Pappus chain and division by zero calculus

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Abstract. We consider circles touching two of three circles forming arbeloi with division by zero and division by zero calculus.

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1. Introduction

Let C be a point on the segment AB such that $|BC| = 2a$ and $|CA| = 2b$ (see Figure 1). For an arbelos configuration formed by the three circles α, β and γ with diameters BC, CA and AB, respectively, we consider circles touching two of the three circles by the definition of division by zero [3]:

$$\frac{z}{0} = 0 \quad \textit{for any real number } z, \tag{1}$$

and division by zero calculus [21]. We use a rectangular coordinates system with origin C such that the point B has coordinates $(2a, 0)$. We call the line AB the baseline.

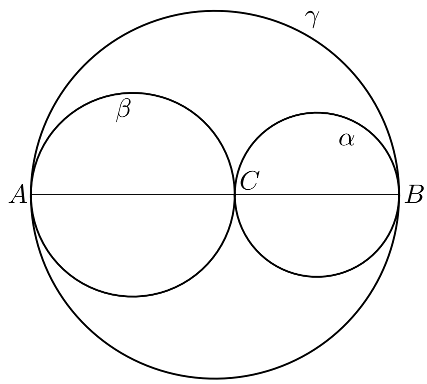

Figure 1.

2. Circles touching two of α, β and γ

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. Let $c = a + b$ and $d = \sqrt{ab}/c$.

Theorem 1. *The following statements hold.*
(i) *A circle touches the circles β and γ in the opposite sense if and only if its has radius r_z^α and center of coordinates (x_z^α, y_z^α) given by*

$$r_z^\alpha = \frac{abc}{a^2z^2 + bc} \quad \textit{and} \quad (x_z^\alpha, y_z^\alpha) = \left(-2b + \frac{b+c}{a} r_z^\alpha, 2zr_z^\alpha\right) \quad \textit{for a real number } z.$$

(ii) *A circle touches the circles γ and α in the opposite sense if and only if its has radius r_z^β and center of coordinates (x_z^β, y_z^β) given by*

$$r_z^\beta = \frac{abc}{b^2z^2+ca} \quad \text{and} \quad (x_z^\beta, y_z^\beta) = \left(2a - \frac{c+a}{b}r_z^\beta, 2zr_z^\beta\right) \text{ for a real number } z.$$

(iii) *A circle touches the circles α and β in the same sense if and only if its has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by*

$$r_z^\gamma = |q_z^\gamma| \quad \text{and} \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c}q_z^\gamma, 2zq_z^\gamma\right), \text{ where } q_z^\gamma = \frac{abc}{c^2z^2-ab}$$

for a real number $z \neq \pm d$.

Proof. Let δ_z be the circle of radius and center described in (iii). Then we have $(x_z^\gamma - a)^2 + (y_z^\gamma)^2 = (a + q_z^\gamma)^2$. Therefore δ_z and α touch internally or externally according as $q_z^\gamma < 0$ or $q_z^\gamma > 0$. Similarly δ_z and β touch internally or externally according as $q_z^\gamma < 0$ or $q_z^\gamma > 0$. Hence δ_z touches α and β in the same sense. Conversely we assume that a circle δ' of radius r touches α and β in the same sense. Then there is a real numbers z such that $r_{\pm z}^\gamma = r$. Therefore we have $\delta' = \delta_z$ or $\delta' = \delta_{-z}$. This proves (iii). The rest of the theorem can be proved similarly. □

Essentially the same formulas as Theorem 1 can be found in [22], not so simple though. Simpler expression in the case z being an integer can be found in [4, 5] and cited in [1] and [12].

We denote the circle of radius r_z^α and center of coordinates (x_z^α, y_z^α) by α_z. The circles β_z and γ_z are defined similarly. Notice that $\alpha_1 = \beta_1 = \gamma_1$ (resp. $\alpha_{-1} = \beta_{-1} = \gamma_{-1}$) is the incircle of the arbelos in the region $y \geq 0$ (resp. $y \leq 0$).

The circle γ_z touches α and β internally (resp. externally) if and only if $|z| < d$ (resp. $|z| > d$). The external common tangents of α and β have following equations [19, 20]:

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{2}$$

which are denoted by $\gamma_{\pm d}$.

Corollary 1. *The following statements hold.*
(i) *The distance between the center of the circle α_z and the baseline equals $2|z|r_z^\alpha$.*
(ii) *The distance between the center of the circle β_z and the baseline equals $2|z|r_z^\beta$.*
(ii) *The distance between the center of the circle γ_z and the baseline equals $2|z|r_z^\gamma$.*

Corollary 2. *The following statements hold.*
(i) *The ratio between the distance from the center of α_z to the perpendicular to the baseline at A and the radius of α_z is constant and equal to $((b+c)/a)r_z^\alpha$.*
(ii) *The ratio between the distance from the center of β_z to the perpendicular to the baseline at B and the radius of β_z is constant and equal to $((c+a)/b)r_z^\beta$.*
(ii) *The ratio between the distance from the center of γ_z to the perpendicular to the baseline at C and the radius of γ_z is constant and equal to $(|a-b|/c)r_z^\gamma$ for $z \neq \pm d$.*

3. Division by zero

The circle α_z has an equation $(x - x_z^\alpha)^2 + (y - y_z^\alpha)^2 = (r_z^\alpha)^2$, which is arranged as

$$\alpha_z(x, y) = \frac{bc((x-a)^2 + y^2 - a^2) - 4abcyz + a^2((x+2b)^2 + y^2)z^2}{a^2z^2 + bc} = 0.$$

Therefore we get $(x-a)^2 + y^2 = a^2$, $y = 0$ and $(x+2b)^2 + y^2 = 0$ in the case $z = 0$ from $\alpha_z(x, y) = 0$, $\alpha_z(x, y)/z = 0$, and $\alpha_z(x, y)/z^2 = 0$, respectively by (1). They represent the circle $\alpha = \alpha_0$, the baseline and the point circle A, respectively. We denote the point circle A and the baseline by α_∞ and $\alpha_{\overline{\infty}}$, respectively, and consider that they also touch α and γ (see Figure 2). Someone may consider that $\alpha_{\overline{\infty}}$ is orthogonal to α and γ and does not touch them. But (1) implies $\tan(\pi/2) = 0$. Therefore we can consider that $\alpha_{\overline{\infty}}$ still touches α and γ. We also consider that α_∞ and $\alpha_{\overline{\infty}}$ touch.

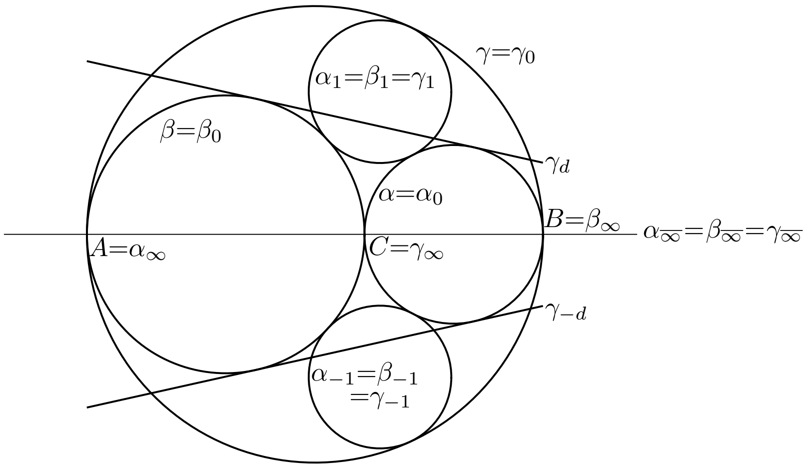

Figure 2.

We have $\beta_0 = \beta$, and denote the point B and the baseline by β_∞ and $\beta_{\overline{\infty}}$, respectively. We also have $\gamma_0 = \gamma$, and denote the point C and the baseline by γ_∞ and $\gamma_{\overline{\infty}}$, respectively.

4. Pappus chain

Let $r_{\mathrm{A}} = ab/(a+b)$. Circles of radius r_{A} are said to be Archimedean.

Theorem 2. *We assume that $a \neq b$, and w and z are real numbers. The two circle of each of the three pairs α_z, α_w; β_z, β_w; γ_z, γ_w touch if and only if $|w - z| = 1$.*

Proof. If $|w| > d$ and $|z| > d$, we get $r_w^\gamma = q_w^\gamma$ and $r_z^\gamma = q_z^\gamma$, and

$$(x_w^\gamma - x_z^\gamma)^2 + (y_w^\gamma - y_z^\gamma)^2 - (r_w^\gamma + r_z^\gamma)^2 = \frac{4a^2b^2c^2((w-z)^2 - 1)}{(c^2w^2 - ab)(c^2z^2 - ab)}.$$

Hence the theorem holds. If $|w| < d$ and $|z| > d$, we get $r_w^\gamma = -q_w^\gamma$ and $r_z^\gamma = q_z^\gamma$. Then γ_w and γ_z touch if and only if γ_z touches γ_w from inside of γ_w. While we have

$$(x_w^\gamma - x_z^\gamma)^2 + (y_w^\gamma - y_z^\gamma)^2 - (r_w^\gamma - r_z^\gamma)^2 = \frac{4a^2b^2c^2((w-z)^2 - 1)}{(c^2w^2 - ab)(c^2z^2 - ab)}.$$

Hence the theorem holds for γ_w and γ_z. If $|w| < d$ and $|z| < d$, both γ_w and γ_z touch α and β internally. Therefore they do not touch. If $w = d$, $z = d \pm 1$ and $a \neq b$, then γ_w and γ_z have only one point in common, whose coordinates equal

$$\left(-2r_{\mathrm{A}}\frac{\sqrt{a} \mp \sqrt{b}}{\sqrt{a} \pm \sqrt{b}}, \pm 2r_{\mathrm{A}}\right). \tag{3}$$

Therefore they touch. Since the figure is symmetric in the baseline, γ_{-d} and $\gamma_{-d\pm1}$ also touch. Now the theorem is proved for the circles γ_w and γ_z. The rest of the theorem is proved in a similar way. □

The theorem holds for the two pairs α_z, α_w; β_z, β_w in the case $a = b$. The theorem shows that any Pappus chain, whose members touch β and γ, is expressed by $\cdots$, α_{z-2}, α_{z-1}, α_z, α_{z+1}, α_{z+2}, $\cdots$ for a real number z. Also it shows that Corollary 1 is a generalization of Pappus chain theorem.

One of the circles γ_{d+1} and γ_{d-1} is the incircle of the curvilinear triangle made by α, β and γ_d, and the other touches the three in the region $y < 0$. Hence y^{γ}_{d+1} and y^{γ}_{d-1} have different signs. While $y^{\gamma}_{d-1} > 0$ shows that γ_{d+1} is the incircle of the curvilinear triangle. Therefore the center of γ_{d-1} lies in the region $y < 0$ (see Figure 3).

Corollary 3. *If $a \neq b$ and $z = d$ or $z = -d$, then the two smallest circles passing through the point of tangency of γ_z and $\gamma_{z\pm1}$ and touching the baseline are Archimedean.*

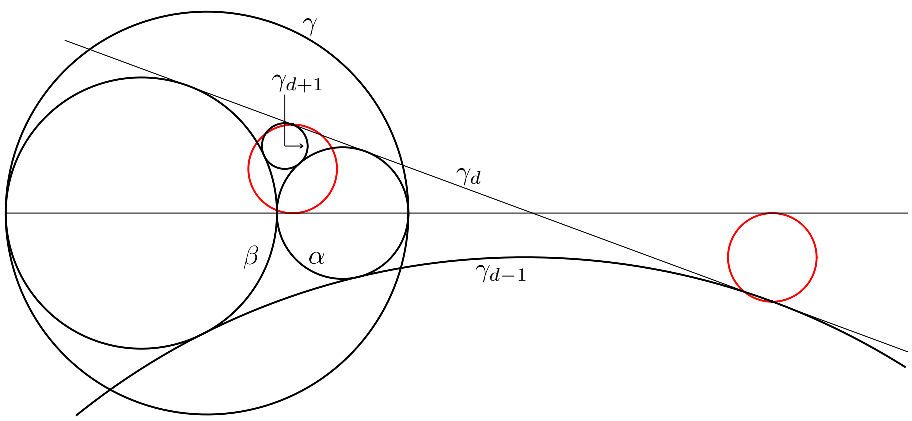

Figure 3: The two circles in red are Archimedean.

5. Division by zero calculus

If $f(z) = \cdots + C_{-2}(z-a)^{-2} + C_{-1}(z-a)^{-1} + C_0 + C_1(z-a) + C_2(z-a)^2 + \cdots$ is the Laurent expansion of a function $f(z)$ around $z = a$, the definition $f(a) = C_0$ is called the division by zero calculus [16], [21].

Let $g_z(x,y) = (x - x^{\gamma}_z)^2 + (y - y^{\gamma}_z)^2 - (r^{\gamma}_z)^2$. Then $g_z(x,y) = 0$ is an equation of the circle γ_z for $z \neq \pm d$. Let

$$g_z(x,y) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + C_1(z-d) + \cdots$$

be the Laurent expansion of $g_z(x,y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$

$$C_{-1} = d((a-b)x - 2\sqrt{ab}y + 2ab),$$

$$C_0 = \left(x - \frac{a-b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2+18ab+b^2}}{4}\right)^2,$$

$$C_n = -\frac{1}{2}\left(\frac{-1}{2d}\right)^n ((a-b)x + 2\sqrt{ab}y + 2ab), \ \textit{for } n = 1, 2, 3, \cdots.$$

Therefore $C_{-1} = 0$ gives an equation of the line γ_d, but $C_0 = 0$ does not. Also $C_n = 0$ gives an equation of the line γ_{-d} for $n = 1, 2, 3, \cdots$.

Let ε be the circle given by the equation $C_0 = 0$. We have considered this circle in [19], which has the following properties (see Figure 4):
(i) The points, where γ_d touches α and β, lie on ε.
(ii) The radical center of the three circles α, β and ε has coordinates $(0, -\sqrt{ab})$, and lies on the line γ_{-d}.
We would like to state one more here:
(iii) The radical axis of the circles ε and γ passes though the points of coordinates $(0, 3\sqrt{ab})$ and $(2ab/(b-a), 0)$, where the latter coincides with the point of intersection of γ_d and γ_{-d}.
The y-axis meets γ and $\gamma_{\pm d}$ in the points of coordinates $(0, \pm 2\sqrt{ab})$ and $(0, \pm\sqrt{ab})$, respectively. Hence the six points, where the y-axis meets γ, $\gamma_{\pm d}$, the baseline, the radical axis of γ and ε, are evenly spaced. Those points are denoted in magenta in Figure 4. Reflecting the figure in the baseline, we also get similar results for the Laurent expansion of $g_z(x, y)$ around $z = -d$.

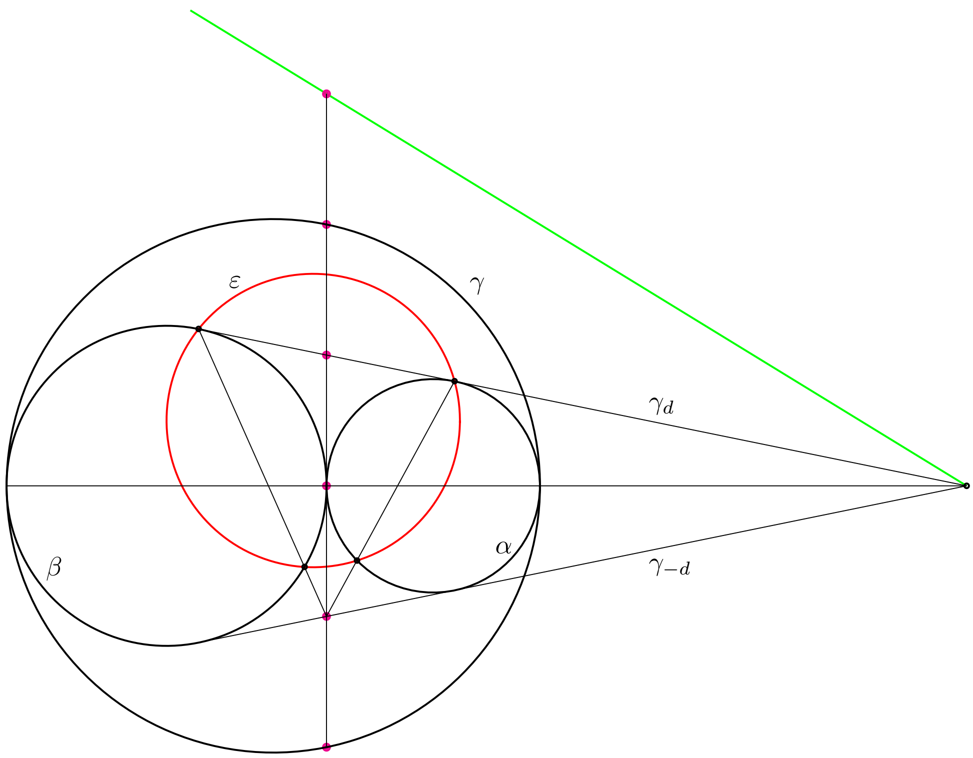

Figure 4: The green line denotes the radical axis of the circles γ and ε.

For more applications of division by zero and division by zero calculus to circle geometry, see [2], [6], [7, 8, 9, 10, 11, 12, 13, 14, 15] [16, 17, 18, 19], and for an extensive reference see [21].

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