On the Existence of Triangles

VOLKER W. THÜREY
Bremen, Germany *

June 4, 2020

We formulate criterions about the existence of triangles depending on its sidelengths.

Keywords and phrases: Triangle
MSC 2010 subject classification: 51P99

It is well-known that a triangle with sides of lengths a,b and c exists if and only if three inequalities are fulfilled. They are called the 'triangle inequalities'.
Proposition 1. *A triangle with sidelenghts a,b and c exists if and only if three inequalities hold.*

$$a+b>c \tag{1}$$
$$a+c>b \tag{2}$$
$$b+c>a \tag{3}$$

These inequalities can be combined by the inequality

$$Minimum\{a+b-c, a+c-b, b+c-a\} > 0 \tag{4}$$

We show that this inequality can be replaced by another inequality with a product of three factors.
Proposition 2. *A triangle with sidelengths a,b and c exists if and only if the following inequality holds.*

$$(a+b-c)\cdot(a+c-b)\cdot(b+c-a) > 0 \tag{5}$$

Proof. We show that out of the three factors, two cannot simultaneously be negative. Assume three real numbers x,y,z such that both $x+y-z$ and $z+x-y$ are negative. This means $x+y<z$ and $z+x<y$. This means $z+x<y<z-x$, hence $x<-x$. This is only possible with a negative x. This proves the proposition, since a,b and c are positive. □

Proposition 3. *Let a and b are line segments which meet only once in a single point called C at the end. The endpoints of a are B and C, while the endpoints of b are A and C. The generated angle is $\angle(ACB)$. ($\angle(XYZ)$ means the angle made by three points X,Y,Z such that Y is the apex.)*
A triangle with sidelengths a,b and c exists if and only if the following equality holds.

$$c = a\cdot\cos\angle(ABC) + b\cdot\cos\angle(BAC) = \sqrt{a^2+b^2-2\cdot a\cdot b\cdot\cos\angle(ACB)} \tag{6}$$

A and B have the distance c.

*49 (0)421 591777, volker@thuerey.de

Proof. The proposition is well-known. It is true due to the law of cosines. □

We show that there are two further inequalities which are equivalent to the triangle inequalities of Proposition 1.
Let a,b,c be positive real numbers such that $b,c \leq a$.
Proposition 4. *A triangle with sidelengths a,b and c exists if and only if one of the following inequalities holds.*

$$a - b < c < a + b \tag{7}$$

$$(a-c)^2 < b^2 \tag{8}$$

Proof. From (1) and (3) follow (7), and from (7) follows (1), (3), and since $b \leq a$ also (2).
Similarly, with $c \leq a$, (3) is equivalent to (8), and since $0 < b,c \leq a$ it holds (1) and (2). □

Remark 1. Note that we get equivalent inequalities if we exchange the variables b and c in (7) and (8).

Furthermore we ask by given positive real numbers v and w for t such that v,w and t are sidelengths of a triangle.
Let v and w be line segments which meet only once at the end. The generated angle is called τ. Let $\tau < 180°$.
Proposition 5. A triangle with sidelengths v,w and t exists if and only if it holds

$$t = \sqrt{v^2 + w^2 - 2 \cdot v \cdot w \cdot \cos\tau} \tag{9}$$

Proof. The proposition is true due to the law of cosines. □

Acknowledgements: We thank Kshitija Jog for a careful reading.

References

[1] Recent Advances in Geometric Inequalities D.S. Mitrinović, J.E. Pečarić, V. Volenec , Kluwer (1989)