The system of equations describing 4 generations with the symmetry group $SU(3)_C \times SU(2)_L \times U(1)^*$

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Abstract

The system of 16-component equations including two equations of the Bethe-Salpeter kind (without an interaction) and two additional conditions are considered. It is shown that the group of the initial symmetry is $SU(3)_C \times SU(2)_L \times U(1)$. The symmetry group is established as the consequence of the field equations; $SU(2)$ should be chiral, the color space has the signature $(++-)$. The structure of permissible multiplets of the group coincides with the one postulated in the $SU(3)_C \times SU(2)_L$-model of strong and electroweak interactions excluding the possible existence of the additional $SU(2)_R$-singlet in a generation.

1 Introduction

The lepton and quark sectors of the model of strong and electroweak interactions based on the group $SU(3)_C \times SU(2)_L \times U(1)$ [1], if one does not take into account differences relatively to $SU(3)_C$, differ insignificantly - one singlet of $SU(2)_R$ is introduced in the lepton sector, and the two singlets in the quark one. If results of measurements of the neutrino mass [2] are confirmed, the simplest way to modify the model to get the non-zero neutrino mass is the introduction of the second $SU(2)_R$ singlet in the lepton sector. It is shown in this paper that the model of field of the two-component fermions admitting the existence of four generations of the same type has namely such the structure

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of multiplets and such the group of the initial symmetry of its solutions. It is important that the types of the symmetry group and of its admitted multiplets are consequences of the model which can be established by the analysis of the field equations. The minimum number of generations cannot be arbitrary, too. The representation about a color of a field state may be connected in the model with the certain inversions affecting internal and external coordinates of the composite fermion. So, it is possible that the similar model of the field of fermions may be used to construct the general description of the fundamental particles and their interactions.

The considered model by zero masses of the components may be interpreted in two ways: as the non-local field theory based on the use of the two-point wave function in the 4-space-time or as the local field theory in the eight-dimensional pseudo-euclidian space with two time axis, and besides the additional coordinates have clear physical interpretation: the ones are the coordinates of the relative position of the composite system's components. It is known that the geometrical description of fields and their interactions in spaces of a dimension greater than 4 permits to consider jointly gravi-electro-weak and gravi-electro-strong interactions in 7-dimensional space with one time [3], and the possibility of the simple interpretation of the additional dimensions in the 8-space seems to be essential.

2 The 16-component field equations of the two-component fermions

Let us use the Dirac way to introduce the field equations for ψ starting from the classical equations of the connection of the energies E, E^1, E^2 and of the momenta $\mathbf{p}, \mathbf{p}^1, \mathbf{p}^2$ of the composite system and its components:

$$E = E^1 + E^2, \tag{1}$$

$$\mathbf{p} = \mathbf{p}^1 + \mathbf{p}^2, \tag{2}$$

where $E^i = (m^{i2} + \mathbf{p}^{i2})^{0.5}$. At first we linearize the non-linear (relatively p^{ik}) equation (1), and after that we replace the classical energies and momenta by their operators. With Eq.(1) we juxtapose the linear equation ($c = h = 1$):

$$i\partial\psi/\partial t = (\beta_1 m^1 + \alpha_{1k}p^{1k} + \beta_2 m^2 + \alpha_{2k}p^{2k})\psi, \tag{3}$$

where t is the time, p^{1k}, p^{2k} are the operators of the constituents momenta, β_i, α_{ik} are matrices, $k = 1, 2, 3$. It may be shown that matrices of the dimension greater than 8×8 are needed to satisfy the conformity principle. The dimension 16×16 is sufficient to construct the matrices with the following algebra containing commutators [ ] and anticommutators { }:

$$\{\alpha_{ik}, \alpha_{il}\} = 2\delta_{kl}, \{\alpha_{il}, \beta_i\} = 0,$$
$$[\beta_i, \beta_j] = [\alpha_{ik}, \beta_j] = [\alpha_{ik}, \alpha_{jl}] = 0,\ \beta_i^2 = I_{16},\ i \neq j, \tag{4}$$

there is not summation on i.

These relations are executed for the following matrix presentation:

$$\beta_1 = I_2 \times \sigma_3 \times I_4,\ \beta_2 = \sigma_3 \times \sigma_3 \times I_4,$$
$$\alpha_{1k} = \sigma_1 \times \sigma_1 \times I_2 \times \sigma_k^{'},\ \alpha_{2k} = \sigma_1 \times I_2 \times \sigma_k^{''} \times I_2, \tag{5}$$

where σ_k are the Pauli matrices, $\sigma_k^{'},\ \sigma_k^{''}$ are two transpositions of them, for example: $\sigma_k^{'} = \sigma_k^{''} = \sigma_k$. Eq. (3) has the same form as the Bethe-Salpeter equation [4] which is used for the description of such composite systems as mesons.

For the momentum operators it will be naturally to postulate instead Eq. (2) the following connection:

$$p^k\psi = p^{1k}\psi + p^{2k}\psi, \tag{6}$$

where p^k is the operator of the system momentum, ψ is the 16-component vector. If we introduce the operators of the component energies $E^1,\ E^2$, the Eq. (3) will have the same view ($E \equiv i\partial/\partial t$):

$$E\psi = E^1\psi + E^2\psi, \tag{7}$$

i.e. the equations of motion of the components are:

$$E^i\psi = \alpha_{ik}p^{ik}\psi + \beta_i m^i\psi, \tag{8}$$

there is not summation on i. If $x_{1\mu},\ x_{1\mu}$ are the coordinates of the first and second components, $\psi = \psi(x_{1\mu}, x_{1\mu})$, then Eqs. (6,7) in the form: $p^\mu\psi = p^{1\mu}\psi + p^{2\mu}\psi$ can be understood as a transition in the eight-dimensional pseudo-euclidian space from the coordinates $x_{1\mu},\ x_{2\mu}$ to the new coordinates $x_\mu,\ y_\mu$, where x_μ are the coordinates of the system center of inertia, and y_μ are still not defined. In the space of the operators $p^{1\mu},\ p^{2\mu}$ by such the transition we can define the operators $\pi_\mu \equiv i\partial/\partial y^\mu$ to be independent of p^μ as:

$$\pi_\mu\psi \equiv p_{1\mu}\psi - p_{2\mu}\psi. \tag{9}$$

Then from Eq. (8) besides Eq. (3) we get the equation independent of it:

$$\pi^0\psi = (\beta_1 m^1 - \beta_2 m^2 + \alpha_{1k}p^{1k} - \alpha_{2k}p^{2k})\psi. \tag{10}$$

Eqs. (3, 10) contain the terms: $\alpha_{1k}p^{1k}\psi \pm \alpha_{2k}p^{2k}\psi \equiv 1/2((\alpha_{1k} \pm \alpha_{2k})p^k\psi + (\alpha_{1k} \mp \alpha_{2k})\pi^k\psi)$, distinguished by the replacement $p^k\psi \leftrightarrow \pi^k\psi$. Such the matrices A_k exist that the additional condition to be accepted:

$$\alpha_{1k}p^{1k}\psi + \alpha_{2k}p^{2k}\psi = A_k p^k\psi, \tag{11}$$

and its consequence (due to the noted symmetry):

$$\alpha_{1k}p^{1k}\psi - \alpha_{2k}p^{2k}\psi = A_k \pi^k\psi \tag{12}$$

lead to the split of the 16-component Eqs. (3, 10) into the four Dirac equations for some sets of four components of ψ. Let us accept Eq. (11) as a postulate; we can choose A_k as: $A_k = I_2 \times \sigma_1 \times \sigma_k \times I_2$. Now we can rewrite Eqs. (3, 10) as:

$$E\psi = (\beta_1 m^1 + \beta_2 m^2)\psi + A_k p^k \psi, \tag{13}$$

$$\pi^0\psi = (\beta_1 m^1 - \beta_2 m^2)\psi + A_k \pi^k \psi, \tag{14}$$

where $\psi = \psi(x_\mu, y_\mu)$, and Eqs. (11, 12) taking into account Eqs. (6, 7) as:

$$(A_k - \alpha_{1k})(p^k + \pi^k)\psi = 0, \tag{15}$$

$$(A_k - \alpha_{2k})(p^k - \pi^k)\psi = 0. \tag{16}$$

By the transition from Eq. (8) to Eqs. (13,14) the conditions (15,16) provide the compatibility of these system of equations for the function ψ given in the different coordinate spaces. These conditions do not contain evidently derivatives with respect to time, to be like the condition on a wave function for particles with spin 3/2, when field equations are written in the Rarita-Schwinger form [5]. Eqs. (13-16) give us the model of the two-component system without interactions describing four sets of fermions in the physical space with the coordinates x_μ which will be named generations.

3 Discrete and continues symmetries of the model

Let us consider the symmetries of the model for the case $m^1 = m^2 = 0$. The structure of the matrices is such that the following sets of components of ψ obey the Dirac equations:

$$\psi_1, \psi_3, \psi_5, \psi_7;\ \psi_2, \psi_4, \psi_6, \psi_8;\ \psi_9, \psi_{11}, \psi_{13}, \psi_{15};\ \psi_{10}, \psi_{12}, \psi_{14}, \psi_{16}.$$

Let us introduce left and right components of these four-component spinors $\psi^{'}$: $\psi^{'}_L = 1/2(1-\gamma_5)\psi^{'}$, $\psi^{'}_R = 1/2(1+\gamma_5)\psi^{'}$; their components of the view $1/2(\psi_k \pm \psi_n)$ we shall write shortly as $k \pm n$, where $(+)$ relates to the right components, $(-)$ relates to the left ones. The existence of pairs of solutions of Eqs. (13,14) leads to the $SU(2)$ symmetry of the model. The analysis of the conditions (15) and (16) shows that namely the ones define the structure of $SU(2)$ multiplets. These conditions put on the following restrictions: 1) doublets and singlets of $SU(2)_L$, as well as $SU(2)_R$, cannot exist together; 2) if ψ^A and ψ^B are two different solutions, then their components separately may form only singlets; 3) the first (second) component of doublets can be formed only from components of one solution; 4) it is possible to form up to 4 doublets of one of groups $SU(2)_L$ or $SU(2)_R$ (it depends on a variant of the model), moreover all doublets are transformed on the interwoven group presentations (i.e. the transformation of one doublet should be accompanied by the same transformation of other doublets), and up to 8 singlets of another $SU(2)$.

Before to prove these propositions let us explain what is mentioned as "model variants". The algebra (4) admits the sign change for any matrix of the chosen presentation. Let $\varepsilon_{ik} = \pm I$ be multipliers for α_{ik} by this change, $\alpha_{ik} \to \varepsilon_{ik}\alpha_{ik}$ (there is not summation by i, k). It turns out that by $\varepsilon_{1k}\varepsilon_{2k} = +I$ for any k only the $SU(2)_R$-doublets are possible, and by $\varepsilon_{1k}\varepsilon_{2k} = -I$ only the $SU(2)_L$-doublets are possible.

To prove the given propositions let us rewrite (15) and (16) in components:

$$D_1\varphi_1 = 0, \; D_2\varphi_2 = 0, \tag{17}$$

where the operators $D_i \equiv (p^{i1}, -p^{i1}, ip^{i2}, -ip^{i2}, p^{i3}, -p^{i3})$, where $p^{i\mu} = p^{i\mu}(p^\mu, \pi^\mu)$, and the matrices $\varphi_1, \; \varphi_2$ have the view:

$$\left|\begin{array}{cccccccc}
3\pm7 & 4\pm8 & 1\pm5 & 2\pm6 & 11\pm15 & 12\pm16 & 9\pm13 & 10\pm14 \\
10\pm14 & 9\pm13 & 12\pm16 & 11\pm15 & 2\pm6 & 1\pm5 & 4\pm8 & 3\pm7 \\
10\pm14 & -(9\pm13) & 1\pm5 & 2\pm6 & 2\pm6 & -(1\pm5) & 9\pm13 & 10\pm14 \\
3\pm7 & 4\pm8 & -(12\pm16) & 11\pm15 & 11\pm15 & (12\pm16) & -(4\pm8) & 3\pm7 \\
1\pm5 & 2\pm6 & -(11\pm15) & 12\pm16 & 9\pm13 & 10\pm14 & -(3\pm7) & 4\pm8 \\
9\pm13 & -(10\pm14) & 3\pm7 & 4\pm8 & 1\pm5 & -(2\pm6) & 11\pm15 & 12\pm16
\end{array}\right|$$

$$\left|\begin{array}{cccccccc}
\pm(3\pm7) & \pm(4\pm8) & \pm(1\pm5) & \pm(2\pm6) & \pm(11\pm15) & \pm(12\pm16) & \pm(9\pm13) & \pm(10\pm14) \\
11\pm15 & 12\pm16 & 9\pm13 & 10\pm14 & 3\pm7 & 4\pm8 & 1\pm5 & 2\pm6 \\
11\pm15 & 12\pm16 & \pm(1\pm5) & \pm(2\pm6) & 3\pm7 & 4\pm8 & \pm(9\pm13) & \pm(10\pm14) \\
\pm(3\pm7) & \pm(4\pm8) & 9\pm13 & 10\pm14 & \pm(11\pm15) & \pm(12\pm16) & 1\pm5 & 2\pm6 \\
\pm(1\pm5) & \pm(2\pm6) & 11\pm15 & 12\pm16 & \pm(9\pm13) & \pm(10\pm14) & 3\pm7 & 4\pm8 \\
9\pm13 & 10\pm14 & \pm(3\pm7) & \pm(4\pm8) & 1\pm5 & 2\pm6 & \pm(11\pm15) & \pm(12\pm16)
\end{array}\right|$$

If ψ is transformed under the action of generators of $SU(2)_L$ or $SU(2)_R$, all columns of the matrices $\varphi_1, \; \varphi_2$ should be saved excluding their rearrangements. Let $\psi^A, \; \psi^B$ be solutions of Eqs. (13,14). The matrices φ_i are such that components of ψ^A (or ψ^B) cannot be the first and the second components of the doublet because by the $SU(2)-$transformations it would lead to the change of some columns of φ_i, i.e. from ψ^A (ψ^B) only singlets may be constructed. The matrix φ_1 allows the single correspondence besides of identical between components of ψ^A and ψ^B: $\psi^B = U\psi^A$, where $U = \sigma_1 \times I_8$, i.e.

$$\begin{array}{cccccccc}
(1+5)^A & (3+7)^A & (2+6)^A & (4+8)^A & (9+13)^A & (11+15)^A & (10+14) & (12+16)^A \\
(9+13)^B & (11+15)^B & (10+14)^B & (12+16)^B & (1+5)^B & (3+7)^B & (2+6)^B & (4+8)^B
\end{array} \tag{18}$$

That gives 4 possible doublets of $SU(2)_L$ or $SU(2)_R$, moreover the component placement in columns of φ_1 is such that the same transform should be carried out above all four sets of components. If we suggest that one of these possible doublets can be replaced on two singlets then components of other possible doublets should be or zero or the pairs of singlets of the corresponding $SU(2)$.

By virtue of the Dirac equation for $R-$ and $L-$components (only by $m_1 = m_2 = 0$) Eq. (17) can be transformed to the view ($E^i \equiv p^{i0}$):

$$D_1^{'}\varphi_1^{'} = 0, \; D_2^{'}\varphi_2^{'} = 0, \tag{19}$$

where the operators $D_i^{'} \equiv (p^{i1}, -ip^{i2}, p^{i3}, p^{i0})$, and the matrices $\varphi_1^{'}$, $\varphi_2^{'}$ have the view:

$$\left|\begin{array}{cccccccc} 10\pm14 & 9\pm13 & 12\pm16 & 11\pm15 & 2\pm6 & 1\pm5 & 4\pm8 & 3\pm7 \\ 10\pm14 & -(9\pm13) & 12\pm16 & -(11\pm15) & 2\pm6 & -(1\pm5) & 4\pm8 & -(3\pm7) \\ 9\pm13 & -(10\pm14) & 11\pm15 & -(12\pm16) & 1\pm5 & -(2\pm6) & 3\pm7 & -(4\pm8) \\ \pm(1\pm5) & \pm(2\pm6) & \pm(3\pm7) & \pm(4\pm8) & \pm(9\pm13) & \pm(10\pm14) & \pm(11\pm15) & \pm(12\pm16) \end{array}\right|,$$

$$\left|\begin{array}{cccccccc} 3\pm7 & 4\pm8 & 1\pm5 & 2\pm6 & 11\pm15 & 12\pm16 & 9\pm13 & 10\pm14 \\ 3\pm7 & 4\pm8 & -(1\pm5) & -(2\pm6) & 11\pm15 & 12\pm16 & -(9\pm13) & -(10\pm14) \\ 1\pm5 & 2\pm6 & -(3\pm7) & -(4\pm8) & 9\pm13 & 10\pm14 & -(11\pm15) & -(12\pm16) \\ 9\pm13 & 10\pm14 & 11\pm15 & 12\pm16 & 1\pm5 & 2\pm6 & 3\pm7 & 4\pm8 \end{array}\right|,$$

that is equivalent in this case to the record of Eq. (8) via components.

The inversion $(x, y) \to (x, -y)$ transfers $D_1^{'} \leftrightarrow D_2^{'}$, $\psi(x, y) \to \psi^{'}(x, -y) = u\psi(x, y)$. By that we have:

$$D_1^{'}\varphi_1^{'}[\psi(x,y)] \to D_2^{'}\varphi_1^{''}[u\psi(x,y)],$$

$$D_2^{'}\varphi_2^{'}[\psi(x,y)] \to D_1^{'}\varphi_2^{''}[u\psi(x,y)],$$

i.e. Eq. (19) will be invariant under this transformation if $\varphi_1^{''} = \varphi_2^{'}$, $\varphi_2^{''} = \varphi_1^{'}$. The matrices $\varphi_1^{'}$, $\varphi_2^{'}$ can be transferred one into another transforming ψ but only for ψ_L or ψ_R: the transform exist for that components to whom the same signs correspond in the bottom rows of $\varphi_1^{'}$, $\varphi_2^{'}$. Then $\varphi_1^{''}(V\psi_{R\vee L}) = \varphi_2^{'}(\psi_{R\vee L})$, $\varphi_2^{''}(V\psi_{R\vee L}) = \varphi_1^{'}(\psi_{R\vee L})$, V is the matrix, $V = V^{-1}$, $R \vee L$ means "R or L". Only for that type of components the existence of the $SU(2)$-doublets is permitted. For the given presentation (5) $SU(2)_R$-doublets exist, $\varepsilon_{1k}\varepsilon_{2k} = +I$; $SU(2)_L$-doublets are permitted by $\varepsilon_{1k}\varepsilon_{2k} = -I$, i.e. after the change of signs of α_{1k} or α_{2k}.

The group $SU(2)$ is not an only symmetry group of the model. Unlike φ_1, $\varphi_1^{'}$ admits 8 types of solutions distinguishing by the transposition of the ψ components; all of them, taking into account the transform $\psi \to V\psi$, are allowed by $\varphi_2^{'}$, too. These eight solutions are splitted in two subsets of the considered type of ψ^A and $\psi^B = U\psi^A$, i.e. four types of doublets of some $\psi_{R\vee L}$ are admitted. Let us show these possible solutions as the substitutions skipping solution indices (their second part can be get by the transform $\psi \to U\psi$):

$$\begin{array}{cccccccc} 1\pm5 & 3\pm7 & 2\pm6 & 4\pm8 & 9\pm13 & 11\pm15 & 10\pm14 & 12\pm16 \\ 3\pm7 & 9\pm13 & 4\pm8 & 10\pm14 & 11\pm15 & 1\pm5 & 12\pm16 & 2\pm6 \\ 1\pm5 & 11\pm15 & 2\pm6 & 12\pm16 & 9\pm13 & 3\pm7 & 10\pm14 & 4\pm8 \\ 3\pm7 & 1\pm5 & 4\pm8 & 2\pm6 & 11\pm15 & 9\pm13 & 12\pm16 & 10\pm14 \end{array} \tag{20}$$

Writing them as: $\psi, U_1\psi, U_2\psi, U_3\psi$, where U_k are the transposition matrices, we have the following algebra for U_k an U:

$$U_1U_2U_3 = U_2U_3U_1 = U_3U_1U_2 = U,$$

$$U_1U_3U_2 = U_2U_1U_3 = U_3U_2U_1 = I,$$
$$U_2U_1 = U_3,\ U_1U_3 = U_2,\ U_2U_3 = U_1,$$
$$U_1U_2 = UU_3,\ U_3U_1 = UU_2,\ U_3U_2 = UU_1,$$
$$U_1^2 = U,\ U_2^2 = U_3^2 = I,\ [U, U_k] = 0. \tag{21}$$

By (21), all solutions are splitted into the two classes if we identify $\psi \sim U\psi$: the ones having only one of three "properties" being an analog of three "colors" of quarks (it is $U_k\psi$), and having all three "properties" (it is ψ). So, we may interpret in this way the concept of "color" for the composite system. By the additional requirement of the conservation of the norm $\psi^+\psi$ for every class of solutions, $SU(3)_C \times SU(2)$ will be the global symmetry group, the chiral properties of solutions are already discussed. The solutions $U_k\psi,\ k = 1,2,3,$ form the $SU(3)$-triplets, while ψ and $U\psi$ form its singlets and the doublet of one of $SU_{R\vee L}$, $U_k\psi$ and $UU_k\psi$ will be the doublets of the last, too. Let us note that $[V, U_k] \neq 0,\ [V, U] = 0$.

The discrete group of 16×16 matrices: $S = \{I, U_k, UU_k, U | k = 1,2,3\}$ forms the representation of the groups of transformations of the coordinates (x, y) by 8×8 matrices $S = \{I, u_k, uu_k, u\}$ and of the coordinates $(x_{1\mu}, x_{2\mu})$ by 8×8 matrices $S^{'} = \{I, u'_k, u'u'_k, u'\}$ with the same algebra (21). To reconstruct the last group by s, let us use the following reduction method. Note that U represents the inversion $(x, y) \to (x, -y)$, and the last is equivalent to the display $(x_{1\mu}, x_{2\mu}) \to (x_{2\mu}, x_{1\mu})$. Let us introduce the continuous numbering of the coordinates $(x_{i\mu}$: $z_{\alpha+1} = (x_{1\alpha},\ z_{\alpha+5} = (x_{2\alpha},\ \alpha = 0,1,2,3,$ their transforms will be written as a substitutions (we will write k instead of z_k). The matrix u' corresponding to the inversion we get if to assign numbers $1,2,...,8$ to the spinors components in the first raw of (18), and to identify the substitution of these numbers by (18) with u':

$$u' = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{vmatrix}.$$

This reduction method lets to write all matrices ${u_k}'$ by the substitutions (20):

$${u_1}' = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 7 & 6 & 1 & 8 & 3 \end{vmatrix},$$

$${u_2}' = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 6 & 3 & 8 & 5 & 2 & 7 & 4 \end{vmatrix},$$

$${u_3}' = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{vmatrix}.$$

By them one can reconstruct the matrices u_k of the group s taking into account relations of $x_\mu,\ y_\mu$ with $x_{1\mu},\ x_{2\mu}$:

$$u_1 = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ x_1 & x_0 & x_3 & x_2 & y_1 & -y_0 & y_3 & -y_2 \end{vmatrix},$$

$$u_2 = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 & y_0 & -y_1 & y_2 & -y_3 \end{vmatrix},$$

$$u_3 = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ x_1 & x_0 & x_3 & x_2 & y_1 & y_0 & y_3 & y_2 \end{vmatrix}.$$

Two of these transforms affect the sector of coordinates x_μ, and all of them affect the sector of y_μ; Eqs. (13 - 16) are invariant under such the transforms. From the geometrical point of view, the sets $\{U_k\}$, $\{u_k\}$, $\{u_k^{'}\}$ are the generatrix sets of some algebras which are homomorphic to the Clifford algebra $C(2,1)$ corresponding to the 3-dimensional color space with the signature $(++-)$ [3]. The discrete groups $S, s, s^{'}$ are isomorphic to the dihedron group D_4 [6].

Some further development of this model can be found in the author's later paper [7].

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