A Proof Of The *ABC Conjecture*.

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Abstract.
In this article, its shown that the *ABC Conjecture* is correct for integers $a+b=c$, and any real number $r>1$. This article proposes that the ABC Conjecture is true *iff*: c>0.

Keywords: Number Theory; *ABC Conjecture*; Square-free Numbers; Diophantine Equations; Prime Numbers; Mathematical Cryptography; Combinatorics.

1. Introduction.

The *ABC Conjecture* has been a controversial topic in Mathematics and was proposed independently by both Joseph Oesterle and David Masser in 1985 – see Scholze & Stix (2018), and Granville & Tucker (2002). The *ABC Conjecture* is defined as follows. Let *a*, *b* and *c* be coprime integers, where a+b=c. A square-free number is a number that cannot be divided by the square of any number. The "square-free part" of a number *n* [formally referred to as "sqp(*n*)" or "rad(*n*)" or "radical(*n*)"] is the largest square-free number that can be formed by multiplying the factors of *n* that are prime numbers.

The Original ABC Conjecture ("**ABC conjecture-I")** states that for every positive real number ε, there exist only finitely many coprime positive integers (*a*,*b*,*c*), with $a+b=c$, such that:

$c > \text{rad}(abc)^{(1+\varepsilon)}$

A second equivalent formulation of the ABC Conjecture ("**ABC conjecture-II")** states that for every positive real number ε, there exists a constant K_ε such that for all triples (*a*, *b*, *c*) of coprime positive integers, with $a+b=c$:

$c < (K_\varepsilon)\text{rad}(abc)^{(1+\varepsilon)}$

A third equivalent formulation of the ABC Conjecture ("**ABC conjecture-III")** states that for co-prime integers $a+b=c$, the ratio $[\text{rad}(abc)^r/c]$ is always greater than zero for any value of *r* greater than one. Its easy to see that *ABC Conjecture-I* is equivalent to *ABC Conjecture-III* (and the following effectively proves *ABC Conjecture-I*) because:

i) $r=(1+\varepsilon)$.

ii) if $c>[\text{rad}(abc)^{(1+\varepsilon)}]$ and $r=(1+\varepsilon)$, then the statement "…*the ratio rad(abc)^r/c is always greater than zero for any value of r*…." automatically implies that there are only *finitely many* triples (*a*, *b*, *c*) of coprime positive integers with $a+b=c$, that satisfy the condition $c > \text{rad}(abc)^{(1+\varepsilon)}$. The "*always-greater-than-zero*" restriction in *ABC Conjecture-III* eliminates all negative-number values (of the ratio $\text{rad}(abc)^r/c$) and also reduces the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* to *only-finitely-many triples*.

iii) As $(a,b,c)\rightarrow 0$, the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* that satisfy $c>[\text{rad}(abc)^{(1+\varepsilon)}]$ also tends to zero. That is as $(a,b,c)\rightarrow +\infty$, the powers of primes that are factors of a,b,c (and that are included in rad[abc]) will typically increase, but the number of "distinct factors" of a, b and c that are primes (and that are included in rad[abc]) will decline. Thus, there exist *only finitely many* triples (a,b,c) of coprime positive integers, with a+b=c, such that: $c > \text{rad}(abc)^{(1+\varepsilon)}$.

iv) As $(a,b,c)\rightarrow +\infty$, the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* that satisfy $c>[\text{rad}(abc)^{(1+\varepsilon)}]$ also tends to zero. That can be partly attributed to the following:

1) That is as (a,b,c)→+∞, the powers of primes that are factors of a,b,c (and that are included in rad[abc]) will typically increase, but the number of "distinct factors" of a, b and c that are primes (and that are included in rad[abc]) may not increase and may decline.
2) As (a,b,c)→+∞, the number of "distinct factors" that of a, b and c that are primes (and that are included in rad[abc]) will generally decline because as (a,b,c)→+∞, the absolute number of primes in any contiguous series of equal intervals (of positive integers), tends to zero. For example, for the series of positive-integer intervals (1,1000), (1001-2000), (2001,3000)……..(200,001;201,000), the number of primes in each interval declines as the positive-integers increase in value.

Thus, there exist *only finitely many* triples (a,b,c) of coprime positive integers, with a+b=c, such that: $c > rad(abc)^{(1+\varepsilon)}$.

It's also easy to see that *ABC Conjecture-II* is equivalent to *ABC Conjecture-III* because:

i) $r=(1+\varepsilon)>1$.
ii) if $c<[(K_\varepsilon)rad(abc)^{(1+\varepsilon)}]$ and $r=(1+\varepsilon)$, then K_ε, $[rad(abc)^r/c)] >0$. That is, the inequality $c<[(K_\varepsilon)rad(abc)^{(1+\varepsilon)}]$ is mathematically equivalent to the statement "…..$[rad(abc)^r/c)]>0$, for any value of the *r*…..".

The *ABC Conjecture* is related to compounding (financial mathematics) because of the exponent $r=(1+\varepsilon)>1$ (see Chapters 4, 5, 7 & 8 in Nwogugu [2017]). Contrary to assertions by mathematics professors, the *ABC Conjecture* isn't related to *Fermat's Last Conjecture* primarily because: i) in Fermat's equation, (a+b) is not required to be equal to *c*; and each of *a*, *b*, and *c* are not required to be co-prime; and ii) there is compounding in both sides (all the variables/bases) of Fermat's equation – see Nwogugu (2020a;b); iii) *Fermat's Last Conjecture* can be proved without reference to the factors of *a*, *b* and *c* – see Nwogugu (2020a;b).

Most or all the attempts to prove the *ABC Conjecture* have been un-necessarily convoluted and remain unverified – for example, see: Mochizuki (2020a;b;c;d), Yamashita (2018), and Silverman (1988). Scholze & Stix (2018) specifically noted that Mochizuki (2020a;b;c;d) was wrong and didn't prove the *ABC Conjecture*. Also see Yirka (April 2020) and Castelvecchi (April 2020).

2. The Theorems.

Theorem-1 ("*ABC conjecture-III*"): for co-prime integers *a*+*b*=*c*, the ratio [rad(*abc*)^r/*c*] is always greater than zero for any value of *r* greater than one.

Proof:

a+*b*=*c*, are integers but their signs can be positive or negative, and any can be zero. *r*>1 is any real number.

Let $0<p(a)<+\infty$ be the product of multiplying the distinct factors of *a* that are prime numbers (ie. but without repeating factors that are primes and occur more than once); and $a\geq p(a)$, *iff* a>0. Thus in the case of a=125 (which is 5x5x5), p(a)=5x1=5. If *a* is a prime number then its divisible by only one and itself, in which case a=p(a); and thus in the case of a=61, p(a)=61x1=61.

Let $0<p(b)<+\infty$ be the product of multiplying the distinct factors of *b* that are prime numbers (ie. but without repeating factors that are primes and occur more than once); and $b\geq p(b)$, *iff* b>0. If *b* is a prime number then its divisible by only one and itself, in which case b=p(b).

Let $0<p(c)<+\infty$ be the product of multiplying the distinct factors of *c* that are prime numbers (but without repeating factors that are primes and occur more than once); and $c\geq p(c)$, *iff* c>0. If *c* is a prime number then its divisible by only one and itself, in which case c=p(c).

Where *a* or *b* or *c* is a negative integer, it can still have a square-free part that is the product of one or more prime numbers (eg. 1).

Each of p(a), p(b), p(c), [p(a)p(b)p(c)] and rad(abc) is the product of prime numbers and will always be a positive integer.

1.1) Thus, rad(abc) = p(a)p(b)p(c)
1.2) If $a+b=c$, then p(a),p(b),p(c)≤c, *iff* c>0.

1.3) [rad(abc)/c]>1, *iff*:
 i) rad(abc) > $|c|$, and both numbers have the same sign.

1.4) [rad(abc)/c] >0, *iff*:
 i) c>0 (*rad(abc)* is derived from prime numbers and will always be a positive integer).

The smallest positive real number at which compounding (financial mathematics) starts is one (such as 1.0000000000000000000000000000000001). Thus, as long as r>1, there will be compounding, and then if [rad(abc)/c]>0 under conditions stated herein and above, then $+\infty>[rad(abc)^r]/c>0$, *iff*: c>0. Thus, the ABC Conjecture **("ABC conjecture-III")** is correct ■

Theorem-2 (The *Original ABC Conjecture* ("*ABC conjecture-I*")): for every positive real number ε, there exist only finitely many coprime positive integers (*a,b,c*), with *a+b=c*, such that:
 $c> rad(abc)^{(1+\varepsilon)}$

Proof:
As mentioned herein and above, r=(1+ε)>1.

As mentioned herein and above, as (a,b,c)→0, the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* that satisfy $c>[rad(abc)^{(1+\varepsilon)}]$ also tends to zero. That is as (a,b,c)→+∞, the powers of primes that are factors of a,b,c (and that are included in rad[abc]) will typically increase, but the number of "distinct factors" of a, b and c that are primes (and that are included in rad[abc]) will decline. Thus, there exist *only finitely many* triples (a,b,c) of coprime positive integers, with a+b=c, such that: $c> rad(abc)^{(1+\varepsilon)}$.

As mentioned herein and above, as (*a*,*b*,*c*)→+∞, the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* that satisfy $c>[rad(abc)^{(1+\varepsilon)}]$ also tends to zero. That can be partly attributed to the following:

1) That is as (*a*,*b*,*c*)→+∞, the powers of primes that are factors of a,b,c (and that are included in rad[abc]) will typically increase, but the number of "distinct factors" of a, b and c that are primes (and that are included in rad[abc]) may not increase and may decline.
2) As (*a*,*b*,*c*)→+∞, the number of "distinct factors" that of a, b and c that are primes (and that are included in rad[abc]) will generally decline because as (a,b,c)→+∞, the absolute number of primes in any contiguous series of equal intervals (of positive integers), tends to zero. For example, for the series of positive-integer intervals (1,1000), (1001-2000), (2001,3000)……..(200,001;201,000), the number of primes in each interval declines as the positive-integers increase in value.

Thus, there exist *only finitely many* triples (*a*,*b*,*c*) of coprime positive integers, with a+b=c, such that: $c> rad(abc)^{(1+\varepsilon)}$.

Although in many or most instances, c < rad[abc] (in fewer instances, c >rad[abc]); as *r*→+∞ (and because of compounding since r=(1+ε)>1; see Chapters 4, 5, 7 & 8 in Nwogugu [2017]), *r* will reach a value where $c<[rad(abc)^r]$ exists for all feasible triples (a,b,c) (that satisfy $c>[rad(abc)^r]$) and this threshold value of *r* is hence forth referred to as r_{max}. This r_{max} effectively limits/caps both: i) the *number-of-feasible-combinations* of coprimes *a*, *b* and *c* that satisfy $c>[rad(abc)^r]$, and ii) the number of "distinct factors" of *a*, *b* and *c* that are primes (and produce rad[abc]).

Thus, there are *only finitely many* coprime positive integers (*a*,*b*,*c*), with *a*+*b*=*c*, such that: $c> rad(abc)^{(1+\varepsilon)}$; and the Original ABC Conjecture **("ABC conjecture-I")** is correct. ■

3. Conclusion.
The *ABC Conjecture* is true for positive coprime integers $a+b=c$, and any real number $r=(1+\varepsilon)>1$.

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