On The Infinity of Twin Primes and other K-tuples

Jabari Zakiya: jzakiya@gmail.com

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Abstract

The paper uses the structure and math of Prime Generator Theory to show there are an infinity of twin primes, proving the Twin Prime Conjecture, as well as establishing the infinity of other k-tuples of primes.

1 Introduction

In number theory ***Polignac's Conjecture*** (1849) [6] states there are infinitely many consecutive primes (prime pairs) that differ by any even number n. The ***Twin Prime Conjecture*** derives from it for prime pairs that differ by 2, the so called twin primes, e.g. (11, 13) and (101, 103).

K-tuples are groupings of primes adhering to specific patterns, usually designated as (k, d) groupings, where k is the number of primes in the group and d the total spacing between its first and last prime [4]. Thus, Polignac's pairs are type (2, n), where n is any even number. Three named (2, n) tuples are Twin Primes (2, 2), Cousin Primes (2, 4), and Sexy Primes (2, 6). The paper shows there are many more Sexy Primes (in fact, always more abundant) than Twins or Cousins, though an infinity of each, and an infinity of any other (2, n) tuple.

I begin by presenting the foundation of ***Prime Generator Theory (PGT)***, through its various components. I start with ***Prime Generators (PG)***, which as their name implies, generate all the primes. Each larger PG is more efficient at identifying primes by reducing the ***number space*** they can possibly exist within. They thus structurally squeeze the primes into a smaller set of integers that contain fewer composites, in a very systematic manner.

Each PG has a characteristic ***Prime Generator Sequence (PGS)***, a repeating pattern of gaps between the ***residue*** elements of its PG. These gap patterns illustrate, and adhere to, a ***deterministic set of properties***. I use them to systematically show once a PGS gap size between residues exists it will be repeated with higher frequency for all larger PGS. I then show every residue gap will, with certainty, become a gap strictly between prime pairs. This will be used to establish the infinity of twin pairs, and other k-tuples. I provide data and graphs to empirically show this.

The ***epistemological model*** for developing PGT is highly visual, and most easily explained and understood through pictures to establish its properties. Some may not find this "rigorous" and insufficient to meet its claims. However, it will be seen its foundation provides a consistent mathematical framework to qualitatively explain, and quantitatively produce, empirically verifiable results derived using other methods and techniques.

At the time of writing, the largest known twin prime is $2996863034895 \cdot 2^{1290000} \pm 1$ [5] (2016), which resides on ***restracks*** P5[29:31] and P7[29:31] for those PG. There are an infinity of larger twin primes, which will reside on some twin pair restracks for every PG. The same will be true for other k-tuples.

I have previously used Prime Generators to construct and implement efficient and very fast ***prime sieves***, to find all the primes up to a finite N, or within a finite range, including the fastest and most efficient prime sieve methods to find all primes and twin|cousin primes. See [1], [2], [3]

2 Prime Generators

A prime generator **Pn** is composed of a modulus **modpn** and a set of residues $\boldsymbol{r_i}$ with residue count **rescntpn** (determined by Euler's Totient Function, $\varphi(n) = n\prod(1 - 1/p_i)$, which have the form:

$$P_n = modpn \cdot k + \{r_i\} \tag{1}$$

$$modpn = p_n\# = \prod p_i = 2 \cdot 3 \cdot 5 \cdot ... \cdot p_n \tag{2}$$

$$rescntpn = (p_n - 1)\# = \prod(p_i - 1) = (2 - 1) \cdot (3 - 1) \cdot (5 - 1) \cdot ... \cdot (p_n - 1) \tag{3}$$

where $\boldsymbol{p_n}$ is the last PG prime. A PG's **residues** are the set of integers $\boldsymbol{r_i}$ **ε {1...modpn-1}** coprime (no common factors) to its **modpn**, i.e. their greatest common divisor is 1: gcd(r_i, modpn) = 1. They exist as ***modular complement pairs***, such that **modpn** $=$ $\boldsymbol{r_i} + \boldsymbol{r_j}$ and therefore **(**$\boldsymbol{r_i} + \boldsymbol{r_j}$**) mod modpn** $\equiv$ **0**. Thus, we only need to generate the residues $\boldsymbol{r_i} <$ **modpn/2**, and the other half are $\boldsymbol{r_j}$ = **modpn -** $\boldsymbol{r_i}$.

For P5 then, modp5 = 2 · 3 · 5 = 30, with rescntp5 = 1 · 2 · 4 = 8. P5's 8 **canonical** residues are {1, 7, 11, 13, 17, 19, 23, 29}, which are used **functionally** as {7, 11, 13, 17, 19, 23, 29, 31}, to always have the first residue in the sequence be prime p_{n+1}, and permute $r_i = 1$ to be the last residue in the sequence, set to **(modpn + 1)** $\equiv$ **1 mod modpn**. Thus we have:

$$P5 = 30 \cdot k + \{7, 11, 13, 17, 19, 23, 29, 31\} \tag{4}$$

We can now construct P5's **prime candidates (pcs) table**, here up to N = 541, the 100th prime, where each **k** $\geq$ **0** index **residue group (resgroup)** contains pc values along each **residue track (restrack|rt)**.

k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
rt0	7	37	67	97	127	157	187	217	247	277	307	337	367	397	427	457	487	517
rt1	11	41	71	101	131	161	191	221	251	281	311	341	371	401	431	461	491	521
rt2	13	43	73	103	133	163	193	223	253	283	313	343	373	403	433	463	493	523
rt3	17	47	77	107	137	167	197	227	257	287	317	347	377	407	437	467	497	527
rt4	19	49	79	109	139	169	199	229	259	289	319	349	379	409	439	469	499	529
rt5	23	53	83	113	143	173	203	233	263	293	323	353	383	413	443	473	503	533
rt6	29	59	89	119	149	179	209	239	269	299	329	359	389	419	449	479	509	539
rt7	31	61	91	121	151	181	211	241	271	301	331	361	391	421	451	481	511	541

Fig 1.

A table of prime candidates can be created for every PG. All the primes $> p_n$ occur mostly in equal numbers (i.e. statistically uniformly) along each restracks. The marked cells in Fig 1. are prime multiples (composites) of the residue primes, that have been sieved out to identify the primes within the range. See [1], [3]. P5 is the largest Pn for which all its residues are prime. All larger will have residues consisting of primes and their consecutive coprime multiples < modpn.

3 Prime Generator Sequences

Each prime generator has a characteristic ***Prime Generator Sequence (PGS)***. This is the sequence of the differences (gaps) between consecutive residues defined over the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn** where $\boldsymbol{r_0}$ is the first residue of Pn, which is the next prime $> p_n$, i.e. p_{n+1}.

Let's construct the first prime generator P2, and its PGS.

For P2: modp2 = 2, with rescntp2 = (2 - 1) = 1, with residue {1}, but use its functional value {3}.

Thus, P2 = 2·k+3, produces the pc sequence: 3 5 7 9 11 13 15 17... ∞, i.e the odd numbers. So for P2, its PGS is a single element of gap size ($\boldsymbol{r_0}$ - 1) = (3 - 1) = 2: PGS P2: [$\boldsymbol{r_0}$ = 3] 2 |

Now let's construct P3: modp3 = 2 · 3 = 6; rescntp3 = (2 - 1) · (3 - 1) = 2, with residues {1, 5}. P3, thus, has the functional form: P3 = 6·k+ {5, 7}. Its pcs table is shown below up to k = 16.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
rt0	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101
rt1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103

Fig 2.

For P3, each resgroup (column) contains prime candidates forming a possible twin pair, extending into infinity. Except for (3, 5), every twin prime can be written as 6n ± 1 for some n ≥ 1 values.

The last two residues for all prime generators > P2 are **modpn ± 1**, thus they have at least one twin pair set of residues. For larger prime generators there are more twin pair residues, and others. To illustrate this, we examine the PGS for increasing prime generators Pn.

For P3 we see its PGS contains the gaps 2 and 4, which occur one each, with the last ($\boldsymbol{r_0}$ - 1) = 4.

PGS P3: 5 7 11 13 17 19 23 25 29 31 35 . . . ∞
2 4 | 2 4 | 2 4 | 2 4 | 2 4 |

For P5 we see from Fig 1. its sequence of prime candidates, with its PGS spacing.

PGS P5: 7 11 13 17 19 23 29 31 37 41 43 47 49 53 59 61 67 . . . ∞
4 2 4 2 4 6 2 6 | 4 2 4 2 4 6 2 6 |

Again we see the gaps 2 and 4 occurring with the same (odd) frequency, with the last three gaps now having the form **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**, where $\boldsymbol{r_0}$ = 7 is the first residue for P5.

We are beginning to see some of the inherent properties of prime generators emerge. Each larger Pn (P7, P11, P13, P17, etc) will conform to these properties, producing an increasing number of gaps, with a defined number of specific gap sizes, systematically distributed within the sequence.

4 Characterizing PGS

Each prime generator sequence is defined over the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn**, therefore the number of gaps equals the number of residues, and the sum of the gap sizes equals the modulus. Let $\boldsymbol{a_i}$ be the frequency coefficients (number of occurrences) for each gap of size $2i$, $i \geq 1$, thus:

$$rescntpn = \sum a_i \tag{5}$$

$$modpn = \sum gap_i = \sum a_i \cdot 2i \tag{6}$$

Therefore for PGS P3: $[\boldsymbol{r_0} = 5]$ 2 4 | $\rightarrow$ modp3 $= 6 = (1) \cdot 2 + (1) \cdot 4$
and PGS P5: $[\boldsymbol{r_0} = 7]$ 4 2 4 2 4 6 2 6 | $\rightarrow$ modp5 $= 30 = (3) \cdot 2 + (3) \cdot 4 + (2) \cdot 6$

For P7, modp7 = modp5 · 7 = 210, and rescntp7 = rescntp5 · (7 - 1) = 48, with the residues:

{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 209, 211}

PGS P7: $[\boldsymbol{r_0} = 11]$ 2 4 2 4 6 2 6 4 2 4 6 6 2 6 4 2 6 4 6 8 4 2 4 2
4 8 6 4 6 2 4 6 2 6 6 4 2 4 6 2 6 4 2 4 2 10 2 10 |

With: modp7 $= 210 = (15) \cdot 2 + (15) \cdot 4 + (14) \cdot 6 + (2) \cdot 8 + (2) \cdot 10$

Again we see for P7, there are an equal odd number of occurrences for gaps 2 and 4. This illustrates a property of every prime generator with modulus of $p_n\#$, coefficients $a_1 = a_2$ have form:

$$a_1 = a_2 = (p_n - 2)\# = \prod (p_{odd} - 2) = (3 - 2) \cdot (5 - 2) \cdot (7 - 2) \cdot \ldots \cdot (p_n - 2) \tag{7}$$

We also see the consistent pattern that the last gap term is **($\boldsymbol{r_0}$ - 1)**, and starting with P5, the last three gaps have the pattern **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**. This occurs because the last two residues are always twin pairs of form **modpn ± 1**, and the second from last is the modular complement of $\boldsymbol{r_0}$, i.e. **(modpn - $\boldsymbol{r_0}$)**.

We now also notice that the number of unique gap sizes for each generator Pn are of order p_{n-1}. This is observed to be the minumum number of gaps for increasing Pn (for nonzero coefficients). Thus the PGS for P3 has two (2) gaps, for P5 three (3) gaps, for P7 five (5) gaps sizes, and so on.

5 PGS Symmetry and Distribution

Because the residues exist as ***modular complement pairs*** they produce a mirror image gap distribution around a midpoint pivot term. The PGS pattern up to the pivot will exist as its mirror image after.

Starting with P5, we know the last 3 gaps for all Pn have the form **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**, thus their sum is **2$\boldsymbol{r_0}$**, and the remaining odd number **(rescntpn - 3)** gaps must equal **(modpn - 2$\boldsymbol{r_0}$)**.

This requires for P5, the (8 - 3) = 5 gaps at the front of its PGS must sum to (30 - 2·7) = 16. If all the gaps were 2 you would need 8, which is too many, if all were 4 you need just 4, which is too few. The gap structure is numerically constrained to generate the unique combination of gap sizes to satisfy both requirements (5) and (6) that represent each Pn.

In addition, these (rescntpn - 3) odd gaps exist with a symmetric mirror image distribution around a mid pivot gap that is always of size 4 for $p_n\#$ moduli.

To show this, excluding the last 3 term of PGS P5 we have the gap sequence: 4 2 **4** 2 4
Here the terms 4 2 are the mirror image of 2 4 and are symmetric around midterm **4**.

For PGS P7 we get: 2 4 2 4 6 2 6 4 2 4 6 6 2 6 4 2 6 4 6 8 4 2 **4**
2 4 8 6 4 6 2 4 6 2 6 6 4 2 4 6 2 6 4 2 4 2

and again see a similar mirror image symmetry of each half around the midterm 4.

For P7, in order for the (48 - 3) = 45 gaps in its PGS front to sum to (210 - 2·11) = 188 we see new gaps of 8 are introduced (mirrored in both halves) close to the middle pivot point.

As the PG moduli increase, new larger gaps will emerge and be included toward the pivot element. This amounts to pushing the preexisting gaps toward the front and back. This ***expansion process*** ensures all preexisting residue gaps will eventually exist for the primes < $\boldsymbol{r_0}^{\mathbf{2}}$ for some Pn.

Each PGS shows $a_1 = a_2$ are odd because gap size 4 is the pivot term and a gap 2 is part of the last three sequence terms. (I provide the numerical basis for this in the Appendix.) Every other gap term is part of each mirror image and therefore occur in even numbers. Thus as similar to the residues, we only need to (computationally) determine the first (rescntpn - 4)/2 gap terms.

6 The Infinity of Primes

Starting with just the first two primes 2 and 3, we can show the ***infinite progression of primes***.

Using the first two primes we create: P3 = 6·k + {5, 7}, k ≥ 0.

From Fig 2. the pcs < $\boldsymbol{r_0}^{\mathbf{2}} = 5^2 = 25$ are prime, which are the values {5, 7, 11, 13, 17, 19, 23}.

We now use the new found primes 5. . . 23 to construct P23, with modp23 = 223092870, whose $\boldsymbol{r_0} = 29$. All the residues between 29 and $29^2 = 841$ will be primes. The primes counting function $\pi(x)$ tells us there are exactly 137 primes from 29...841, the last being 839. ***We now have a repeatable deterministic process to identify all the primes, into infinity***.

Thus, any prime p can be treated as $\boldsymbol{r_0}$ to a Pn modulus composed of all the primes < p, whose residues from p to p^2 are new primes. We can repeat this ***progression of primes*** process forever, to always generate new primes. Thus from this exact process, we can generate a list of consecutive primes for any Pn, from which we can then exactly determine their prime gaps distribution.

In fact, an estimate of the number of new primes generated in any range p to p^2 will be of order:

$$\pi_{est}(p, p^2) = \frac{p^2}{log(p^2)} - \frac{p}{log(p)} = \frac{p \cdot (p-2)}{2 \cdot log(p)} \tag{8}$$

For $p = 29$, this produces an estimate of 116 primes from 29 to 841, compared to the actual of 137. (See Appendix for fuller elaboration.)

7 Prime Generator Properties

Given what we've observed, and now know about prime generators and their sequences, we can codify their inherent and immutable properties, and use them in a logically consistent manner to empirically establish and project the nature, numbers, and distribution of all prime gap k-tuples.

Though mathematically simple expressions, prime generators reveal an astounding breadth of knowledge about the nature of prime numbers, embedded in their inherent immutable properties. When I refer to their properties as being 'inherent' these are natural aspects and characteristics of their structure that are discernible easily through visual observation. Once observed they could be mathematically described and characterized to formulate a consistent framework for application.

As an example, it is an inherent property of base ten numbers that the *least significant digit* (*lsd*) of an even integer must (only) be the digits, 0, 2, 4, 6, 8, and conversely 1, 3, 5, 7, 9 for odd. However when we change the base system, say to a binary (base two) system, even|odd has a different expression, i.e. the *least significant bit* (*lsb*) of an even number is a '0' and a '1' for odd. We performed no calculation to determine this, these are observable characteristics that are inherently associated with the concepts of even and odd for each base system.

Using these inherent properties of even|odd for base ten numbers, we can apply them through observation to 'prime' numbers. It is an inherent property of prime numbers that, other than for the prime 2, all others are odd, which means their *lsd* aren't 0, 2, 4, 6, or 8. So by mere observation you know 341786 isn't prime. You didn't *need* to perform a calculation to confirm this, if you understood this natural inherent property of prime numbers it's observably obvious.

Also, other than for the prime 5, all other primes *lsd* can only be 1, 3, 7, or 9. This means at minimum 60% of all integers (those with *lsd* of 0, 2, 4, 5, 6, and 8) can't be primes. This is an inherent property of numbers. If you know a little bit more number theory, you also know that while 11 and 101 could be primes (they are) 111, 1011, and 1101 observably could not. Why? Because for base ten numbers, if the sum of their digits is a multiple of 3 then it's divisible by 3, and thus not prime.

Thus it is an inherent property of Twin Primes their *lsd* can only be $\{1, 3\}$, $\{7, 9\}$, or $\{9, 1\}$ e.g. for (11, 13), (17, 19), and (29, 31). It's also inherent for all prime numbers > 2, the gaps between them are even because each is odd. You don't have to 'prove' this (though the proof is simple), it is an inherent property of odd numbers.

Thus, when I refer to the inherent properties of prime generators, these are observable characteristics and patterns that emerge naturally from their structure which I have mathematically codified. They are also immutable because they are the same for all generators constructed as shown, and can't change.

Constructing the Pn modulus as the primorial of primes p_n totally determines its structure, as the residues count is determined by the Euler Totient Function, their values by the *gcd* test, and the residue values determine their gap sizes, whose distribution is determined by the symmetric properties of their modular forms. There is nothing random in this process.

So while there is a clear deterministic numerical foundation for PGT, visualization of its elements reveal and explains it best. ***You have to draw pictures***, e.g. Fig 1. and generator sequences, and produce enough examples to visually reveal their patterns. ***You cannot imagine these properties into existence just from numerical analysis, you have to observe them first.***

Now that I have described and given examples of prime generators and their sequences, I will list their observable inherent properties, which I have codified into a mathematically consistent framework for application.

Major Properties of Prime Generators

- the modulus of every prime generator with last prime $\boldsymbol{p_n}$ has primorial form: **modpn** $= \boldsymbol{p_n}\#$
- the number of residues are even with form: **rescntpn** $=$ **(**$\boldsymbol{p_n}$ **- 1)**$\#$
- the residues occur as ***modular complement pairs*** to its modulus: **modpn** $= \boldsymbol{r_i} + \boldsymbol{r_j}$
- the last two residues of a generator are constructed as: **(modpn - 1) (modpn + 1)**
- the residues include all the ***coprime primes*** up to **modpn**
- the first residue $\boldsymbol{r_0}$ is the next prime $> \boldsymbol{p_n}$
- the residues from $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^2$ are primes
- each prime generator has a ***characteristic sequence*** of even sized residue gaps
- the last 3 sequence gaps have form: **(**$\boldsymbol{r_0}$ **- 1) 2 (**$\boldsymbol{r_0}$ **- 1)**
- the gaps are distributed in a ***symmetric mirror image*** around a pivot gap of size $\underline{\mathbf{4}}$
- the residue gaps sum from $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn** equals the modulus: **modpn** $= \boldsymbol{\Sigma a_i \cdot 2i}$
- the coefficients $\boldsymbol{a_i}$ are the ***frequency*** of each gap of size $\boldsymbol{2i}$
- the sum of the coefficients $\boldsymbol{a_i}$ equal the number of residues: **rescntpn** $= \boldsymbol{\Sigma a_i}$
- coefficients $\boldsymbol{a_1} = \boldsymbol{a_2}$ are odd and equal with form: $\boldsymbol{a_1} = \boldsymbol{a_2} =$ **(**$\boldsymbol{p_n}$ **- 2)**$\#$
- the coefficients $\boldsymbol{a_i}$ are even for $\boldsymbol{i} > \mathbf{2}$
- the minumum number of nonzero coefficients $\boldsymbol{a_i}$ in the sequence for **Pn** is of order $\boldsymbol{p_{n-1}}$

These inherent and immutable properties form a bounded set of constraints which characterize the formation and distribution of primes, and thus also the distribution of all their prime k-tuples.

These discrete mathematical properties and operations form a striking correlation to calculus, where for distance x(t) its first derivative is velocity = dx(t)/dt and its second derivative is acceleration = dv(t)/dt. For prime generators, distance is the number span covered by modpn, and its derivative are the number of residues|gaps. Taking the derivative of the number of gaps gives us the actual gap size coefficients.

Calculus	Prime Generators
x(t) = ∫ v(t)dt	modpn $= \Sigma a_i \cdot 2i = \prod p_i$
v(t) = ∫ a(t)dt	rescntpn $= \Sigma a_i = \prod (p_i$ - 1)
a(t) = ∫ A(t)dt	$a_1 = a_2 = \prod (p_i$ - 2)

While calculus integration is analogous to discrete summation, it is not intuitive that discrete summation correlates to ***primorial operators*** for prime generators. Or is it? Actually we see a similar relationship with the Riemann Zeta series and its equivalent Euler primes product form.

$$\sum \frac{1}{n^s} = \prod \left(1 - p^{-s}\right)^{-1} = \frac{\prod p^s}{\prod (p^s - 1)} \Rightarrow \frac{modp^s}{rescntp^s} \tag{9}$$

8 Proof of The Infinity of Twin Primes and other k-tuples

Theory of Proof

For every Pn with largest modulus primorial prime p_n, its residues contain the consecutive primes p_i from $\boldsymbol{r_0} \leq p_i \leq p_n\# + 1$, and their coprime composites, whose total is $(p_n - 1)\#$. In general, we don't know which residues are primes over the whole range. However, if we limit the range of interest to $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ we know those residues are consecutive primes (as $\boldsymbol{r_0} = p_{n+1}$ is the first prime $> p_n$, the residues from p_{n+1} to p_{n+1}^2 are the consecutive primes $> p_n$ and $< p_{n+1}^2$ coprime to $p_n\#$). Thus the gaps between these prime residues constitute the distribution of their prime pair k-tuples. Since we know the residue gap distribution over the whole range, we can estimate with high accuracy their distribution in this range. We find as the residue gaps increase in size and frequency as p_n increases, the prime gaps from p_{n+1} to p_{n+1}^2 similarly increase, for any gap size n as $p_n \to \infty$. Thus, for the infinity of residue gaps sizes n there are an infinity of (2, n) prime tuples.

Thus the ***simplest*** *and elementary proof* of the infinity of k-tuples establishes their endless progression in the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$, for as Pn increases: 1) the residue gaps coefficients a_i (for gap sizes $n = 2i$) increase for size and frequency, without end, and 2) as there are an infinity of $\boldsymbol{r_0} = p$ primes, and ranges p to p^2, they will contain an increasing number of prime pairs for any gap size n, without end, as $p_n \to \infty$.

We start by noting again for all Pn:

$$modpn = p_n\# = \sum a_i \cdot 2i \tag{10}$$

$$rescntpn = (p_n - 1)\# = \sum a_i \tag{11}$$

Proposition 1. *As Pn increases, residue gap coefficients a_i increase infinitely in size and frequency.*

Proof. From (11) as modpn increases by p_n the number of residues increase by $(p_n - 1)$, which equal the number of residue gaps. From (10) we also know the sum of occurrences for each gap size equals the modulus value. The smaller a_i gaps occur first, and in highest frequency, as a function of increasing p_n, while larger gap sizes a_k are functions of the smaller ones, and also systematically increase in frequency with p_n. Thus as Pn increases by p_n, the number of unique residues gap sizes and their frequency of occurrence increase, without end as $p_n \to \infty$.

Proposition 2. *As $p_n \to \infty$, within $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ the a_i gaps increase infinitely in size and frequency.*

Proof. Because the residues exist as modular complement pairs, they have a mirror image symmetry distribution. Smaller residue gaps generally occur with much higher frequency, and large gaps systematically lower, among their total, and sub ranges. As p_n increases, the residues become less dense and have more separation, and thus larger gaps, in higher frequencies, will be reflected within the primes $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$. As the range grows by p^2 the number of primes grows $\sim p^2/log(p^2)$ and contain proportionally more k-tuples, which increase without end as $p_n \to \infty$.

Fig 3. empirically shows the systematic increase in the size and frequency of the residue gaps for increasing Pn, required by (10) and (11). Fig 4. shows the slow initial, but then rapid, growth of the primes in $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$, while Fig 5. shows the steady growth of their k-tuples as p_n increases.

Because coefficients $a_1 = a_2$ have a clear deterministic expression for all Pn, we can formulate a good estimate for prime gaps 2 and 4 (Twins|Cousins) for all Pn. We can simply say it's the percentage of their gaps to its residue count times the number of primes from $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$, i.e. $\pi(p, p^2)$. For computational simplicity we can use $\pi_{est}(p, p^2) = p \cdot (p-2)/2 \cdot log(p)$, for a weaker estimate.

$$Twins|Cousins\ count \simeq \lceil (a_1/rescntpn) \cdot \pi(p, p^2) \rceil \tag{12}$$

If we substitute the expressions for $\boldsymbol{a_1}$, **rescntpn**, and $\boldsymbol{\pi_{est}(p, p^2)}$ we get:

$$Twins|Cousins\ count \simeq \left\lceil \frac{\prod(p_i - 2)}{\prod(p_i - 1)} \cdot \frac{p \cdot (p-2)}{2 \cdot log(p)} \right\rceil \tag{13}$$

To verify it works, let's first use the parameters for P7, with $\boldsymbol{r_0} = p = 11$, rescntp7 = 48, and a_1= 15. The actual primes count π(11,121) = 26, thus: Twins|Cousins count $\simeq \lceil 15 \cdot 26/48 \rceil = 9$. Using the weaker primes estimate of $\lceil$(11)(11 - 2) / 2$\cdot log$(11)$\rceil$, we get $\lceil$(15)(11)(9) / 96$\cdot log$(11)$\rceil$ = 7 Twins|Cousins primes. We see previously for P7 (and Fig 5.) the actual Twins|Cousins counts are 8|9 in the range 11 to 121, thus we get accurate estimates from both calculations.

To test for a larger range, let's use P97, whose $\boldsymbol{r_0} = p = 101$.

rescntp97 $= \prod(p_i - 1) = (2-1)\cdot(3-1)\cdot(5-1)\cdot \ldots \cdot(97-1) = 2773996904277378399530788061184000000$

$a_{1|2} = \prod(p_{odd} - 2) = (3-2)\cdot(5-2)\cdot(7-2)\cdot \ldots \cdot(97-2) = 4414821554294015162827496791260937 5$

$\pi(101, 101^2) = 1227$ $\qquad\qquad$ $\pi_{est}(101, 101^2) = \lceil (101)\cdot(99)/2 \cdot log(101) \rceil = \lceil 1083.3 \rceil = 1084$

Strong estimate: Twins|Cousins $\simeq \lceil (a_1/\text{rescntp97}) \cdot 1227 \rceil = \lceil 195.3 \rceil = 196$
Weaker estimate: Twins|Cousins $\simeq \lceil (a_1/\text{rescntp97}) \cdot 1084 \rceil = \lceil 172.5 \rceil = 173$

From Fig 5. we see the computed Twins|Cousins counts are 202|197 in the range 101 to 101^2.

To *establish with certainty* an infinity of Twins|Cousins, et al, it's only necessary to show at least one additional larger pair continually exists for some set of (not even all) Pn as $p_n \to \infty$. Here it's established there is an estimable increasing large number of pairs ***for every Pn*** as $p_n \to \infty$.

The computational forms for gap coefficients a_3 to a_{34} (see Appendix) have also been determined, and reveal the structured deterministic relationship between all gap sizes. Each larger gap size frequency is a function of all smaller gaps. Thus their values can also be calculated for all Pn, and estimated within the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for them. Once an a_i comes into existence it can not then vanish (go to 0), or even decrease, as that would violate the constraints on the PGS gap structure.

Thus we've *established with certainty*, prime gaps will always increase in frequency and size, precluding a last prime pair for any gap n as $p_n \to \infty$. Thus there are an infinity of all k-tuples.

9 Proof By Contradiction

To say there are ***not an infinity*** of all k-tuples (i.e. a finite number) means empirically for all a_i they become and remain zero (0) starting with some Pn. This mathematically requires the residues structure starting with this Pn to change in a mathematically permissible manner. Is this possible?

The structure of this proof is applicable for every gap coefficient a_i, but I need only demonstrate it for $a_1 = a_2$, as all other gaps are numerically related to them.

Let's imagine for some unknown Pn? with modulus $p_{n?}\#$, $a_1 = a_2$ reach some constant value, as p_n increases. Under this scenario we know there still would be an infinity of Twins|Cousins, because all there needs to be at minimum is one additional larger pair continually found for just some Pn (let alone every Pn) as $p_n \to \infty$.

Thus for there to be a finite number of Twins|Cousins, et al, we must have $a_1 = a_2 = 0$ starting with some Pn, and remaining so forever. But we know (see Appendix) that a_3 is a function of $a_{1|2}$, a_4 a function of a_3, etc, etc, thus it's mathematically impermissible for this scenario to occur. It's a mathematical absurdity for all the gap coefficients be zero, as there would be no residue gaps.

Thus we have a clear contradiction. In addition, $a_1 = a_2$ conform to a deterministic relationship solely based on the modulus primes, and rapidly increase as $p_n \to \infty$. Thus $a_1 = a_2$ are never zero, and in fact increase within the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for every Pn, precluding a last Twin|Cousin prime.

To require an existing a_i to permanently vanish creates a set of mathematically contradictory scenarios. For some Pn, its residues count would no longer be determined by the Euler Totient Function (so there are either more|fewer residues per modulus), and|or the residues are no longer modular complements (so their residues gaps distribution symmetry has changed). But the residue gaps cannot change without the residue values changing, which are the coprimes to modpn.

Every conceivable scenario to establish a finite number for any gap size requires mathematical contradictions or absurdities. In fact, it's easier to imagine by intuition alone there must be an infinity of k-tuples, than somehow mathematically envision and numerically establish their finality.

Thus, to have a finite number for any prime gap requires its a_i to become and remain zero, requiring a Pn's structure to change in multiple impossible ways, which will affect every other gap. As there can be no finite number for any residue gap then consequently so too for any prime gap.

10 Predictive Results

Ultimately, any proof must be able to explain known empirical results, and predict future ones. It's shown we can compute a good minimum estimate for Twins|Cousins (and others) for any Pn. We can also establish when any residue gap first appears in some Pn, and then determine when it appears within the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for some larger Pn.

For example, a_{50}, which denotes residues gaps of 100, first occurs for P59 (because its PGS has on order 53 coefficients). Fig 5. shows a prime gap size of 100 first occurs for $503 < p < 1009$. The exact value is $p = 631$; i.e. between 631 and 631^2 the first prime pair of gap size 100 occurs among those 33,599 primes. Thus, while in general gaps of 100 start occurring between residues with P59, it takes until P619 to establish with certainty the first *prime residue pair* of this size, a span of 98 prime generators. While this simple process may not seem rapid, it is mathematically certain.

The following list are the first prime pairs with the first multiple of 100 gaps sizes shown.

- first instance of prime gap of 100 is (396,733; 396,833)
- first instance of prime gap of 200 is (378,043,979; 378,044,179)
- first instance of prime gap of 300 is (4,758,958,741; 4,758,959,041)
- first instance of prime gap of 400 is (47,203,303,159; 47,203,303,559)
- first instance of prime gap of 500 is (303,371,455,241; 303,371,455,741)

(It should be noted, the gaps don't necessarily occur in linear order, as the first prime gap for 210, for the pair (20,831,323; 20,831,533), occurs well before the first prime pair gap 200.)

Because their are an infinity of primes p_n there are no theoretical upper bounds on this process. As the gap sizes increase their first, etc, prime residue pairs will become unimaginably large. But that's OK. We need not determine their actual values, but merely establish with certainty (with this simple process) that they exist, and that there are an infinity of them of any size.

11 Conclusion

The properties of Prime Generators allow for direct examination of the structure of the gaps between primes. They empirically show prime numbers, and their gaps, conform to a deterministic structure that determines their nature, numbers, and distribution. Residue gaps of any size n will first exist for some Pn, and occur in larger numbers for all larger generators. These residue gaps will ultimately appear and remain in the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$, becoming prime gaps for some Pn, and all larger. Thus, this simple process establishes the residue gaps only increase in size|frequency, and with ultimate certainty will appear as strictly prime gaps, whose k-tuples only increase without end as $p_n \to \infty$.

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Data

The following data was derived using Ruby|Crystal scripts to generate and count the residue gaps.

Listed here are all the residue gap coefficients a_i for the first few prime generators. We observe: the sum of the columns for each Pn equals its residues count; the sum of the products of each a_i by its gap size $2i$ equals modpn; and for each Pn there are on order p_{n-1} unique coefficients. Also for the Pn shown, the first instance for a_{prime} (a_3, a_5, a_7, etc) equal 2.

We also see the gaps frequency values oscillate up and down as they increase in size, with the smaller gaps numerically dominant in their frequency, and larger gaps initially occur with relatively much much lower frequency. This characteristic is a function of the computational forms of the a_i, where each larger gap has a defined numerical relationship with the preceding smaller gaps and p_n for its generator.

Residue gap coefficients a_i for all gaps $2i$ for given Pn											
Pn	2	3	5	7	11	13	17	19	23	29	31
$a_1 \cdot 2$	1	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025
$a_2 \cdot 4$		1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025
$a_3 \cdot 6$			2	14	142	1,690	26,630	470,630	10,169,950	280,323,050	8,278,462,850
$a_4 \cdot 8$				2	28	394	6,812	128,810	2,918,020	83,120,450	2,524,575,200
$a_5 \cdot 10$				2	30	438	7,734	148,530	3,401,790	97,648,950	2,985,436,650
$a_6 \cdot 12$					8	188	4,096	90,124	2,255,792	68,713,708	2,206,209,208
$a_7 \cdot 14$					2	58	1,406	33,206	871,318	27,403,082	903,350,042
$a_8 \cdot 16$						12	432	12,372	362,376	12,199,404	423,955,224
$a_9 \cdot 18$						8	376	12,424	396,872	14,123,368	512,670,088
$a_{10} \cdot 20$						0	24	1,440	61,560	2,594,160	106,604,280
$a_{11} \cdot 22$						2	78	2,622	88,614	3,324,402	126,682,650
$a_{12} \cdot 24$							20	1,136	48,868	2,100,872	88,337,252
$a_{13} \cdot 26$							2	142	7,682	386,554	18,298,102
$a_{14} \cdot 28$								72	5,664	324,792	16,461,600
$a_{15} \cdot 30$								20	2,164	154,220	9,169,532
$a_{16} \cdot 32$								0	72	10,128	833,688
$a_{17} \cdot 34$								2	198	15,942	1,075,458
$a_{18} \cdot 36$									56	7,228	620,632
$a_{19} \cdot 38$									2	570	77,042
$a_{20} \cdot 40$									12	1,464	128,988
$a_{21} \cdot 42$										272	40,636
$a_{22} \cdot 44$										12	3,516
$a_{23} \cdot 46$										2	1,794
$a_{24} \cdot 48$											1,296
$a_{25} \cdot 50$											504
$a_{26} \cdot 52$											20
$a_{27} \cdot 54$											84
$a_{28} \cdot 56$											12
$a_{29} \cdot 58$											2

Fig 3.

As new larger gaps appear within a PGS, it takes some time (i.e. some progression of generators) for them to appear within the range p to p^2 of larger Pn where they become strictly prime gaps. The number of these residues constitute a dwindling percentage of the residue count for larger Pn, as shown below. This affects the rate of progression of Pn necessary to identify the strictly primes gaps.

Pn	7	11	13	17	19	23	29
residues count	48	480	5,760	92,160	1,658,880	36,495,360	1,021,870,080
r_0 to ${r_0}^2$ count	26	34	55	65	91	137	152
% of total residues	54.2	7.08	0.955	0.071	0.055	0.000375	0.0000149

Fig 4.

Below shows the progression of gaps frequency within p to p^2 for gap sizes shown, and the max gap.

Frequency of prime gaps (not complete) between p and p^2									
p	11	53	101	503	1,009	5,003	10,007	50,021	100,003
max gap	8	34	36	86	114	210	220	320	354
gaps of 2	8	74	202	2,585	8,278	130,543	440,666	7,816,170	27,412,929
gaps of 4	9	78	197	2,575	8,239	130,201	440,606	7,816,884	27,410,258
gaps of 6	7	99	296	4,165	13,715	224,001	769,338	13,979,458	49,393,480
gaps of 8	1	37	103	1,692	5,643	96,432	334,491	6,221,667	22,161,302
gaps of 10		39	121	2,120	7,169	123,641	430,458	8,059,613	28,765,142
gaps of 12		27	107	2,267	8,134	151,420	530,008	10,420,167	37,589,303
gaps of 14		15	54	1,199	4,302	81,767	293,529	5,774,452	20,944,700
gaps of 16		6	33	795	2,929	59,224	216,032	4,347,314	15,888,865
gaps of 18		8	40	1,283	4,995	104,769	385,207	7,933,971	29,190,859
gaps of 20		2	15	601	2,433	53,704	203,194	4,366,505	16,296,757
gaps of 22		4	18	555	2,211	46,822	176,170	3,748,342	13,954,841
gaps of 24		2	15	604	2,278	66,815	257,882	5,701,980	21,488,356
gaps of 26		1	3	274	1,195	30,588	119,624	2,720,294	10,348,264
gaps of 28		0	6	271	1,261	32,971	129,739	2,963,462	11,288,578
gaps of 30		0	11	414	1,959	55,436	223,137	5,345,019	20,707,409
gaps of 32		0	1	97	558	16,563	68,384	1,695,929	6,641,679
gaps of 34		1	3	113	563	17,262	71,351	1,785,000	6,997,115
gaps of 36			1	149	779	27,127	114,180	2,927,973	11,593,976
gaps of 38				75	337	12,068	51,843	1,38,1811	5,518,125
gaps of 40				90	436	14,320	60,853	1,640,477	6,576,788
gaps of 42				83	486	19,568	86,754	2,438,771	9,920,126
gaps of 44				23	205	7,745	34,939	1,001,765	4,107,209
gaps of 46				24	158	6,514	29,372	866,337	3,580,246
gaps of 48				29	203	10,790	49,904	1,501,630	6,251,179
gaps of 50				16	110	5,803	27,544	857,165	3,607,941

Fig 5.

Here I use the data for $p = 101$ to visually show the oscillatory behavior of the gap sizes. We see from the data in Fig 5. this characteristic becomes more pronounced for larger p gap ranges. Larger ranges will have more local maxima|minima as they will generate more larger gaps. Each generator, thus, will have its own signature curve. We also see the local maxima|minima gap sizes exhibit an interesting characteristic: most of these a_i indices are primes, $i = 2, 3, 5, 11, 13, 17$, or are powers of 2 or 3, $i = 2, 4, 8, 9, 16, 27$. It will be interesting to see the pattern for much larger gap sizes for increasing Pn.

Prime gaps from p to p^2 for $p = 101$																		
gaps	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
freq	202	197	296	103	121	107	54	33	40	15	18	15	3	6	11	1	3	1

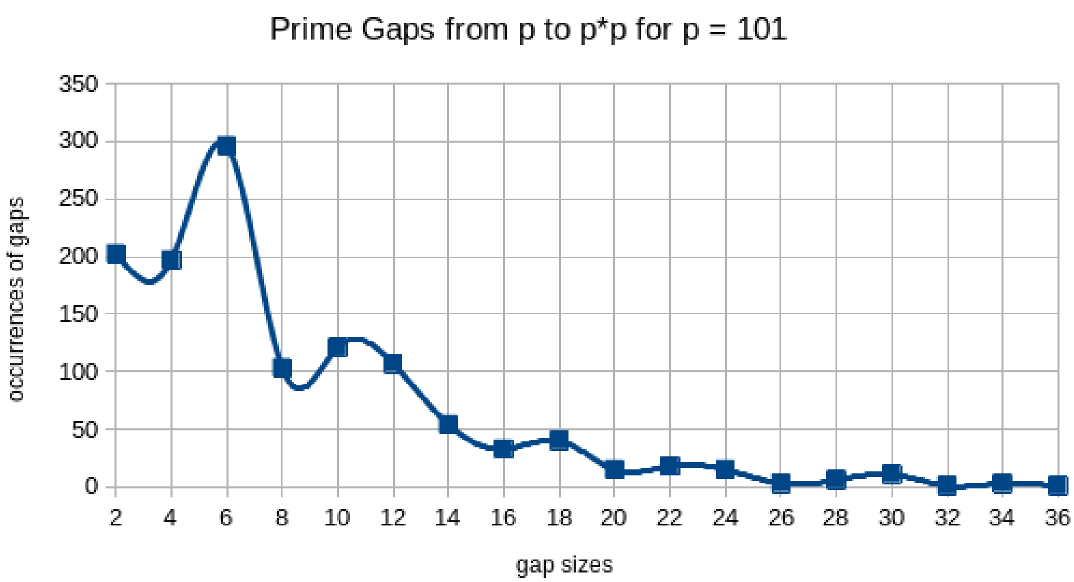

Fig 6.

We also clearly see the prominence of the smaller gaps and their expansion property. All the preexisting gaps are pushed toward the front for the first half mirrored gaps (as larger ones are included within the structure) and they will appear first, and in greater frequency than larger gaps, for each larger generator. But to be clear, we are observing the number of ***atomic gaps*** (between consecutive primes) not ***composite gaps*** (over multiple primes).

The data shows, as expected, the ratio of Twins|Cousins is near unity (1) as their residue gaps are the same (providing the modular form framework to explain the Hardy-Littlewood Conjectures). We also see there will always be more Sexy Primes than Twins|Cousins, or any other individual k-tuple for the ranges shown. But with $p = 503$ we start to see gaps of multiples of 6 become the dominate local maxima of the gaps curves. In fact, the 1999 paper *Jumping Champions*, by Odlyzko, Rubinstein, and Wolf [9], suggests as we increase the number range, the most frequent prime gaps increase from 6, to 30, to 210, etc, i.e. are primorial gap sizes 3#, 5#, 7#, etc.

Here I show in more detail the slow growth rate of max gap sizes for increasing ranges p to p^2.

Max prime gap sizes from p to p^2												
p	11	19	31	59	101	179	317	563	1,009	1,783	3,163	5,623
$log10(p)$	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0	3.25	3.5	3.75
max gap	8	14	20	34	36	72	72	86	114	148	154	210

p	10007	17783	31627	56237	100003	177823	316233	526337	1000003
$log10(p)$	4.0	4.25	4.5	4.75	5.0	5.25	5.5	5.75	6.0
max gap	220	248	282	320	354	456	464	486	540

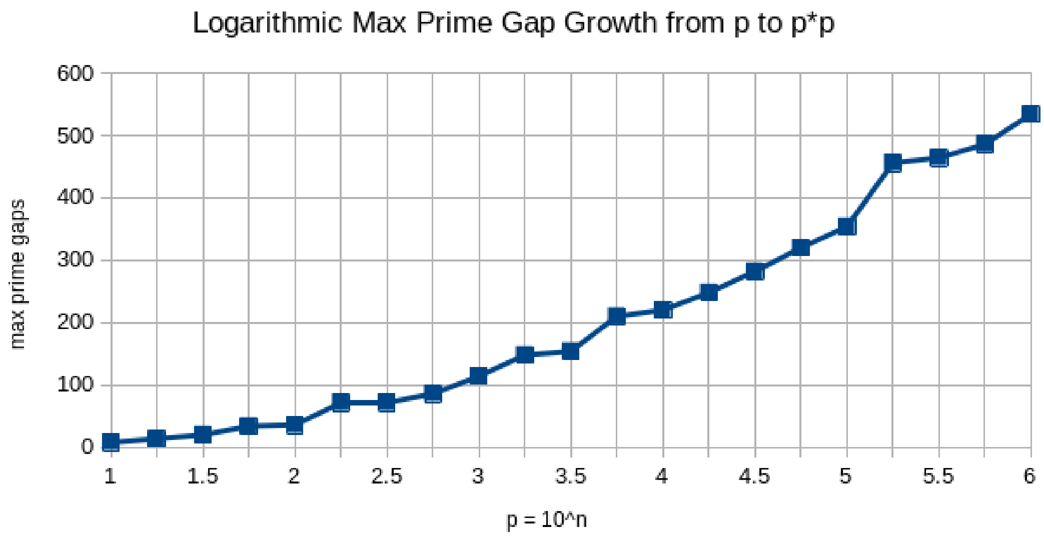

Fig 7.

This graph quantifies the slow expansion. As p increases orders of magnitude its PGS max gap grows much slower. For p of order 10^3 the max gap reaches 10^2, but only increases to $5 \cdot 10^2$ for p of order 10^6. We can create growth curves for all the other gap sizes to see their growth rate.

It should be noted again, though while this graph is technically accurate, it doesn't tell the whole story, as the gaps don't always occur in linear order. For example, the first prime gaps for 210, 220, 248, etc, occur for prime values much smaller than for the first prime pair with gap 200.

Also, primes gaps seem to occur in clusters. Primes with (relatively) small gaps seem to cluster in progression. As we journey higher into the number space we start to observe more and larger prime gaps (in fact an infinity of them), regions I call ***prime vacuums*** (or ***deserts***). The smaller gap clusters exist around the vacuums, which using classical numerical techniques makes searching for extremely large Twins, Merseene Primes, etc harder. You ideally want to be able to identify where the vacuums are and avoid them. We can use the residue gaps profiles for PGS to confine searches accordingly based on the goals. See [1].

Thus the data illustrate the distribution of primes is not random, but in fact deterministic, and conform to the described properties manifested within the structure of prime generators.

Appendix

Infinite Progression of Primes

From the Prime Number Theorem (PNT) (https://en.wikipedia.org/wiki/Prime_number_theorem) it has been proved the number of primes up to any value x is on order $x/log(x)$, or better $Li(x)$ (log integral x). I use equation (8) (for computational simplicity) $x/log(x)$ to estimate the number of primes between any random prime p (or really any value x) and p^2, per the PNT.

The Pn residues are the integers $p_n < r_i <$ modpn coprime to modpn. The Euler Totient Function (ETF) tells us their exact number. Thus it's clear, the $\{r_i\}$ must include all the primes, and their coprime multiples $<$ modpn, necessary to satisfy the ETF residues count.

Each Pn eliminates all its modulus primes multiples from consideration. Since the first residue $\boldsymbol{r_0}$ of every Pn is the next prime $> p_n$, its first multiple in its residue set (pcs table) is the multiple with itself, i.e. $\boldsymbol{r_0^2}$. Therefore, the residues between $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ can only be the consecutive primes in that interval, as they are not multiples (the only non-multiples) of the modulus primes $< \boldsymbol{r_0^2}$. And the PNT estimates their numbers are of order $p^2/log(p^2)$ - $p/log(p)$, or better $Li(p^2)$ - $Li(p)$.

However, for each specific generator Pn we can compute easier a simpler estimate. We know the number of modulus primes for any Pn, I'll note as $\pi(modpn)$. Thus the primes $< \boldsymbol{r_0^2}$, for $\boldsymbol{r_0} = p$ are: $p^2/log(p^2)$ - $\pi(modpn)$. For the previous example for P23, with $\boldsymbol{r_0} = 29$, a simpler calculation is then: $\lceil(841)/log(841) - 9\rceil = \lceil 115.87\rceil = 116$ as before. In fact, we can just use $p^2/log(p^2)$, here $\lceil 841/log(841)\rceil = \lceil 124.88\rceil = 125$, as $\pi(modpn)$ is relatively so much smaller as p^2 becomes larger.

Thus, since we know each generator Pn always generates the consecutive primes $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$, we can use these primes to construct a larger Pn, and keep repeating this process as many times as we want to generate as many consecutive primes groups we want, and thus can also then observe, record, and count, the exact gap structure of all the primes, into infinity.

The graph below shows the growth in the number of new primes in $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for each of the first 100 primorial P_n generators. We see it has the classic x^2 parabolic curve, as the number of primes will grow without end as more primes are used to construct larger primorial generators.

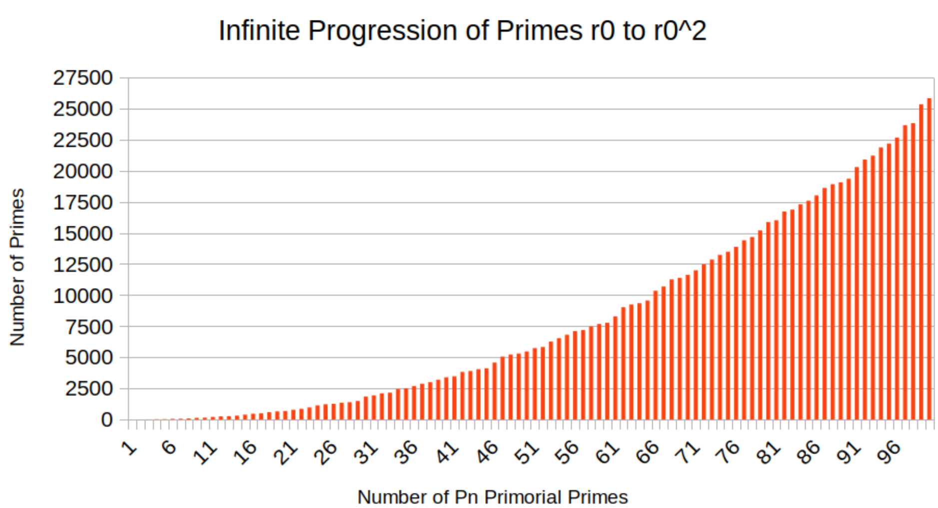

Fig 8.

Here we see the ratio of the number of new primes to primorial primes. It has a much more linear profile, as their growth appears fairly constant for the first 100 primorials. It'll be interesting to see if it approaches some asymptotic limit as the primorial primes increase by orders of magnitude.

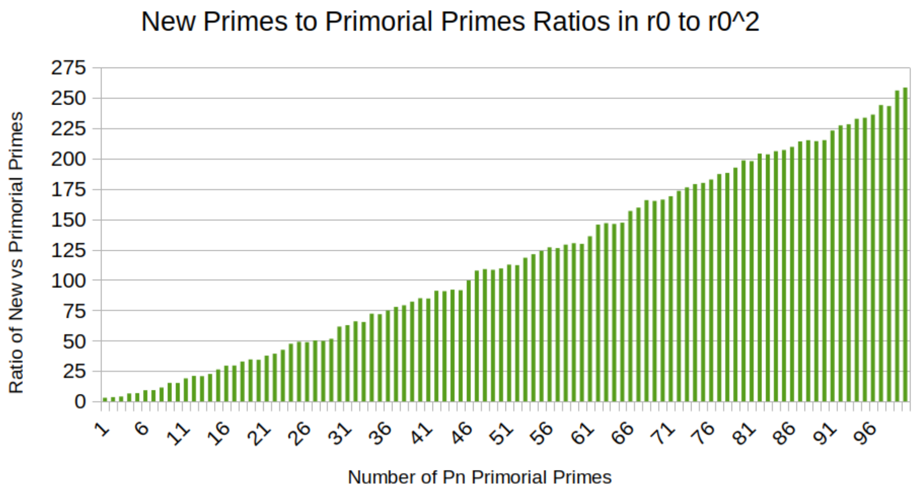

Fig 9.

Modular Complement Property

Using ***clock math***, we see residues exist as ***modular complement pairs***, and ***prime generator sequences*** have ***mirror image symmetry***, as a direct property of their ***modular forms***.

Any even n can be the modulus for a ***cyclic integer generator*** (a ring Zn) we can visualize as a clock of n hours. A 12 hour clock has a modulus of 12 with mod values $0 - 11$. We see if we draw horizontal lines between the hours left-to-right, their sums equals 12, and also see this if we fold the clock on its vertical axis. Moduli with multiple factors of 2 (as here) have even midpoint|bottom values, thus the bottom gap is 2. For primorial moduli, et al, with one factor of 2, the midpoint is odd, and the bottom (pivot) gap is 4 between the odd values on each side. The top gap is 2, so primorials have equal odd numbers of gaps of 2 and 4, while all others occur evenly on the clock.

When we form the prime generator P12, for mod12 we only use the residues coprime to 12, i.e. $\{1, 5, 7, 11\}$, where (1, 11) and (5, 7) are modular complement pairs. Eliminating the non-coprime values creates the P12 generator with these 4 residues, and its mirror image gap distribution. Any even $n > 2$ will have a ***modular form*** with these modular complement properties, for every Pn.

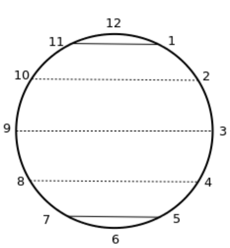

Fig 10.

Reduced Primorials

The principal, and reduced primorials of rank r, play a fundamental role in the construction of the a_i residue gap coefficients values. They have form: $(p_n - r)\# = p_n^{-r}\# = \prod_{p_i \geq r}^{n}(p_i - r)$,where for $p_i = r$, $(p_i - r)\# = 0\#$=1, similar to 0!= 1. Below is a table of the reduced primorials for the first 10 primorials.

Reduced Primorial Values $P_n^{-r}\#$										
n	1	2	3	4	5	6	7	8	9	10
Pn	2	3	5	7	11	13	17	19	23	29
r=0	2	6	30	210	2,310	30,030	510,510	9,699,690	223,092,870	6,469,693,230
r=1	1	2	8	48	480	5,760	92,160	1,658,880	36,495,360	1,021,870,080
r=2	1	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725
r=3		1	2	8	64	640	8,960	143,360	2,867,200	74,547,200
r=4			1	3	21	189	2,457	36,855	700,245	17,506,125
r=5			1	2	12	96	1,152	16,128	290,304	6,967,296
r=6				1	5	35	385	5,005	85,085	1,956,955
r=7				1	4	24	240	2,880	46,080	1,013,760
r=8					3	15	135	1,485	22,275	467,775
r=9					2	8	64	640	8,960	179,200
r=10					1	3	21	189	2,457	46,683
r=11					1	2	12	96	1,152	20,736
r=12						1	5	35	385	6,545
r=13						1	4	24	240	3,840
r=14							3	15	135	2,025
r=15							2	8	64	896
r=16							1	3	21	273
r=17							1	2	12	144
r=18								1	5	55
r=19								1	4	40
r=20									3	27
r=21									2	16
r=22									1	7
r=23									1	6
r=24										5
r=25										4
r=26										3
r=27										2
r=28										1
r=29										1

Fig 11.

Gap Coefficients

It was previously established: $a_1 = a_2 = (p_n - 2)\# = \prod(p_{odd} - 2) = (3-2)\cdot(5-2)\cdot(7-2)\cdot\ldots\cdot(p_n - 2)$. I have also determined the recursive forms for a_1 - a_7. For any generator Pn, with last modulus prime p_n, its gap coefficients a_i are a function of p_n and the preceding generator coefficients a'_i.

$a_1 = a'_1 \cdot (p_n - 2)$
$a_2 = a'_2 \cdot (p_n - 2)$
$a_3 = a'_3 \cdot (p_n - 3) + a'_2 + a'_1$
$a_4 = a'_4 \cdot (p_n - 4) + a'_3$
$a_5 = a'_5 \cdot (p_n - 5) + a'_4 \cdot 2 + a'_3$
$a_6 = a'_6 \cdot (p_n - 5) + a'_5 \cdot 6 - a'_4 \cdot 2$
$a_7 = a'_7 \cdot (p_n - 7) + a'_6 \cdot 3 - a'_5 \cdot 3 + a'_4 \cdot 4$

The P37 gap coefficients distribution has now also been directly generated, and is shown below.

a_1 = 217,929,355,875	a_2 = 217,929,355,875	a_3 = 293,920,842,950	a_4 = 91,589,444,450
a_5 = 108,861,586,050	a_6 = 83,462,164,156	a_7 = 34,861,119,734	a_8 = 16,996,070,868
a_9 = 21,218,333,416	a_{10} = 4,814,320,320	a_{11} = 5,454,179,550	a_{12} = 4,073,954,144
a_{13} = 918,069,454	a_{14} = 857,901,000	a_{15} = 535,673,924	a_{16} = 58,664,256
a_{17} = 69,404,898	a_{18} = 46,346,428	a_{19} = 7,381,190	a_{20} = 10,176,048
a_{21} = 4,153,336	a_{22} = 526,596	a_{23} = 291,342	a_{24} = 239,760
a_{25} = 91,392	a_{26} = 8,912	a_{27} = 25,320	a_{28} = 2,952
a_{29} = 1,654	a_{30} = 452	a_{31} = 26	a_{32} = 48
a_{33} = 24			

We can now calculate the gap estimates within the range p to p^2 for a_1 - a_7. Comparing data from Fig 5. let's calculate the estimates for a_1 - a_7 for p to p^2 for $\boldsymbol{r_0} = p = 53$. This means we have to find all those coefficients values up to P47. Below are their calculated values starting from P37.

Calculated residue gap coefficients a_i for gaps $2i$ for given Pn			
p_n	41	43	47
a_1	8,499,244,879,125	348,469,040,044,125	15,681,106,801,985,625
a_2	8,499,244,879,125	348,469,040,044,125	15,681,106,801,985,625
a_3	11,604,850,743,850	481,192,519,512,250	21,869,408,938,627,250
a_4	3,682,730,287,600	155,231,331,960,250	7,156,139,793,803,000
a_5	4,396,116,829,650	186,022,750,845,750	8,604,610,718,954,250
a_6	3,474,628,537,016	151,047,124,809,308	7,149,653,083,144,936
a_7	1,475,437,583,074	65,082,209,263,162	3,119,286,820,258,154

Fig 12.

We can now use P47's calculated a_i to find their range estimates: $gaps_i \simeq \lceil a_i{\cdot}\pi(p, p^2)/rescntp47\rceil$

For $p = 53$, $\pi(p, p^2) = 394$, and rescntp47 $= \prod_{p2}^{p47}(p_n - 1) = 85287729364992000$, gives values:

gaps 53 to 53^2	$a_1 \vert a_2$	a_3	a_4	a_5	a_6	a_7
actual	74|78	99	37	39	27	15
estimated	73	102	34	40	34	15

Fig 13.

There's also an algebraic way to generate the a_i values without recursion, using reduced primorials. The table below shows the first 20 $c_{i,j}$ rational coefficients, which when multiplied by the respective reduced primorial $P_n^{-r}\#$ column values, and summed across each row, will compute the a_i resdue gap values for any P_n generator.

$a_i(n) = c_{i,j}P_n^{-r}\#$										
$P_n^{-r}\#$	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10	r=11
c_1	1									
c_2	1									
c_3	2	-2								
c_4	1	-2	1							
c_5	$\frac{4}{3}$	-3	2							
c_6	2	-7	10	-2						
c_7	$\frac{6}{5}$	-5	$\frac{28}{3}$	-3						
c_8	1	-5	12	-6	1					
c_9	2	$\frac{-23}{2}$	$\frac{100}{3}$	-22	6					
c_{10}	$\frac{4}{3}$	$\frac{-39}{4}$	$\frac{116}{3}$	-40	24	-2				
c_{11}	$\frac{10}{9}$	$\frac{-63}{8}$	$\frac{632}{21}$	$\frac{-175}{6}$	$\frac{72}{5}$					
c_{12}	2	-17	$\frac{1738}{21}$	$\frac{-344}{3}$	108	-21				
c_{13}	$\frac{12}{11}$	$\frac{-209}{20}$	$\frac{11090}{189}$	$\frac{-1563}{16}$	$\frac{4224}{35}$	$\frac{-119}{3}$	$\frac{28}{5}$			
c_{14}	$\frac{6}{5}$	$\frac{-185}{16}$	$\frac{456}{7}$	$\frac{-325}{3}$	$\frac{662}{5}$	-42	$\frac{16}{3}$			
c_{15}	$\frac{8}{3}$	$\frac{-1207}{40}$	$\frac{12986}{63}$	$\frac{-5221}{12}$	$\frac{25376}{35}$	$\frac{-4285}{12}$	$\frac{482}{5}$	$\frac{-45}{4}$		
c_{16}	1	$\frac{-127}{10}$	$\frac{18833}{189}$	$\frac{-2999}{12}$	$\frac{3651}{7}$	$\frac{-2107}{6}$	$\frac{2287}{15}$	$\frac{-83}{2}$	$\frac{20}{3}$	
c_{17}	$\frac{16}{15}$	$\frac{-765}{56}$	$\frac{1406}{13}$	$\frac{-6565}{24}$	$\frac{220944}{385}$	$\frac{-15411}{40}$	$\frac{4424}{27}$	$\frac{-675}{16}$	$\frac{128}{21}$	
c_{18}	2	$\frac{-15543}{560}$	$\frac{592204}{2457}$	$\frac{-391649}{576}$	$\frac{125544}{77}$	$\frac{-155981}{120}$	$\frac{94618}{135}$	$\frac{-4151}{16}$	$\frac{1516}{21}$	$\frac{-14}{3}$
c_{19}	$\frac{18}{17}$	$\frac{-2499}{160}$	$\frac{1777744}{12285}$	$\frac{-48925}{112}$	$\frac{801716}{715}$	$\frac{-684359}{720}$	$\frac{262864}{495}$	$\frac{-60093}{320}$	$\frac{340}{9}$	
c_{20}	$\frac{4}{3}$	$\frac{-1139}{56}$	$\frac{7658}{39}$	$\frac{-89845}{144}$	$\frac{658414}{385}$	$\frac{-94969}{60}$	$\frac{135476}{135}$	$\frac{-1745}{4}$	$\frac{2848}{21}$	-8

We directly compute each $a_i(n)$ value as: $a_i(n) = \sum_{j=1}^{k} c_{i,j} P_n^{-(1+j)}\#$, for their k row values.
Thus as before: $a_1(n) = a_2(n) = c_{1|2,1}P_n^{-2}\# = (1)(p_n - 2)\#$, are the 2nd reduced primorials.
For a computaionally longer example, let's compute a_8 (residue gaps of 16) for P23, i.e.P_9.

To compute it for any P_n we have:

$$a_8(n) = c_{8,1}P_n^{-2}\# + c_{8,2}P_n^{-3}\# + c_{8,3}P_n^{-4}\# + c_{8,4}P_n^{-5}\# + c_{8,5}P_n^{-6}\#$$

For P23, $p_9 = 23$ (for the 9th primorial) and we use the $P_9^{-r}\#$ reduced primorial table values.

$$\begin{aligned} a_8(9) &= (1)P_9^{-2}\# + (-5)P_9^{-3}\# + (12)P_9^{-4}\# + (-6)P_9^{-5}\# + (1)P_9^{-6}\# \\ &= (7952175) - 5(2867200) + 12(700245) - 6(290304) + (85085) \\ &= 362376 \end{aligned}$$

We see this matches the value in Fig 3., which were obtained by brute force computation.
At time of writing, the values up to c_{80} have been determined, to compute $a_1 - a_{80}$ for any P_n.
However, a presentation of their derivation is beyond the scope of this paper.

Numerical Gap Derivations

The a_i coefficients can be numerically determined by the constrained system of equations for Pn:

$$modpn = p_n\# = \sum a_i \cdot 2i = 2 \cdot a_1 + 4 \cdot a_2 + 6 \cdot a_3 + ... + 2n \cdot a_n \tag{14}$$

$$rescntpn = (p_n - 1)\# = \sum a_i = a_1 + a_2 + a_3 + ... + a_n \tag{15}$$

As $p_n\#$ is an even value c_1 and $(p_n - 1)\#$ an even value c_2 we can reduce the equations to:

$$\begin{aligned} c_1/2 &= a_1 + 2 \cdot a_2 + 3 \cdot a_3 + ... + n \cdot a_n \\ c_2 &= a_1 + a_2 + a_3 + ... + a_n \end{aligned} \tag{16}$$

Oddness of a_1 and a_2

For P2 we only need to use:

$$2\# = 2 = 2a_1 \tag{17}$$

This numerically establishes $a_1 = 1$ for P2 as the single (odd) value for gap size 2 for its PGS.

For P3 we have $c_1 = 3\# = 6$ and $c_2 = (2 - 1) \cdot (3 - 1) = 2$, and we are constrained to only having the two nonzero coefficients a_1 and a_2, which gives:

$$\begin{aligned} 3 &= a_1 + 2a_2 \\ 2 &= a_1 + a_2 \end{aligned} \tag{18}$$

The only solution is $a_1 = a_2 = 1$, matching the known odd occurrences for gaps 2 and 4 for P3.

For P5 we have $c_1 = 5\# = 30$ and $c_2 = 8$, and are constrained to only having the nonzero coefficients a_1, a_2, and a_3 which gives:

$$\begin{aligned} 15 &= a_1 + 2a_2 + 3a_3 \\ 8 &= a_1 + a_2 + a_3 \end{aligned} \tag{19}$$

We now create the system of equations: 2R2 - R1 and 3R2 - R1,

$$\begin{aligned} 1 &= a_1 - a_3 \\ 9 &= 2a_1 + a_2 \end{aligned} \tag{20}$$

which after rearranging gives:

$$\begin{aligned} a_3 &= a_1 - 1 \\ a_2 &= 9 - 2a_1 \end{aligned} \tag{21}$$

We solve this by picking the value for a_1 that produces a_2 and a_3 that satisfy equations (19). Notice a_2 is odd for any value of a_1, and because we know a_3 is even a_1 must be odd, and constrained to 1 or 3 (5 makes a_2 negative). Only $a_1 = 3$ works, producing $a_2 = 3$ and $a_3 = 2$, the known PGS values for P5. Again we see, now purely through numerical methods, that $a_1 = a_2$ and numerically required to be odd, which matches the computational form for these Pn: $a_{1|2} = a'_{1|2} \cdot (p_n - 2)$.

Let's continue for P7, with $c_1 = 7\# = 210$ and $c_2 = \prod_{p2}^{p7}(p_n - 1) = 48$.

$$105 = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 \tag{22}$$
$$48 = a_1 + a_2 + a_3 + a_4 + a_5$$

Now do R1 - R2, to eliminate a_1, and R1 - 2R2, to eliminate a_2, and after rearranging gives:

$$a_2 = 57 - 2a_3 - 3a_4 - 4a_5 \tag{23}$$
$$a_1 = -9 + a_3 + 2a_4 + 3a_5$$

Again, a_1 and a_2 are odd as $a_{3|4|5}$ are even (due to their mirror symmetry). This problem is solvable using linear programming algorithms e.g. the Simplex Method. It can be characterized using their prime generators properties to produce the a_i values for P7, i.e. $a_1 = a_2 = 15$, $a_3 = 14$, $a_4 = 2$, and $a_5 = 2$.[1] For all larger Pn, $a_1|a_2$ will have similar forms as (23) with more a_i terms.

Solving for larger a_i

However, we really want a system of equations where the larger gap coefficients are functions of the smaller ones, to reflect the order of their relational structure we see empirically expressed in their computational forms. Thus, because we know $a_1 = a_2$ we can transform (22) to:

$$(105 - 3a_1) = c_3 = 3a_3 + 4a_4 + 5a_5 \tag{24}$$
$$(48 - 2a_1) = c_4 = a_3 + a_4 + a_5$$

We now create a new system, solving for a_5, and performing 5R2 $-$ R1 and solving for a_4:

$$a_5 = c_4 - a_4 - a_3 \tag{25}$$
$$a_4 = 5c_4 - c_3 - 2a_3$$

We can now pick a_3 to determine a_4, and then a_5, which gives us all the a_i. For P7, $a_1 = a_2 = 15$ gives $c_3 = 60$ and $c_4 = 18$ creates:

$$a_5 = 18 - a_4 - a_3 \tag{26}$$
$$a_4 = 30 - 2a_3$$

Because $a_{3|4|5}$ are > 0 and even, requires $2 \leq a_{3|even} \leq 14$, the only solution is, again, $a_3 = 14$, $a_4 = 2$, and $a_5 = 2$.

Creating the equations in this order provides for computation of the lower values for larger gaps. As the gaps become larger we'll see more of the oscillating nature of their values as functions of smaller gaps, as shown in Fig 6. Thus we illustrate again using numerical methods, the properties of prime generators determine the unique solution to the system of constraints for each Pn, which show the gap coefficients a_i will only increase in frequency value for all gap sizes, as the Pn moduli $p_n\#$ increase as $p_n \to \infty$.

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[1] Using Simplex Calculator at http://cbom.atozmath.com/CBOM/Simplex.aspx?q=is with following constraints, produces known a_i values: MIN Z = $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$ subject to: $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 105$; $x_1 + x_2 + x_3 + x_4 + x_5 = 48$; $x1 <= 15$; $x_2 <= 15$; $x_3 <= 17$; $x_4 <= 13$; $x_5 <= 10$; $x_1 >= 3$; $x_2 >= 3$; $x_3 >= 2$; $x_4 >= 2$; $x_5 >= 2$; and $x_1, x_2, x_3, x_4, x_5 >= 0$.

Closing Thoughts

Since the 2013 release of Yitang Zhang's paper[2] that for some integer N $<$ 70 million there are infinitely many pairs of primes that differ by N, there has been a fury of activity to reduce its bound to a smallest gap size. Included now is the quest to solve problems regarding questions of small and large gaps.[3] The work presented here proposes to *establish with certainty* there are an infinity of prime pairs that differ by any gap size, large and small.

Using strictly numerical approaches will likely continue to be fruitless to definitively answer questions about prime gaps. If you want to understand and characterize the nature of prime gaps the most direct (and easiest) approach is to strictly work within the domain of prime gaps. Prime Generator Theory (PGT) provides the theoretical, philosophical, and numerical framework to do this, which current analytical and numerical methods alone are not equipped to do.

At the beginning of the 20th Century, Relativity Theory was imagined by Einstein to provide both a qualitative and quantitative framework to better understand and explain how nature works. Initially it was resisted, but ultimately was (had to be) embraced because it worked. It could quantitatively answer questions about the known behavior of nature other theories couldn't, and accurately predict and explain previously uncontemplated behavior. And continual experimental testing has reaffirmed its validity (for the reality we are aware of), over and over.

Here at the start of the 21st Century, I believe PGT shares a similar role in the field of math. It provides a better framework to qualitatively and quantitatively understand, characterize, explain, and predict the behavior of primes. Resistance has run mostly along the lines of questioning language, the meaning of terminology, being too simplistic, the perceived lack of rigor, etc. These are complaints more about its qualitative nature, and|or epistemological basis for knowing, than a refutation of its theoretical foundations or its empirical results and predictions.

The content herein is a major revision of the earlier versions, to present its findings in a clearer and more "mathematician friendly" format, and to present new information and findings. I would ask whatever it may seem to lack in traditional mathematical rigor not be a deterrent from recognition of its mathematically sound theoretical under girding. Judge it on the merits of the evidence of its findings and results, which I contend overwhelmingly *establish with certainty* it claims.

Undoubtedly the work presented here touches just the surface of a body of knowledge begging to be explored and revealed. Hopefully the curious will take up the challenge to do just that, and share their findings, and apply them to the myriad of known problems waiting to be solved, while contemplating and proposing new ones heretofore unimagined.

[2]**Bounded gaps between primes**; https://annals.math.princeton.edu/2014/179-3/p07
[3]Small and Large Gaps Between the Primes; https://www.youtube.com/watch?reload=9&v=pp06oGD4m00&t=425