. Then $\forall t \in J^+$ the points c and $D(t)$ belong to the same connected component of A_S.*

Proof. Let $C_1 \subset A_S$ the connected component that contains c. Using the continuity of D it follows that $C_2 \stackrel{def}{=} D(J^+)$ is a connected set and $c \in C_2 \subset A_S$. Thus $C_1 \cup C_2$ is a connected subset of A_S and, from the maximality of C_1, we infer that $C_2 \subset C_1$ etc. □

3.3. The T_ε transforms

Let $S(e,s,k,n)$ be an (S)–system, $1 \le i < j < k \le n$, $c \in A_S$ and $D = D[c_i,c_j,c_k] : I'_1 \to \mathbb{R}^n$ defined as in previous section.

We have seen that if c satisfies the $A^+_{i,j,k}$ condition then exists a largest interval $J^+ = [c_i, c_i + \varepsilon^*_T]$ ($\varepsilon^*_T > 0$) with the property that $D(J^+) \subset A_S$.

Similarly, if c satisfies the $A^-_{i,j,k}$ condition then exists a largest interval $J^- = [c_i - \varepsilon^*_B, c_i]$ ($\varepsilon^*_B > 0$) with the property that $D(J^-) \subset A_S$.

DEFINITION 6. Let c satisfying the $A^+_{i,j,k}$ condition and $\varepsilon \in [0, \varepsilon^*_T]$. We say that the n-tuple $c' \in A_S$ is a $T^+_\varepsilon(i,j,k)[c]$ transform of c and we write $c' = T^+_\varepsilon(i,j,k)[c]$ if $c' = D(c_i + \varepsilon)$.

The $T^-_\varepsilon(i,j,k)[c]$ transforms are similarly defined.

We notice that when we apply to c a $T^+_\varepsilon(i,j,k)[c]$ transform (for example) then c_i and c_k "increase" and c_j "decreases" (the precise meaning is that $c'_i > c_i$, $c'_k > c_k$ and $c'_j < c_j$). This follows, of course, from the monotony of the u and v functions (u is strictly decreasing and v strictly increasing). We can also observe that $c'_i + c'_j = 3s - c'_k < 3s - c_k = c_i + c_j$ so, by applying a T^+_ε transform, the sum $c_i + c_j$ (or $c_j + c_k$) "decreases".

A $T^+_\varepsilon | T^-_\varepsilon$ transform is called *strict* if $\varepsilon \in (0, \varepsilon^*_T)$, respectively $\varepsilon \in (0, \varepsilon^*_B)$. We notice that if $c' = T^+_\varepsilon(i,j,k)[c]$ is a strict transform then c' still satisfies the $A^+_{i,j,k}$ condition (respectively $A^-_{i,j,k}$ in the T^-_ε case).

LEMMA 13. *(a) If $x \in A_S$ satisfies the $A^+_{i,j,k}$ condition then there is a chain of strict transforms of type T^+_ε that map x to an $y \in A_S$ with $y_n > x_n$.*

(b) If $x \in A_S$ satisfies the $A^-_{i,j,k}$ condition then there is a chain of strict transforms of type T^-_ε that map x to an $y \in A_S$ with $y_1 < x_1$.

Proof. (a) Case 1 $k=n$. We can apply to x a strict transform $y=T_{\varepsilon}^{+}(i,j,n)[x]$ and, obviously, $y_n > x_n$.

Case 2 $k<n$. We start by applying to x a strict transform $x'=T_{\varepsilon}^{+}(i,j,k)[x]$ for which, obviously, $x'_k > x_k$ and so we are sure that we also have $x'_k > x'_{k+1} = x_{k+1}$. If $k+1=n$ we continue exactly as in the case 1. If not, we apply to x' a strict transform $x''=T_{\varepsilon}^{+}(i,j,k+1)[x']$ for which $x''_{k+1} > x''_{k+2} = x_{k+2}$ and so on.

For (b) the proof is similar to the above. □

3.4. The poles ω, Ω

Let $S(e,s,k,n)$ be an (S)–system. Because A_S is a compact set it follows that $P_k \overset{def}{=} \mathrm{Pr}_k(A_S)$ $(k=1,2\ldots n)$ are also compact sets and let $m_k=\min(P_k)$, $M_k=\max(P_k)$ $(k=1,2\ldots n)$, hence $P_k \subseteq I_k \overset{def}{=} [m_k, M_k]$ $(k=1,2\ldots n)$

In particular, we deduce that there exists points $\omega \in A_S$ for which $\omega_1 = m_1$ (or points $\Omega \in A_S$ for which $\Omega_n = M_n$).

LEMMA 14. *Let $\Omega \in A_S$ for which $\Omega_n = M_n$. Then Ω is of the form*

$$\Omega = (\underbrace{M,\ldots M}_{r\geq 0}, a, b\ldots b)$$

where $r \geq 0$ and $a,b \in I_S$ with $a \geq b = M_n$

Proof. We can start, obviously, by writing Ω in the form $\Omega = (\underbrace{M,\ldots M}_{r\geq 0}, \Omega_{r+1}, \ldots \Omega_n)$.

If $r \geq n-2$ our problem is solved, so we can assume $r \leq n-3$ with $\Omega_{r+1} \neq M$. If there exists $r+1 < i < n$ with $\Omega_i > \Omega_{i+1}$ then, considering that $\Omega_{r+1} < M$, we infer that Ω satisfies the $A^{+}_{r+1,i,i+1}$ condition hence, according to Lemma 13, there is a chain of strict transforms of type T_{ε}^{+} that map Ω to an $\Omega' \in A_S$ with $\Omega'_n > \Omega_n = M_n$, a contradiction. Therefore $\Omega_{r+2} = \ldots = \Omega_n$ etc.

□

LEMMA 15. *If $\Omega, \Omega' \in A_S$ are of the form* $\begin{cases} \Omega = (\underbrace{M,\ldots M}_{r\geq 0}, a, b\ldots b) \\ \Omega' = (\underbrace{M,\ldots M}_{r'\geq 0}, a', b'\ldots b') \end{cases}$ *where $a \geq b$, $a' \geq b'$ then $\Omega = \Omega'$.*

Proof. Without loss of generality we may assume that $b \geq b'$ and from this we infer

$$\begin{cases} T_k(\Omega) \leq T_k(\Omega') \ \forall k = 1\ldots r \\ B_k(\Omega) \geq B_k(\Omega') \ \forall k = r+2\ldots n \end{cases}$$

and this means $\Omega \preccurlyeq \Omega'$ (according to Remark 2). Suppose $\Omega \neq \Omega'$. Then $\Omega' \prec \Omega$ (strictly) and applying Karamata to the strictly convex function e we get $kn < kn$, a contradiction. □

THEOREM 11. *Let* $S(e,s,k,n)$ *an* (S)*–system and* $m=\inf(I_S)$, $M=\sup(I_S)$. *Then:*

(a) There exists a unique point $\Omega \in A_S$ *for which* $\Omega_n = M_n$. *Moreover, it is of the form*

$$\Omega = (\underbrace{M,\dots M}_{r\geq 0},a,b,\dots b)$$

Conversely, $\forall \Omega' \in A_S$ *of the form* $\Omega' = (\underbrace{M,\dots M}_{r'\geq 0},a',b',\dots b') \Rightarrow \Omega' = \Omega$.

(b) There exists a unique point $\omega \in A_S$ *for which* $\omega_1 = m_1$. *Moreover, it is of the form*

$$\omega = (a,\dots a,b,\underbrace{m,\dots m}_{r\geq 0})$$

Conversely, $\forall \omega' \in A_S$ *of the form* $\omega' = (a',\dots a',b',\underbrace{m,\dots m}_{r'\geq 0}) \Rightarrow \omega' = \omega$.

Proof. (a) Let $\Omega,\Omega' \in A_S$ two points for which $\Omega_n = \Omega'_n = M_n$. Then, according to Lemma 14, Ω and Ω' are of the form $\begin{cases} \Omega = (\underbrace{M,\dots M}_{r\geq 0},a,b\dots b) \\ \Omega' = (\underbrace{M,\dots M}_{r'\geq 0},a',b'\dots b') \end{cases}$ and applying Lemma 15 we infer $\Omega = \Omega'$. The converse follows, obviously, from Lemma 15.

(b) The lemmas 14 and 15 has similar versions for the ω case and after that the proof is similar to the above. □

REMARK 7. We call these two points Ω, ω *the poles of the system* (upper and lower) and we can show that $[m_1,M_1]=[\omega_1,\Omega_1]$ and $[m_n,M_n]=[\omega_n,\Omega_n]$. For the first equality, for example, we observe that, by definition, $\omega_1 = m_1$. On the other hand, Ω is of the form $(\underbrace{M,\dots M}_{r\geq 0},a,b,\dots b)$. If $r>0$ then, $\Omega_1 = M = M_1$ and if $r=0$ then $\Omega = (a_1|b_1)_S$ but, in this case, $a_1 = M_1$ (according to Theorem 3a) and so again $\Omega_1 = M_1$.

REMARK 8. If $x \neq \Omega$ we can prove that there exist $1 \leq i < j < n$ such that x satisfies the $(A^+_{i,j,j+1})$ condition. According to Theorem 11, x is *not* of the form $(\underbrace{M,\dots M}_{r\geq 0},a,b,\dots b)$ (*). It's clear then that $\exists i \leq n-2$ with $x_i < M$ and, supposing i minimal with this property, we also find $i<j<j+1\leq n$ with $x_j > x_{j+1}$, otherwise x would be of the form (*).

Similarly, if $x \neq \omega$ we deduce that there exist $1 \leq i < i+1 < j \leq n$ such that x satisfies the $(A^-_{i,i+1,j})$ condition.

THEOREM 12. *Let* $S(e,s,k,n)$ *be a non-empty* (S)*–system. The following assertions are equivalent:*

(a) $|A_S| = 1$ *(that is, S is trivial)*

(b) $\omega = \Omega$

(c) $\exists x \in A_S$ *of the form* $x = (\theta, \theta, \dots, \theta)$ *or* $x = (\underbrace{M, \dots M}_{r \geq 0}, \theta, \underbrace{m, \dots m}_{t \geq 0})$

Proof. $(a) \Rightarrow (b)$ it's obvious.

$(b) \Rightarrow (a)$ If $\omega = \Omega$ then, according to remark 7, we infer that $m_1 = M_1$ and so, for an arbitrary $x \in A_S$ we deduce that $x_1 = m_1$. But this means, according to Theorem 11, that $x = \omega$. Hence $A_S = \{\omega\}$ etc.

$(c) \Rightarrow (b)$ From Theorem 11 we know that for any point $\Omega' \in A_S$ of the form $\Omega' = (\underbrace{M, \dots M}_{r' \geq 0}, a', b', \dots b') \Rightarrow \Omega' = \Omega$. But x, in either of the two variants, is also of that form and so $x = \Omega$. In a similar manner we deduce that $x = \omega$ hence $\Omega = \omega$.

$(b) \Rightarrow (c)$ Let $\Omega = (\underbrace{M, \dots M}_{r \geq 0}, a, b, \dots b)$, $\omega = (a', \dots a', b', \underbrace{m, \dots m}_{r' \geq 0})$.

Case 1 $r > 0$. We know that $\omega = \Omega$ hence $a' = M$ and $\omega = (M, \dots M, b', \underbrace{m, \dots m}_{r' \geq 0})$

Case 2 $r' > 0$. Using $\omega = \Omega$ it follows that $b = m$ hence $\Omega = (\underbrace{M, \dots M}_{r \geq 0}, a, m, \dots m)$

Case 3. $r = 0$, $r' = 0$. Then $\omega = \Omega \Leftrightarrow (a, b \dots b) = (a', \dots a', b')$ hence $a = a' = b = b' = \theta$ and $\omega = (\theta, \theta, \dots, \theta)$.

□

3.5. A_S is a connected set

THEOREM 13. *Let $S(e,s,k,n)$ be an (S)–system. Then A_S is a connected set.*

Proof. Suppose that A_S is not connected, hence there exist at least two connected components that are also compact sets, because A_S is compact. Let C_1 be the connected component that contains the point Ω and let $C_2 \neq C_1$ be another one. Using the compactness of C_2, we can choose a point $x = (x_1, x_2, \dots x_n) \in C_2$ with maximal x_n.

According to Remark 8 $\Rightarrow$ there exist indices $i < j < k$ such that x satisfies the "ascending" condition $A^+_{i,j,k}$ and applying Lemma 13a, we get a chain of strict T^+_ε transforms that map x to an y with $y_n > x_n$.

On the other hand, according to Lemma 12, for any $w' = T^+_\varepsilon(i,j,k)[w]$ transform, the point w' belongs to the same connected component as w, hence x and y are both contained in C_2. But $y_n > x_n$ and this contradicts the maximality of x_n.

□

COROLLARY 4. *Let $S(e,k,s,n)$ be an (S)–system and $I_r = [m_r, M_r], 1 \leq r \leq n$. If $P_r = \mathrm{Pr}_r(A_S)$ then $P_r = I_r$, hence I_r is exactly the set of all possible values of the x_r component ($x \in A_S$).*

4. Extension of the Karamata's inequality and related results

4.1. The $\preccurlyeq_p$ and $\trianglelefteq$ relations

Fix $1 \le p \le n-1$ and let $x, y \in A_S$.

$$y = (\overbrace{y_1,\ y_2,\ \dots\ y_{p-1}}^{T\ \text{zone}},\ y_p,\ y_{p+1},\ \overbrace{y_{p+2},\ \dots\ y_{n-1}, y_n}^{B\ \text{zone}})$$
$$x = (\underbrace{x_1,\ x_2,\ \dots\ x_{p-1}}_{T\ \text{zone}},\ x_p,\ y_{p+1},\ \underbrace{y_{p+2},\ \dots\ x_{n-1}, x_n}_{B\ \text{zone}})$$

By definition,

$$x \preccurlyeq_p y \Leftrightarrow \begin{cases} T_k(x) \le T_k(y) & \forall 1 \le k \le p-1 \\ B_k(x) \le B_k(y) & \forall p+2 \le k \le n \end{cases} \tag{5}$$

where $T_k(x) = x_1 + \ldots + x_k$ (top sums) and $B_k(x) = x_k + \ldots + x_n$ (bottom sums).

Note that for $p = 1$ the definition is equivalent to $B_k(x) \le B_k(y) \quad \forall 3 \le k \le n$ (that is, the T zone is empty) and for $p = n-1$ the definition is equivalent to $T_k(x) \le T_k(y) \quad \forall 1 \le k \le n-2$ (so B zone is empty).

We also consider the strict version of this relation, that is, we say that $x \prec_p y$ if $x \preccurlyeq_p y$ and at least one of the inequalities (5) is strict.

LEMMA 16. *Let $x, y \in A_S$. If $x \preccurlyeq_p y$ then $x_1 \le y_1$ and $x_n \le y_n$.*

Proof. If $p \ge 2$ the definition (5) implies in particular that $T_1(x) \le T_1(y)$ so $x_1 \le y_1$. If $p = 1$ then (5) $\Leftrightarrow$ $B_k(x) \le B_k(y) \quad \forall 3 \le k \le n$ and if $x_1 > y_1$ we infer $x \succ y$ but, applying Karamata to e, we arrive to the contradiction $kn > kn$. Hence $x_1 \le y_1$ and we can prove similarly that $x_n \le y_n$. □

DEFINITION 7. Let $x \in A_S$ and $1 \le i_1 < i_2 < \ldots < i_r \le n$ (fixed indices).

(a) We define $x \setminus (x_{i_1}, \ldots x_{i_r})$ as being that $(n-r)$ tuple constructed from x by removing the components $x_{i_1}, \ldots x_{i_r}$.

(b) We define a "reduced" system $S'(e, k', s', n')$ (where $n' = n - r$) by:

$$\begin{cases} t_1' + \ldots + t_{n'}' = ns - (x_{i_1} + \ldots + x_{i_r}) = n's' \\ e(t_1') + \ldots + e(t_{n'}') = nk - (e(x_{i_1}) + \ldots + e(x_{i_r})) = n'k' \\ t_1' \ge t_2' \ge \ldots \ge t_{n'}' \end{cases}$$

denoted also by $\hat{S}[x_{i_1}, x_{i_2} \ldots x_{i_r}]$.

Notice that $x \setminus (x_{i_1}, \ldots x_{i_r}) \in \hat{S}[x_{i_1}, x_{i_2} \ldots x_{i_r}]$

LEMMA 17. *Let $x, y \in A_S$ with $x \preccurlyeq_p y$ and suppose $\exists r$ with $x_r = y_r$. If $x' = x \setminus (x_r)$ and $y' = y \setminus (y_r)$ then $\exists 1 \le p' \le n'-1$ such that $x' \preccurlyeq_{p'} y'$ (where $n' = n-1$).*

Proof. It' clear that $x, y \in \hat{S}[x_r]$ and we can choose $p' = p-1$ (if $p \ge 2$) or $p' = 1$ (if $p = 1$) so, in general, we can choose $p' \in \{p-1, p\}$. □

LEMMA 18. *Let $x, y \in A_S$ and $1 \leq p, q \leq n-1$. If $x \preccurlyeq_p y$ and $y \preccurlyeq_q x$ then $x = y$.*

Proof. The proof is by induction on n. Using Lemma 16, we first observe that $x \preccurlyeq_p y \Rightarrow x_1 \leq y_1$ and $y \preccurlyeq_q x \Rightarrow y_1 \leq x_1$, hence $x_1 = y_1$.

For $n = 3$ the conclusion follows directly from Lemma 6.

If $n > 3$ we consider the points $x' = x \setminus (x_1)$ and $y' = y \setminus (y_1)$ so, according to Lemma 17, $\exists 1 \leq p', q' \leq n' - 1$ with $x' \preccurlyeq_{p'} y'$ and $y' \preccurlyeq_{q'} x'$ (where $n' = n - 1$). But $x', y' \in A_{S'}$ where $S'(e, s', k', n')$ is the reduced system $\hat{S}[x_1]$ and so, by the induction hypothesis, it follows that $x' = y'$, hence $x = y$. □

THEOREM 14. *$\preccurlyeq_p$ is an order relation on A_S*

Proof. The reflexivity and transitivity are evident and antisymmetry follows from Lemma 18. □

COROLLARY 5. *Let $x, y \in A_S$ with $x \preccurlyeq_p y$. Then $x \prec_p y \Leftrightarrow x \neq y$*

Proof. If $x \prec_p y$ then it's clear that $x \neq y$.

If $x \neq y$ then at least one of the inequalities (5) is strict. Otherwise, we would have at the same time $x \preccurlyeq_p y$ and $y \preccurlyeq_p x$, hence $x = y$. □

LEMMA 19. *Let $x, y \in A_S$ with $x \prec_p y$. Then $\exists\ r \leq p < p+1 \leq t$ such that $x_r < y_r$, $x_t < y_t$.*

Proof. We will show that $\exists r \leq p$ such that $x_r < y_r$. Otherwise, $x_i > y_i \ \forall 1 \leq i \leq p$ hence $T_i(x) > T_i(y) \ \forall 1 \leq i \leq p$ but this, together with the $B_i(x) \leq B_i(y)$ $(i = p+2 \ldots n)$ inequalities, implies that $x \succ y$ (strictly) and, applying Karamata to the strictly convex function e, we get $nk > nk$, a contradiction. □

THEOREM 15. *Let ω, Ω be the poles of the $S(e, s, k, n)$ and let $x \in A_S$ be an arbitrary point. Then there exists $1 \leq p, q \leq n-1$ such that $\Omega \succcurlyeq_p x \succcurlyeq_q \omega$.*

Proof. We will show that $\exists 1 \leq p \leq n-1$ such that $\Omega \succcurlyeq_p x$. We know that Ω is of the form $\Omega = \begin{matrix} 1 & \ldots & r-1 & r & r+1 & r+2 & \ldots & n \\ (M, & \ldots & M, & a, & b, & b, & \ldots & b) \end{matrix}$ for some $r \geq 1$

and, by the definition of Ω, we know that $x_n \leq b$.

It's clear that $T_k(\Omega) \geq T_k(x) \ \forall 1 \leq k \leq r-1$ and, if it happens that $B_k(\Omega) \geq B_k(x)$ $\forall r+2 \leq k \leq n$, then it follows trivially that $\Omega \succcurlyeq_r x$. If not, there exists an index $r+2 \leq k \leq n$ such that $B_k(\Omega) < B_k(x)$ and we suppose k largest with this property. Because $\Omega_n = b \geq x_n$ we see that $k < n$.

So we have, for now, $B_i(\Omega) \geq B_i(x) \ \forall k+1 \leq i \leq n$ and $B_k(\Omega) < B_k(x)$. We will prove that $\Omega \succcurlyeq_{k-1} x$ and for this we need that $T_j(\Omega) \geq T_j(x) \ \ \forall 1 \leq j \leq k-2$. We already know that $T_j(\Omega) \geq T_j(x) \ \forall 1 \leq j \leq r-1$, so we can assume $r \leq j \leq k-2$. If, by reductio ad absurdum, there exists $r \leq j \leq k-2$ such that $T_j(\Omega) < T_j(x)$ then

$$M(r-1) + a + (j-r)b < x_1 + \ldots + x_j \tag{6}$$

But $B_k(\Omega) < B_k(x) \Rightarrow$

$$(n-k+1)b < x_k + \ldots + x_n \tag{7}$$

and from (6) and (7) we infer

$$M(r-1) + a + [n-r-(k-j-1)]b < (x_1 + \ldots + x_j) + (x_k + \ldots + x_n)$$
$$\Rightarrow ns - (k-j-1)b < ns - (x_{j+1} + \ldots + x_{k-1})$$
$$\Rightarrow (k-j-1)b > x_{j+1} + \ldots + x_{k-1}$$

Hence $b > x_{k-1}$ but from (7) it also follows that $b < x_k \le x_{k-1}$, a contradiction. The proof for $x \succcurlyeq_q \omega$ is similar to the above. □

DEFINITION 8. If $x, y \in A_S$ we say that $x \trianglelefteq y$ if $\exists 1 \le p \le n-1$ with $x \preccurlyeq_p y$

REMARK 9. The $\trianglelefteq$ relation is, obviously, reflexive and antisymmetric (according to Lemma 18) but, unfortunately, it's not also transitive so, in general, $\trianglelefteq$ is not an order relation.

The fact that it is not transitive follows from a counterexample. We consider the system $S(e, \frac{2}{5}, \frac{44}{5}, 5)$ where $e : \mathbb{R} \to \mathbb{R}, \quad e(x) = x^2$ and we will arrive at a counterexample by a convenient deformation of the following points in A_S:

$$z = (3 + \frac{\sqrt{35}}{2},\ 3 - \frac{\sqrt{35}}{2},\ 0,\ -\frac{3}{2},\ -\frac{5}{2})$$
$$y = (3 + 2\sqrt{2},\ 3 - 2\sqrt{2},\ 0,\ -1,\ -3)$$
$$x = (3 + \frac{3\sqrt{3}}{2},\ 3 - \frac{3\sqrt{3}}{2},\ 0,\ -\frac{1}{2},\ -\frac{7}{2})$$

First, observe that $\begin{cases} x_1 < y_1 < z_1, \quad x_5 < y_5 < z_5 \\ x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = 6 \\ x_4 + x_5 = y_4 + y_5 = z_4 + z_5 = -4 \end{cases}$.

Next, we see that $x_1 > x_2 > x_3$ so there exist strict transforms $x' = T_\varepsilon^-(1,2,3)[x]$. We have $x'_1 < x_1$ and $x'_1 + x'_2 > x_1 + x_2 = 6$.

Similarly, we can apply to z a transform $z' = T_\varepsilon^+(3,4,5)[z]$, we have $z'_5 > z_5$ and also $z'_4 + z'_5 < z_4 + z_5 = -4$.

Finally, we see that $x' \preccurlyeq_2 y$, $y \preccurlyeq_3 z'$ but it's not possible to choose an index $1 \le p \le 4$ with $x' \preccurlyeq_p z'$ because $x'_1 + x'_2 > 6 = z'_1 + z'_2$ and $x'_4 + x'_5 = -4 > z'_4 + z'_5$.

4.2. The perturbation lemmas

DEFINITION 9. Fix $1 \le p \le n-1$ and let $x, y \in A_S$ with $x \preccurlyeq_p y$. We say that:

(a) there exist equal sums in (T) if $\exists 1 \le k \le p-1$ with $T_k(x) = T_k(y)$.

(b) all (T)-sums are distinct if $T_k(x) \ne T_k(y)\ \ \forall 1 \le k \le p-1$.

(and similarly for B-zone)

$$y = (\overbrace{y_1,\ y_2,\ \dots\ y_{p-1}}^{T\text{ zone}},\ y_p,\ y_{p+1},\ \overbrace{y_{p+2},\ \dots\ y_{n-1}, y_n}^{B\text{ zone}})$$
$$x = (\underbrace{x_1,\ x_2,\ \dots\ x_{p-1}}_{T\text{ zone}},\ x_p,\ y_{p+1},\ \underbrace{y_{p+2},\ \dots\ x_{n-1}, x_n}_{B\text{ zone}})$$

If there exist equal sums in (T), we also consider the extreme indices $a \le b$ such that $\begin{cases} T_a(x) = T_a(y), & T_b(x) = T_b(y) \\ T_k(x) < T_k(y), & \forall k \in \{1 \dots a-1\} \cup \{b+1 \dots p-1\} \end{cases}$

Similarly, if there exist equal sums in (B), we consider the extreme indices $c \le d$ such that $\begin{cases} B_c(x) = B_c(y), & B_d(x) = B_d(y) \\ B_k(x) < B_k(y), & \forall k \in \{p+2 \dots c-1\} \cup \{d+1 \dots n\} \end{cases}$

LEMMA 20. *Fix $1 \le p \le n-1$ and let $x, y \in A_S$ with $x \preccurlyeq_p y$.*

A) *1) If $x_1 < y_1$ then $\exists 2 \le i \le n-1$ with $x_i > x_{i+1}$*

2) If $x_n < y_n$ then $\exists 1 \le i \le n-2$ with $y_i > y_{i+1}$

B) *1) If $x_1 < y_1$ and there exist equal sums in (T) then $\exists 1 \le i \le a-1$ with $y_i > y_{i+1}$*

2) If $x_n < y_n$ and there exist equal sums in (B) then $\exists d \le i \le n-1$ with $x_i > x_{i+1}$

Proof. (A) If (1) is not true, then $x_i = x_{i+1}\ \ \forall 2 \le i \le n-1 \ \Rightarrow\ x_2 = x_3 = \ldots = x_n$ and so $x = (a_1|b_1)_S \Rightarrow x_1 = a_1$. But, from the extremal properties of invariants we know that $y_1 \le a_1$ hence $y_1 \le x_1$, a contradiction. For (2) the proof is similar.

$B)$ If (1) is not true, then $y_i = y_{i+1}\ \ \forall 1 \le i \le a-1 \ \Rightarrow\ y_1 = \ldots = y_a$ and so $y_1 = \frac{T_a(y)}{a}$. On the other hand, $T_a(x) = T_a(y)$ and, obviously, $x_1 \ge \frac{T_a(x)}{a} = \frac{T_a(y)}{a}$ hence $x_1 \ge y_1$, a contradiction. The proof of (2) is similar.

□

LEMMA 21. *Fix $1 \le p \le n-1$ and let $x, y \in A_S$ with $x \preccurlyeq_p y$ and $x_1 < y_1$*

A) If all (T)-sums are distinct and, also, all (B)-sums are distinct then there exist strict transforms $z = T_\varepsilon^+(1, i, i+1)[x]$ with $2 \le i \le n-1$ such that $z \preccurlyeq_p y$

B) If all (T)-sums are distinct but there exists equal sums in (B) then there exist strict transforms $z = T_\varepsilon^+(1, i, i+1)[x]$ with $d \le i \le n-1$ such that $z \preccurlyeq_p y$

C) Suppose there exists equal sums in (T)

(a) If $T_{a+1}(x) \le T_{a+1}(y)$ then there exist strict transforms $z = T_\varepsilon^+(1, a, a+1)[y]$ such that $z \preccurlyeq_p y$

(b) If $T_{a+1}(x) > T_{a+1}(y)$ *then* $p \geq 2$ *and there exist strict transforms* $z = T_\varepsilon^+(1,i,i+1)[y]$ *such that* $z \preccurlyeq_{p-1} y$.

Proof. $A)$ By hypothesis, we have $\begin{cases} T_k(x) < T_k(y) & \forall 1 \leq k \leq p-1 \\ B_k(x) < B_k(y) & \forall p+2 \leq k \leq n \end{cases}$ and, according to Lemma 20 $(A1)$ we know that $\exists 2 \leq i \leq n-1$ with $x_i > x_{i+1}$. Because the above inequalities are strict, there exists an $\varepsilon > 0$ such that the transform $z = T_\varepsilon^+(1,i,i+1)[x]$ still verify the strict inequalities $\begin{cases} T_k(z) < T_k(y) & \forall 1 \leq k \leq p-1 \\ B_k(z) < B_k(y) & \forall p+2 \leq k \leq n \end{cases}$ hence $z \preccurlyeq_p y$.

$B)$ According to Lemma 20 $(B2)$ we know that $\exists d \leq i \leq n-1$ such that $x_i > x_{i+1}$. Because $i+1 > d$ we have $B_{i+1}(x) < B_{i+1}(y)$ $[*]$ and, by hypothesis, we also have $T_k(x) < T_k(y)$ $\forall 1 \leq k \leq p-1$ $[**]$

Because the inequalities $[*]$ and $[**]$ are strict there exists an $\varepsilon > 0$ such that the transform $z = T_\varepsilon^+(1,i,i+1)[x]$ still verify the strict inequalities

$$\begin{cases} T_k(z) < T_k(y) & \forall 1 \leq k \leq p-1 \\ B_{i+1}(z) < B_{i+1}(y) \end{cases}$$

and so it only remains to show that $B_k(z) < B_k(y)$ $\forall p+2 \leq k \leq n,\ k \neq i+1$

We notice that for $k \neq i+1$ a $B_k(x)$ sum can contains either the both terms x_i and x_{i+1}, either none of them. In the first case it's clear that by the $z = T_\varepsilon^+(1,i,i+1)[x]$ transform the sum $x_i + x_{i+1}$ can only decrease to $z_i + z_{i+1}$ and definitely $B_k(z) < B_k(y)$. In the second case, the sum $B_k(x)$ obviously remains unaffected by the $z = T_\varepsilon^+(1,i,i+1)[x]$ transform, hence $B_k(z) = B_k(x) \leq B_k(y)$.

$C1)$ We first show that $x_a > x_{a+1}$. Because $T_1(x) < T_1(y)$ it's clear that $a \geq 2$. We have $T_{a-1}(x) < T_{a-1}(y)$ and $T_a(x) = T_a(y)$, therefore $x_a > y_a$. On the other hand, $T_{a+1}(x) \leq T_{a+1}(y)$ and using again $T_a(x) = T_a(y)$ we have $x_{a+1} \leq y_{a+1}$. Hence $x_a > y_a \geq y_{a+1} \geq x_{a+1} \Rightarrow x_a > x_{a+1}$ (so there exists transforms of type $T_\varepsilon^+(1,a,a+1)[z]$).

Furthermore, we know that $T_k(x) < T_k(y)$ $\forall 1 \leq k \leq a-1$ and because all these inequalities are strict it is clear that we can find an $\varepsilon > 0$ small enough so that the $z = T_\varepsilon^+(1,a,a+1)[y]$ transform still verify the inequalities $T_k(z) < T_k(y)$ $\forall 1 \leq k \leq a-1$.

The remaining $T_k(x)$ sums can either contain the terms x_1, x_a (if $k = a$), either all x_1, x_a, x_{a+1} terms. In the first case the sum $x_1 + x_a$ can only decrease to $z_1 + z_a$ so definitely $T_k(z) < T_k(y)$ and in the latter the sum $T_k(x)$ obviously remains unchanged, so $T_k(z) = T_k(x) \leq T_k(y)$.

Regarding the sums B_k with $p+2 \leq k \leq n$ it is obvious that they are unaffected by the $z = T_\varepsilon^+(1,a,a+1)[x]$ transform, hence $B_k(z) = B_k(x) \leq B_k(y)$ $\forall p+2 \leq k \leq n$.

$C2)$ In this case it's clear that $a = p-1$ (if $a < p-1 \Rightarrow a+1 < p \Rightarrow T_{a+1}(x) < T_{a+1}(y)$, impossible) and so $T_p(x) > T_p(y)$ (because $p = a+1$) $\Rightarrow ns - T_p(x) < ns - T_p(y) \Rightarrow B_{p+1}(x) < B_{p+1}(y)$, hence

$$\begin{cases} T_k(x) < T_k(y) & \forall 1 \le k \le p-2 \\ B_k(x) \le B_k(y) & \forall p+1 \le k \le n \end{cases} \Rightarrow x \prec_{p-1} y$$

Because all T_k sums $(1 \le k \le p-2)$ are distinct we can apply Lemma 21 $A1)$ or $B1)$ to find a strict transform $z = T_\varepsilon^+(1,i,i+1)[x]$ such that $z \preccurlyeq_{p-1} y$.

□

THEOREM 16. *Let $x,y \in A_S$ with $x \trianglelefteq y$ and $x_1 < y_1$. Then there exists a strict transform $z = T_\varepsilon^+(1,i,i+1)[x]$ with $z \trianglelefteq y$.*

Proof. The conclusion follows from Lemma 21. □

4.3. The Karamata's inequality for (S)–systems

THEOREM 17. *Let $S(e,s,k,3)$ be a non-empty 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $f : I_S \to \mathbb{R}$ strictly 3-convex with respect to e. Then*

$$\forall x,y \in A_S,\ \ x_1 < y_1 \Rightarrow f(x_1)+f(x_2)+f(x_3) < f(y_1)+f(y_2)+f(y_3)$$

Proof. Because f is strictly 3-convex with respect to e $\Rightarrow$ $\exists g : J \to \mathbb{R}$ strictly convex with $e'(\mathring{I}_S) \subset J$ such that $f' = g \circ e'$.

Case 1. (S) is a 2-convex system. We will prove this case using a proof scheme similar to the one in [1] or [2], adapted to our more general framework.

According to Theorem 9 and 10 we know that $\exists! u : I_1 \to I_2, v : I_1 \to I_3$ continuous on I_S, differentiable in $\mathring{I}_S$, bijective, strictly monotonic (u decreasing, v increasing) and such that $A_S = \{(t,u(t),v(t)) | t \in I_1\}$. We can, certainly, assume that (S) is nontrivial, hence (see Remark 5) $\mathring{I}_k \neq \emptyset$ $(k = 1,2,3)$ and $\forall x \in A_S$ with $x_1 \in \mathring{I}_1$ $\Rightarrow$ $x_2 = u(x_1) \in \mathring{I}_2$, $x_3 = v(x_1) \in \mathring{I}_3$ and $x_1 > x_2 > x_3$. For such a $x_1 \in \mathring{I}_1$ we can write:

$$\begin{cases} x_1 + u(x_1) + v(x_1) = 3s \\ e(x_1) + e(u(x_1)) + e(v(x_1)) = 3k \end{cases} \Rightarrow \begin{cases} u'(x_1) + v'(x_1) = 0 \\ e'(x_1) + e'(u(x_1))u'(x_1) + e'(v(x_1))v'(x_1) = 0 \end{cases}$$

and infer immediately that

$$u'(x_1) = \frac{e'(x_1) - e'(x_3)}{e'(x_3) - e'(x_2)},\ \ v'(x_1) = \frac{e'(x_1) - e'(x_2)}{e'(x_2) - e'(x_3)} \tag{8}$$

Let $S : \mathring{I}_1 \to \mathbb{R} \Rightarrow S(x_1) = e(x_1) + e(u(x_1)) + e(v(x_1))$. By differentiating we get

$$\forall x_1 \in \mathring{I}_1,\ \ S'(x_1) = f'(x_1) + f'(u(x_1))u'(x_1) + f'(v(x_1))v'(x_1)$$

$$S'(x_1) = f'(x_1) + f'(x_2)\frac{e'(x_1) - e'(x_3)}{e'(x_3) - e'(x_2)}) + f'(x_3)\frac{e'(x_1) - e'(x_2)}{e'(x_2) - e'(x_3)} \tag{9}$$

(noticing that $x_1 > x_2 > x_3 \Rightarrow e'(x_1) > e'(x_2) > e'(x_3)$ because e' is strictly increasing)

We have $f'(x_k) = g(e'(x_k))$ $(k = 1,2,3)$ and, using the notation $e'(x_k) = y_k$, we can write (9) as

$$\frac{S'(x_1)}{(y_1-y_3)(y_1-y_2)} = \frac{g(y_1)}{(y_1-y_3)(y_1-y_2)} + \frac{g(y_2)}{(y_2-y_1)(y_2-y_3)} + \frac{g(y_3)}{(y_3-y_1)(y_3-y_2)}$$

By the strictly convexity of g we deduce that the right side of the above relation is strictly positive and because $(y_1-y_3)(y_1-y_2) > 0$ we infer that $S'(x_1) > 0\ \forall x_1 \in \mathring{I}_1$ so S is strictly increasing on $\mathring{I}_1$, in fact on I_S (because S is continuous on I_S) and we conclude that $\forall x,y \in A_S,\quad x_1 < y_1 \Rightarrow S(x_1) < S(x_2) \Rightarrow f(x_1)+f(x_2)+f(x_3) < f(y_1)+f(y_2)+f(y_3)$.

Case 2. (S) is a 2-concave system, so now e is a strictly concave function on I_S. We consider the dual system $S'(h,s,k',3)$ where $k' = -k$ and $h: I_S \to \mathbb{R},\quad h = -e$ is strictly convex and clearly $A_S = A_{S'}$.

By hypothesis, we know that $\exists g : J \to \mathbb{R}$ strictly convex with $e'(\mathring{I}_S) \subset J$ such that $f' = g \circ e'$. Let $g_1 : -J \to \mathbb{R},\quad g_1(y) = g(-y)$ and it's clear that g_1 is also strictly convex and $f'(x) = g(e'(x)) = g_1(-e'(x)) = g_1(h'(x))$, hence $f' = g_1 \circ h'$.

In this way, we can apply the Case 1 to the system (S') and we conclude again that $\forall x,y \in A_S = A_{S'},\quad x_1 < y_1 \Rightarrow f(x_1)+f(x_2)+f(x_3) < f(y_1)+f(y_2)+f(y_3)$. □

REMARK 10. If I_S is an open interval, we can give a more direct proof (not based on the functional dependence), using an interesting technique from [5] and [6].

Let $x,y \in A_S$ with $x_1 < y_1$. According to Lemma 5 we have $y_1 \geq x_1 \geq x_2 \geq y_2 \geq y_3 \geq x_3$ and let $A_1 = [x_1,y_1]$, $A_2 = [y_2,x_2]$, $A_3 = [x_3,y_3]$ and $B_k = e'(A_k)$ $(k = 1,2,3)$. We observe that the intervals A_k have mutual disjoint interiors and so the intervals B_k also have mutual disjoint interiors (because e' is a strictly increasing function).

Next, we consider the linear function $L : \mathbb{R} \to \mathbb{R},\ L(r) = \alpha + \beta r$ that agree with g at the endpoints of B_2 and because g is convex we have $\begin{cases} g(r) \leq L(r)\ \forall r \in B_2 \\ g(r) \geq L(r)\ \forall r \in B_1 \cup B_3 \end{cases}$

and so $E_1 \overset{def}{=} \int_{A_1} g(e'(t))dt + \int_{A_3} g(e'(t))dt \geq \int_{A_1} L(e'(t))dt + \int_{A_3} L(e'(t))dt = \alpha(l(A_1) + l(A_3)) + \beta\left[\int_{A_1} e'(t)dt + \int_{A_3} e'(t)dt\right]$ and we observe that $l(A_1)+l(A_3) = l(A_2)$ because $x_1+x_2+x_3 = y_1+y_2+y_3$ and $\int_{A_1} e'(t)dt + \int_{A_3} e'(t)dt = \int_{A_2} e'(t)dt$ because $e(x_1)+e(x_2)+e(x_3) = e(y_1)+e(y_2)+e(y_3)$. Hence

$$E_1 \geq \alpha l(A_2) + \beta \int_{A_2} e'(t)dt = \int_{A_2} L(e'(t))dt \geq \int_{A_2} g(e'(t))dt \overset{def}{=} E_2$$

But $g(e'(t)) = f'(t)\ \forall t \in I_S$ so $E_1 = \int_{A_1} f'(t)dt + \int_{A_3} f'(t)dt = f(y_1) - f(x_1) + f(y_3) - f(x_3)$ and $E_2 = \int_{A_2} f'(t)dt = f(x_2) - f(y_2)$ etc.

THEOREM 18. *Let $S(e,s,k,n)$ be a non-empty 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $f : I_S \to \mathbb{R}$ strictly 3-convex with respect to e. Then*

$$\forall x,y \in A_S,\ \ x \trianglelefteq y \Rightarrow E_f(x) \leq E_f(y) \tag{10}$$

where $E_f(x) = f(x_1)+f(x_2)+\ldots+f(x_n)$. The equality holds if and only if $x = y$.

Proof. First we will prove the inequality (10) by induction on n and next we will discuss the equality case.

If $n=3$ then $x \preccurlyeq_p y \Rightarrow x_1 \le y_1$ (according to Lemma 16) and the inequality (10) follows directly from Theorem 17. Suppose now that $n>3$.

Case 1) $x_1 = y_1$. Let $x' = (x_2, \ldots, x_n)$, $y' = (y_2, \ldots, y_n)$. It's clear (according to Lemma 17) that $x' \trianglelefteq y'$ and that $x', y' \in A_{S'}$ where $S'(e, s', k', n-1)$ is the reduced system $\hat{S}[x_1]$ (see Definition 7). By induction hypothesis, $E_f(x') \le E_f(y')$ hence $E_f(x) = f(x_1) + E_f(x') \le f(y_1) + E_f(y') = E_f(y)$.

Case 2) $x_1 \ne y_1$, that is, according to Lemma 16, $x_1 < y_1$.
Let $M_x = \{z \in A_S | z \trianglelefteq y \text{ and } E_f(z) \ge E_f(x)\}$, $\lambda = \sup\{z_1 | z \in M_x\}$ and $(z^m)_{m\ge1} \subset M_x$ with $z_1^m \to \lambda$. Because A_S is a compact set it follows that $(z^m)_{m\ge1}$ has convergent subsequences and so we can assume $(z^m)_{m\ge1}$ is convergent (if not, we replace it with a convergent subsequence). Let $z^m \longrightarrow \tilde{z} \in A_S$. Notice that $\tilde{z}_1 = \lambda \le y_1$ (because $z^m \trianglelefteq y \ \forall m$ and so, according to Lemma 16, $z_1^m \le y_1$).

We will prove that $\tilde{z} \in M_x$. Knowing that $E_f(z^m) \ge E_f(x) \ \forall m \ge 1$ and using the continuity of f we infer that $E_f(\tilde{z}) \ge E_f(x)$. It remains to show that $\tilde{z} \trianglelefteq y$. But $z^m \trianglelefteq y \Rightarrow \exists 1 \le p_m \le n-1$ with $z^m \preccurlyeq_{p_m} y$ and clearly we can find an index p that appears an infinite number of times, so we can consider a subsequence $(m_l)_{l\ge1}$ such that $z^{m_l} \preccurlyeq_p y$ for any $l \ge 1$. But

$$z^{m_l} \preccurlyeq_p y \Leftrightarrow \begin{cases} T_k(x) \le T_k(z^{m_l}) & \forall 1 \le k \le p-1 \\ B_k(x) \le B_k(z^{m_l}) & \forall p+2 \le k \le n \end{cases}$$

By passing to the limit as $l \to \infty$ it follows that $\tilde{z} \preccurlyeq_p y$, hence $\tilde{z} \trianglelefteq y$ and so $\tilde{z} \in M_x$.

Next we will prove that $\tilde{z}_1 = y_1$. Suppose that $\tilde{z}_1 < y_1$. Then, using the fact that $\tilde{z} \trianglelefteq y$ we can apply Theorem 16 to get a strict transform $w = T_\varepsilon^+(1, i, i+1)[\tilde{z}]$ with $w \trianglelefteq y$. Observe that $E_f(w) > E_f(\tilde{z}) \Leftrightarrow f(w_1) + f(w_i) + f(w_{i+1}) > f(\tilde{z}_1) + f(\tilde{z}_i) + f(\tilde{z}_{i+1})$ and this is true according to Theorem 17 because $w_1 > \tilde{z}_1$. Thus $E_f(w) > E_f(\tilde{z}) \ge E_f(x)$ and it follows that $w \in M_x$. But $w_1 > \tilde{z}_1 = \lambda$ and this contradicts the maximality of λ.

Hence $\tilde{z}_1 = y_1$. But $\tilde{z} \trianglelefteq y$ and applying the induction hypothesis exactly as in Case 1 we deduce that $E_f(y) \ge E_f(\tilde{z})$. But $E_f(\tilde{z}) \ge E_f(x)$ and our inequality (10) is proved.

We discuss now the equality case. We will show that if $x \prec_p y$ (strictly) then $E_f(x) < E_f(y)$. Let r be the first index $1 \le r \le p$ with the property that $x_r < y_r$ (see Lemma 19), hence $x_i = y_i \ \ \forall 1 \le i \le r-1$. Let $x' = (x_r, \ldots x_n)$, $y' = (y_r, \ldots y_n)$ and clearly $x', y' \in A_{S'}$ where $S'(e, s', k', n')$ is the reduced system $\hat{S}[x_1, \ldots x_{r-1}]$ (see Definition 7), $n' = n - r + 1$.

Using Lemma 17 it follows that $x' \trianglelefteq y'$. We observe that $E_f(y) - E_f(x) = E_f(y') - E_f(x')$, so it's enough to prove that $E_f(y') - E_f(x') > 0$. Because $x'_1 = x_r < y_r = y'_1$ we find, according to Theorem 16 applied to (S'), a strict transform $z' = T_\varepsilon^+(1, i, i+1)[x']$ with $z' \trianglelefteq y'$. But, according to Theorem 17,

$$E_f(z') - E_f(x') = f(z'_1) + f(z'_i) + f(z'_{i+1}) - (f(x'_1) + f(x'_i) + f(x'_{i+1})) > 0$$

because $z'_1 > x'_1$ and so $E_f(z') > E_f(x')$. But, according to inequality (10) previously proved, we also have $E_f(y') \ge E_f(z')$, therefore $E_f(y') - E_f(x') > 0$. □

REMARK 11. Our Karamata type theorem doesn't have a converse (in contrast to the classical Karamata's theorem) because $\trianglelefteq$ is not an order relation. To remedy this situation, we can try to define a relation $x \preccurlyeq\!\preccurlyeq y \Leftrightarrow \exists z_0, \dots z_r \in A_S$ with $x = z_0 \trianglelefteq z_1 \dots \trianglelefteq z_{r-1} \trianglelefteq z_r = y$ and it's easy to prove that this is actually an order relation and, obviously, Theorem 18 remains true if we use $\preccurlyeq\!\preccurlyeq$ instead $\trianglelefteq$. Moreover, it's plausible to think that this version of Theorem 18 has a corresponding converse, but this is only our conjecture.

THEOREM 19. (extended version of the V. Cîrtoaje equal variable theorem) *Let $S(e,s,k,n)$ be a non-empty 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $f : I_S \to \mathbb{R}$ strictly 3-convex with respect to e. Then $\forall x \in A_S$ the following inequality holds*

$$E_f(\omega) \leq E_f(x) \leq E_f(\Omega)$$

where $E_f(x) = f(x_1) + f(x_2) + \ldots + f(x_n)$ and ω, Ω are the poles of the (S). The equality occurs if and only if $x = \omega$ or $x = \Omega$.

Proof. Follows immediately by Theorem 15 and 18. □

REMARK 12. V. Cîrtoaje's original theorems correspond to the particular case of an $S(e,s,k,n)$ system where e is of the form $e(x) = x^r$ (see [1] and [2]).

REMARK 13. Let $S(e,s,k,n)$ be a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$. We can further extend the previous theorems by replacing E_f by more general classes of functions. More precisely, we will say that $E : I_S^n \to \mathbb{R}$ satisfies the Schur-Ostrowski (SO) condition with respect to $S(e,s,k,n)$ if E is continuous on I_S^n, differentiable on $\mathring{I}_S^n$ and verifies the condition:

$$\left[\frac{\partial_i E(x) - \partial_j E(x)}{e'(x_i) - e'(x_j)} - \frac{\partial_k E(x) - \partial_j E(x)}{e'(x_k) - e'(x_j)}\right](e'(x_i) - e'(x_k)) > 0 \;\; \forall x \in \mathring{I}_S^n, x_i \neq x_j \neq x_k \quad (11)$$

If $S(e,s,k,n)$ is a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $f : I_S \to \mathbb{R}$ is strictly 3-convex with respect to e, we can show that E_f actually satisfies (SO) with respect to (S). We know that $f' = g \circ e'$ (g strictly convex) and we see that $\partial_l E_f(x) = f'(x_l) = g(e'(x_l))$, $l = 1,2,3$ hence, using the notation $y_l = e'(x_l)$ we can write the condition (11) as

$$\left[\frac{g(y_i) - g(y_j)}{y_i - y_j} - \frac{g(y_k) - g(y_j)}{y_k - y_j}\right](y_i - y_k) > 0$$

and this is true because g is a strictly convex function and so the first factor of the above expression has the sign of $(y_i - y_k)$.

If $S(e,s,k,n)$ is a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $E : I_S^3 \to \mathbb{R}$ satisfies (SO) with respect to (S) we can also get a more general version of Theorem 17. The proof is largely the same. We similarly define $S : \mathring{I}_1 \to \mathbb{R}$ given by $S(x_1) = E(x_1, u(x_1), v(x_1)) \;\Rightarrow\; S'(x_1) = \partial_1 E(x) + \partial_2 E(x) u'(x_1) + \partial_3 E(x) v'(x_1)$ and using the equivalent expressions (8) for u', v' we can further write:

$$S(x_1)\frac{e'(x_1) - e'(x_3)}{e'(x_1) - e'(x_2)} = \left[\frac{\partial_1 E(x) - \partial_2 E(x)}{e'(x_1) - e'(x_2)} - \frac{\partial_3 E(x) - \partial_2 E(x)}{e'(x_3) - e'(x_2)}\right](e'(x_1) - e'(x_3))$$

and so, using the condition (11), we infer that $S'(x_1) > 0$ etc.

The proof of the theorem 18 can also be adapted, leading to the following more general version:

THEOREM A. *Let $S(e,s,k,n)$ be a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $E : I_S^n \to \mathbb{R}$ that satisfies (SO) with respect to (S). Then:*

$$\forall x, y \in A_S, \quad x \trianglelefteq y \Rightarrow E(x) \leq E(y)$$

Equality holds if and only if $x = y$

We have also the following version of Theorem 19:

THEOREM B. *Let $S(e,s,k,n)$ be a 2-convex (or 2-concave) system with e differentiable on $\mathring{I}_S$ and $E : I_S^n \to \mathbb{R}$ that satisfies (SO) with respect to (S). Then $\forall x \in A_S$*

$$E(\omega) \leq E(x) \leq E(\Omega)$$

where ω, Ω are the poles of the (S). Equality holds if and only if $x = \omega$ or $x = \Omega$.

REMARK 14. The idea of a Schur criterion of type (11) can already be found in [7] where systems of type (S) are discussed under the particular hypothesis $e : \mathbb{R} \to \mathbb{R}$, $e(x) = x^2$, but with a different definition of the majorization on (S), more precisely $a \succ_3 b \Leftrightarrow \forall f : \mathbb{R} \to \mathbb{R},\ f^{(3)} \geq 0 \Rightarrow \sum_{i=1}^n f(a_i) \geq \sum_{i=1}^n f(b_i)$.

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