Essence of Millennium Problems

Vaibhav Gupta

Abstract

This article contains seven Millennium problem simple explanation. The most of things are from different sources and I just arranged it in simple manner so that even general person can understand that. The problems contain diverse fields including Pure Mathematics (Topology and Number Theory), Theoretical Computer Science, Physics etc.

Essence of Millennium Problems

If you observe closely, Number 7 appears in many places like 7 periods of periodic table, 7 wonders of world, 7 colors of rainbow and many more. The Millennium Prize Problems (MPP) are 7 unsolved (now one is solved) problems in mathematics. The Clay Mathematics Institute, in May 2000, announced a prize of $1M for solving the MPP each. The problems encompass a diverse group of topics, including theoretical computer science and physics, as well as pure mathematical areas such as number theory, algebraic geometry, and topology.

One of the seven problems, Poincaré conjuncture has been solved in 2006 by mathematician Grigori Perelman, but he did not accept the prize money of $1M. In sept 2018, mathematician Michael Atiyah claimed to have a solution for Riemann Hypothesis, but the validity of his solution remains to be seen.

1. Poincaré conjuncture: If a compact three-dimensional manifold M^3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3?

Explanation: It deals with topology of spheres. Topology means study of shapes and surfaces. Two objects are homeomorphic if they can be deformed into each other by an elastic deformation like cup and doughnut. The meaning of manifold is written in Hodge conjecture section.

Suppose we have a string loop which both ends are to be glued and we stretched this to a surface of orange. Now we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, if we imagine that the same string loop has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the string or the doughnut. So, there is a difference in shapes of orange and a doughnut. In other words, the shape which have holes (doughnut) are much more complicated properties. (for simplicity we called sphere as orange and torus as doughnut)

We say the surface of the orange is "simply connected," but that the surface of the doughnut is not. Poincaré, almost a hundred years ago, knew that a two-dimensional sphere is essentially characterized by this property of simple connectivity, and asked the corresponding question for the three-dimensional sphere.

2. Riemann hypothesis: The nontrivial zeros of ζ(s) have real part equal to 1/2 or for which s (the complete set) ζ(s)= 0

They Explanation: Prime numbers are building block of many fields like mathematics, internet security etc. play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all-natural numbers does not follow any regular pattern. However, the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function called the Riemann Zeta function (This function can be written in multiplication of prime numbers so it encodes the secret about it.) The Riemann hypothesis asserts that all interesting solutions of the equation ζ(s) = 0 lie on a certain vertical straight line that is passing from 1/2 on real number line.

The Riemann Zeta function: $\zeta(s) = 1 + (1/2)^s + (1/3)^s + (1/4)^s + \ldots$

Note: All the negative even numbers give the value of zero to zeta functions, they are known as trivial zeros. The only point which does not give any number to zeta function is 1 called as singularity or pole.

3. P versus NP: Does P = NP? (If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem?)

Explanation: It is related to the mysteries of computer science. Finding a needle in a haystack is the extremely difficult task, despite the fact that a needle has very little in common with straw. If you were given what was claimed to be a needle, you could very quickly tell whether it was actually a piece of straw or not. Finding the needle in the haystack is difficult to do, but it is very easy to check whether a result is correct. Here is another example: consider your institute timetable, it is easy to check the whether it has clashes or not, but finding a clash free timetable is just harder. This is the essence of the P vs NP problem.

4. Yang-Mills Existence and the Mass Gap: Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on R^4 and has a mass gap $\Delta > 0$.

Explanation: As there are Newton's laws for macroscopic world, there exists some law which explain the behavior of microscopic world, the quantum laws. The quantum theory should be able to describe the classical behavior as well (as we see & feel). Mass gap means that, there can not be arbitrary small or close to zero masses particle within the framework of this theory OR in other words mass gap is the difference in energy between the lowest energy state (vacuum, assume to be 0) and the next lowest energy state i.e. the mass gap is the mass of lightest particle. The classical theory predicts the massless particles (interaction of electromagnetic force via photons), long-range forces (like gravitational force) but quantum theory tells about massive particles and short-range forces. So mass gap can explain the non-existence of massless particle in observation of strong interaction.

This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view. It can tell us about quarks confinement (which create the neutron and proton) and strong force that binds the nucleus.

5.Navier-stokes equation: Existence and Smoothness of the Navier-stokes Equation (It means for fluids in 3D, can we determine if the solution of this equation exist? Ans if they exist, are they smooth i.e. differentiable everywhere?)

Explanation: Fluids move in mysterious ways. Mathematicians are not even sure the equations that describe them will work in every situation, so if you can derive that equation which predicts the fluid movement, you can surely solve this one. Now a simple example is if you pour the water in your glass, you will see different movement of water every time. Mathematicians and physicists believe that an explanation for and prediction of both the breeze and the turbulence can be found through an understanding of the solutions to the Navier-stokes equations. In simple words the Navier-stokes equation are differential form of Newton's second law (F=ma) and conservation of mass.

Note: Turbulence is a chaotic motion of fluids resulting from eddies like smoke rising from source or in airplane cases.

6. Hodge Conjecture: On a projective non-singular algebraic variety over C, any Hodge class is a rational linear combination of classes cl(Z) of algebraic cycles.

Explanation: You can describe the geometry by the use of algebraic equations or vice versa like $y^2=4x$ is a kind of parabola and all the points on Cartesian plane which make the boundary of curve is a solution for this equation and a line parallel to y axis is written as x= a(constant). Also, you can solve the both equations to see where it crosses the curve. So, if you write more complicated equation, you will get a unique representation of that but there may be some shapes which we can not visualize (in spite of writing equations). So, if we deal with objects beyond what we can visualize, these "shapes" became known in general as "algebraic cycles". If an algebraic cycle was a nice smooth and generally well-behaved shape, it also earned the title of manifold.

All of the shapes that could be distorted from one to the other (without being lifted off the manifold surface) into a "homology class" – a kind of generalized shape. e.g. If you draw a shape on the surface of doughnut, you can make other shape by distorting of previous one. So, the problem is if you drew a random shape on a manifold, how would know its homology class will equivalent to algebraic cycles. In one sentence, "Given a random shape, when is it homeomorphic to a shape described by polynomial?"

7. Birch and Swinnerton-Dyer Conjecture (BSD conjecture): The genus of C_0 is greater than or equal to 2, then $C_0(Q)$ is finite?

Explanation: First we discuss some terms used in statement. C_0 is a curve defined by a polynomial equation in two variables, f (x, y) =0 and Q is notation/set of rational numbers. In commonly genus means number of holes it has like sphere has genus 0 and torus has genus 1. This conjecture describes the set of rational solutions to equation defining an elliptical curve. The elliptical curve does not mean it contains ellipses only, but it relates to that integral which describe the planetary motion in space.

Now as we have known about zeta function (used in Riemann hypothesis), it is easy to describe in terms of it. Riemann zeta function has only one singularity or pole i.e. where it does not give any value, s=1 so this conjecture suggests if ζ(1) is equal to 0, then there are an infinite number of rational points (solution), and conversely, if ζ(1) is not equal to 0, then there is only a finite number of such points.

Source: Clay Mathematics Institute (http://www.claymath.org/)