The Vertices of a Graph and its Dimension

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Abstract

We show that the dimension of a graph is less or equal to the cardinality of the set of its vertices

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1 Introduction

From [1] or the book [2], p.92, we know the inequality $dim(G) \leq 2 \cdot \chi(G)$ for every graph G, where $\chi(G)$ means the chromatic number of G. Here we show a further inequality. For the sake of clarity we repeat the definition of the dimension of a graph. Please see [1] and [2], p.88.
Here an *embedding* means an injective map of an isomorphic graph, different from [1]. A *display* is less. Note that a display is also an isomorphic graph and the number of intersection points of different edges is finite.

Definition 1. Let G be an arbitrary graph. We define the *dimension* of G, in symbols $dim(G)$, as the minimum number n such that G can be displayed in the Euclidean space $\mathbb{R}^n$ by an isomorphic graph and all edges have length one.

Theorem 1. *Let G be an arbitrary graph, and let vert be the set of its vertices. It holds the inequality*

$$dim(G) \leq cardinality(vert)$$

This is an improvement in many cases. For instance, if G is the complete graph $K_r, r > 1$, we have

$$dim(G) = r - 1 < r = cardinality(vert(G)) < 2 \cdot r = 2 \cdot \chi(G).$$

The proof of the theorem is yielded in the following section.

2 Construction

We prove the theorem only for $cardinality(vert) < \infty$. Hence we assume a graph G with a finite set of vertices. Let $\{v_1, v_2, \ldots v_{n-1}, v_n\}$ be the set of vertices of G. We construct an embedding of G in the Euclidean space $\mathbb{R}^n$.

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Let $\vec{e}_i := (0,0,\dots,0,0,1,0,0,\dots,0)$ be the the i^{th} unit vector of the $\mathbb{R}^n$, i.e. $1 \le i \le n$ and $\vec{e}_i$ has $n-1$ zeros and a single one at place i. We construct a graph $H \subset \mathbb{R}^n$ which is isomorphic to G. We take n vectors

$$\vec{w}_i := \frac{1}{2} \cdot \sqrt{2} \cdot \vec{e}_i, \tag{1}$$

$1 \le i \le n$, as the vertices of H. Note that the Euclidean distance between two different vertices of H is one, i.e. length $(\vec{w}_i - \vec{w}_j) = 1$ for $i \neq j$. Now we add edges. We add the straight line between $\vec{w}_i$ and $\vec{w}_j$ if and only if there is an edge between v_i and v_j. The constructed graph H is an embedding of G.

Lemma 1. *The above construction of H has solely straight lines as edges. If v_i, v_j and v_m are three different vertices of G, and if there is an edge beween v_m and v_i, and if there is another edge between v_m and v_j, the two constructed edges between $\vec{w}_m$ and $\vec{w}_i$ and between $\vec{w}_m$ and $\vec{w}_j$ are straight lines, and they meet only once. The intersection point is their common vertex $\vec{w}_m$.*

Proof. The graph H has straight lines as edges due to the construction. The edges between $\vec{w}_m$ and $\vec{w}_i$ and between $\vec{w}_m$ and $\vec{w}_j$, respectively, are given by

$$\alpha \cdot \vec{w}_m + (1-\alpha) \cdot \vec{w}_i \;\text{ and }\; \beta \cdot \vec{w}_m + (1-\beta) \cdot \vec{w}_j \;\text{ for } \alpha, \beta \in \; [0,1]. \tag{2}$$

They meet once in $\vec{w}_m$. They intersect only if $\alpha = \beta = 1$. □

3 Definitions

We create further definitions of dimensions in graphs besides dim. We define 'dimensions' with names $straight\ dim$, Dim, $straight\ Dim$, $\mathbf{k}\ double\ points$, $polygon\ \mathbf{k}\ double\ points$, $straight\ \mathbf{k}\ double\ points$, $straight\ \mathbf{k}\ different\ lengths$ and $straight\ \mathbf{k}\ Different\ Lengths$.
Let G be an arbitrary graph.

Definition 2. For $straight\ dim$ we use the same definition as for dim, except that only straight lines are allowed. We define $Dim(G)$ as the minimum number n such that G can be embedded in the Euclidean space $\mathbb{R}^n$, and all edges have the same length.

Proposition 1. It holds

$$dim(G) \le Dim(G)$$

The definition of $straight\ Dim$ is equal the definition of Dim, except that only straight lines are allowed as edges. Trivially we have the inequalities

$$Dim(G) \le straight\ Dim(G) \quad \text{ and } \quad dim(G) \le straight\ dim(G) \le straight\ Dim(G) \tag{3}$$

Remark 1. *Since we use only straight lines in our construction and since our graph is an embedding, we also have proven for each graph G*

$$straight\ Dim(G) \le cardinality(vert)$$

Definition 3. Let H be a display in $\mathbb{R}^n$ of a graph for any n. We call a *double point* a point $\vec{x}$ such that $\vec{x}$ is not a vertex and $\vec{x}$ is an element of at least two edges of H.

Let $\mathbf{k}$ be a natural number. We define $\mathbf{k}\ double\ points(G)$ as the minimum number n such that there is a display called H in $\mathbb{R}^n$ such that H is isomorphic to G and H has exactly $\mathbf{k}$ double points. For the natural number $straight\ \mathbf{k}\ double\ points(G)$ we take the same definition, except that only straight lines are allowed.

Definition 4. We define a *polygon line* as a line that consists of a finite number of straight lines, and that is homeomorphic to a line segment.

We define *polygon* **k** *double points*(G) to be the smallest number n such that there is a display called H in $\mathbb{R}^n$, where H is isomorphic to G and H has exactly **k** double points and all edges are polygon lines.

Note that a line segment is a polygon line.

Proposition 2. For every $\mathbf{k} \in \mathbb{N}$ there exists a natural number **k** *double points*(G) and a natural number *polygon* **k** *double points*(G) for each graph G with more than one edge. Let *vert* be the set of vertices of G. It holds

$$polygon\ \mathbf{k}\ double\ points(G) \leq cardinality(vert)$$

Proof. We construct an isomorphic graph of G in $\mathbb{R}^n$, which we call H. Let $vert := \{v_1, v_2, v_3, \ldots v_{n-1}, v_n\}$ be the set of vertices of G. We shall go a similar way as in the section ‘Construction’. The set of vertices of H is taken from the finite set $\{\vec{e}_i \mid 1 \leq i \leq n\}$, where $\vec{e}_i$ is the i^{th} unit vector of $\mathbb{R}^n$. Since G has more than one edge, we have at least two edges. We call these edges l and k. The vertices of l are without restriction of generality v_1 and v_2, while k has the vertices v_3 and v_4. If $v_1 = v_3$ it holds $v_2 \neq v_4$. If $v_1 \neq v_3$ we assume that v_1, v_2, v_3, v_4 are pairwise different. On the line $\alpha \cdot \vec{e}_3 + (1-\alpha) \cdot \vec{e}_4$, where $\alpha \in [0,1]$, we fix **k** points. Let $p_i := \frac{i}{\mathbf{k+1}} \cdot \vec{e}_3 + (1 - \frac{i}{\mathbf{k+1}}) \cdot \vec{e}_4$, where $1 \leq i \leq \mathbf{k}$. On the line $\beta \cdot \vec{e}_1 + (1-\beta) \cdot \vec{e}_2$, where $\beta \in [0,1]$, we fix $\mathbf{k}+1$ points. Let $q_j := \frac{j}{\mathbf{k+2}} \cdot \vec{e}_1 + (1 - \frac{j}{\mathbf{k+2}}) \cdot \vec{e}_2$, where $1 \leq j \leq \mathbf{k}+1$. Now we define a polygon line in ‘zig-zag’ shape, starting from $\vec{e}_1$ and alternating between the points q_j and p_i and ending in $\vec{e}_2$. The first and the last piece of the polygon line are parts of the line segment which connects $\vec{e}_1$ and $\vec{e}_2$. The two pieces are defined as $\gamma \cdot \vec{e}_1 + (1-\gamma) \cdot q_1$, and $\gamma \cdot q_{\mathbf{k}+1} + (1-\gamma) \cdot \vec{e}_2$, respectively, where $\gamma \in [0,1]$. Please see the picture. There we assume $\vec{e}_1 = \vec{e}_3$, i.e. $v_1 = v_3$, and $\mathbf{k} = 2$.

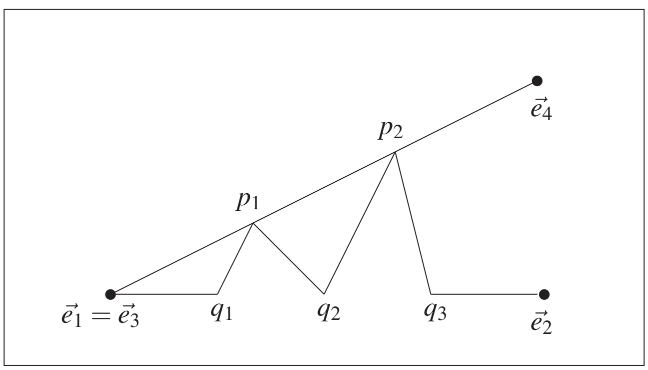

Figure 1:

We assume $v_1 = v_3$

and **k** = 2.

The line $\delta \cdot q_j + (1-\delta) \cdot p_j$, $\delta \in [0,1]$ connects q_j and p_j, while $\varepsilon \cdot p_j + (1-\varepsilon) \cdot q_{j+1}$, $\varepsilon \in [0,1]$ connects p_j and q_{j+1}, where $1 \leq j \leq \mathbf{k}$. We define the polygon line from $\vec{e}_1$ to $\vec{e}_2$ through the points p_i and q_j as an edge of H. Further we add the segment $\beta \cdot \vec{e}_3 + (1-\beta) \cdot \vec{e}_4$, where $\beta \in [0,1]$, which connects $\vec{e}_3$ and $\vec{e}_4$ as an edge of H. From this construction we get **k** double points p_i, $1 \leq i \leq \mathbf{k}$. The rest of the graph is constructed as in section ‘Construction’. We add the straight line in H between $\vec{e}_s$ and $\vec{e}_t$ if and only if there is an edge between v_s and v_t in G, $(s,t) \notin \{(1,2),(2,1),(3,4),(4,3)\}$. By this construction we add no more double points to H. We get that the cardinality of the set of the double points of H is **k** and that H is isomorphic to G. □

Let us assume that the graph G has a finite set of edges.

Proposition 3. It holds **k** *double points*$(G) =$ *polygon* **k** *double points*(G) for each number **k**.

Proof. Every line of finite length can be replaced by a polygon line, such that the old intersection points are kept and no new intersection points are generated. □

Definition 5. We define $straight\ \mathbf{k}\ different\ lengths(G)$ as the smallest natural number n such that G can be displayed in the $\mathbb{R}^n$, where the edges are straight lines, for each graph G. These edges have exactly **k** different lengths. For $straight\ \mathbf{k}\ Different\ Lengths(G)$ we take the same definition, but here the display has to be an embedding.

Remark 2. In the case that there is no realization with the corresponding conditions in $\mathbb{R}^n$ for any n, we define $xxx(G) = \infty$, where 'xxx' stands for *straight* **k** *different lengths, straight* **k** *Different Lengths or straight* **k** *double points.*

We have the inequality

$$polygon\ \mathbf{k}\ double\ points(G) \leq straight\ \mathbf{k}\ double\ points(G)$$

for every graph G.

Note $straight\ dim = straight\ \mathbf{1}\ different\ lengths$ and $straight\ Dim = straight\ \mathbf{1}\ Different\ Lengths$.

4 Pictures

We show displays of the graph W_4 and the Petersen graph. Note that both graphs can not be displayed in $\mathbb{R}$ by isomorphic graphs.
See two displays of the graph W_4, which consists of five vertices and eight edges. The first proves $straight\ \mathbf{2}\ different\ lengths(W_4) = straight\ \mathbf{2}\ Different\ Lengths(W_4) = 2$.

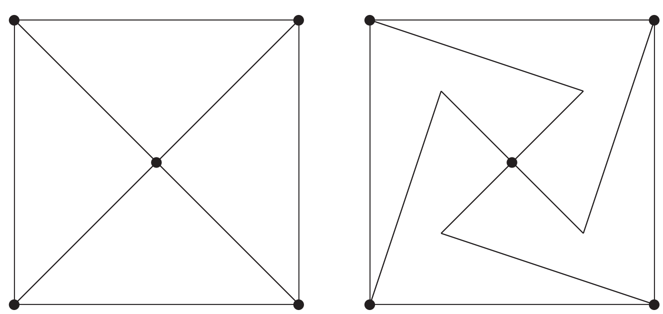

Figure 2:

On the left hand side we show

two embeddings of the 'wheel' W_4.

From [2], p.91 we know $dim(W_4) = 3$. An embedding of W_4 in $\mathbb{R}^3$ with only edges of length one yields a pyramid with quadratic base and the right sidelength. Here we show a display of W_4 with three double points and an embedding of W_4 with equal edgelengths. This demonstrates $\mathbf{3}\, double\ points(W_4) = 2$ and $dim(W_4) = Dim(W_4) = 2$. If in the second display the square of W_4 has the corners $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$, the kink on the edge from $\left(-\frac{1}{2}, \frac{1}{2}\right)$ to $(0,0)$ is (s,t), where $s = t = \frac{1}{8} \cdot \sqrt{2}$.

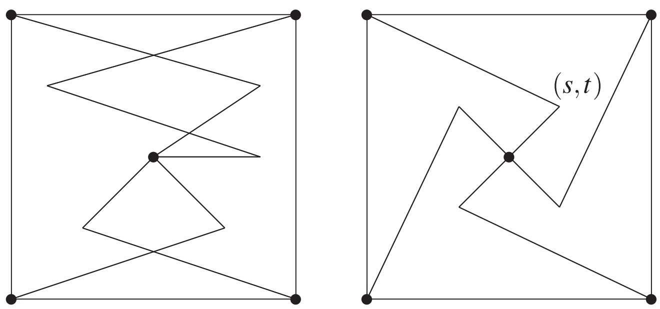

Figure 3:

We show two displays of W_4.

The second is an embedding of W_4 with edges of equal length.

Now we consider two displays of the Petersen graph P. The website [3] was helpful by generating the displays. The first display shows $straight\ \mathbf{3}\ different\ lengths(P) = straight\ \mathbf{5}\ double\ points(P) = 2$. The second demonstrates again $straight\ \mathbf{5}\ double\ points(P) = 2$ and $dim(P) = straight\ dim(P) = 2$.

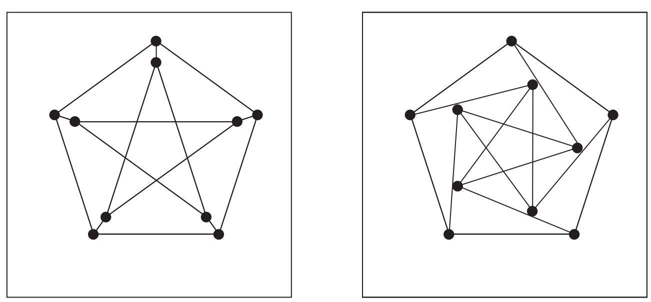

Figure 4:

We show two displays of the Petersen graph. The second is a display of P with edges of equal length.

5 Example

As an example we consider the complete graph called K_3 with three vertices. From [2], p.88, we have $dim(K_3) = 2$. An embedding in $\mathbb{R}^2$ is shown by each triangle. With our theorem we get $dim(K_3) = 2 < 3 = cardinality(vert(K_3)) < 6 = 2 \cdot \chi(K_3)$.
We add a further theorem.

Theorem 2. *Let G be a graph. We assume a nonempty set of edges of G called edges. Let the cardinality of the set of vertices of G does not overrun the cardinality of $\mathbb{R}$. It holds*

$$dim(G) \leq 2 \cdot cardinality(edges)$$

Proof. If the cardinality of the edges is infinite, we have nothing to show. Hence we assume a finite set of edges of G. An edge connects two vertices. Hence there are at most $n := 2 \cdot cardinality(edges)$ vertices, which are part of an edge. With these vertices we go the same way as in the section ‘Construction’. We embed G in $\mathbb{R}^n$. □

The theorem may be an improvement in some cases. We only know a single example. For the complete graph K_2 it holds

$$dim(K_2) = 1 < 2 = 2 \cdot cardinality(edges) < 4 = 2 \cdot \chi(K_2).$$

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References

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[2] Alexander Soifer: *The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of its Creators*, New York Springer(2009), ISBN 978-0-387-74640-1

[3] http://cosypanther.blogspot.com/2010/11/koordinaten-eines-regelmaigen-funfecks.html

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