AN IMPROVED LOWER BOUND OF HEILBRONN'S TRIANGLE PROBLEM

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ABSTRACT. Using the method of compression we improve on the current lower bound of Heilbronn's triangle problem. In particular, by letting $\Delta(s)$ denotes the minimal area of the triangle induced by s points in a unit disc. Then we have the lower bound

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

1. Introduction

Let $\mathcal{D}$ denotes any convex shape in the plane and $\Delta(S)$ denotes the minimal area of the triangle induced by a set of s points in $\mathcal{D}$ so that $\Delta(s)$ denotes the supremum of all the $\Delta(S)$. Then Heilbronn conjectured what is now known as Heilbronn's triangle problem, which states

Conjecture 1.1. The minimal area of the triangle induced by s points in $\mathcal{D}$ satisfies

$$\Delta(s) = O\bigg(\frac{1}{s^2}\bigg).$$

Indeed Erdős had shown earlier to the effect of Heilbronn's conjecture the lower bound

$$\Delta(s) \gg \frac{1}{s^2}.$$

This lower bound would have vindicated Heilbronn's conjectured upper bound as the sharpest if it had been proven to be true. Heilbronn's triangle problem had long remained open and it was indeed a breakthrough in 1982 when the first chunk of this problem was solved by Komlos, Pintz and Szemeredi [1]. In particular, they constructed a set of points in $\mathcal{D}$ whose minimal area of their induced triangles, denoted $\Delta(s)$, satisfies the lower bound (see [1])

$$\Delta(s) \gg \frac{\log s}{s^2}.$$

What remains apparently open now is the asymptotic growth rate of the minimal area of the triangle determined by a finite set of points in $\mathcal{D}$. To that effect the quest for improved lower and upper bounds are of worthy pursuit. The first non-trivial upper bound was obtained by Roth [4] of the form

$$\Delta(s) \ll \frac{1}{s\sqrt{\log\log s}}.$$

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A slight refinement of a method in [3] eventually yields the best currently known upper bound (see [2])

$$\Delta(s) \ll \frac{e^{c\sqrt{\log s}}}{s^{\frac{8}{7}}}.$$

Using a completely new idea which is very fundamental, we obtain an improved lower bound for the minimal area of the triangle induced by s points in a unit disc, by considering a particular type of configuration:

Theorem 1.1. *Let $\Delta(s)$ denotes the minimal area of the triangle formed by s points in the unit disc. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

2. Preliminaries and background

Definition 2.1. By the compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$, we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of rescaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin. Intuitively, compression induces some kind of motion on points in the Euclidean space $\mathbb{R}^n$ for $n \geq 2$.

Proposition 2.1. *A compression of scale $1 \geq m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. The mass of compression.

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\text{Inf}(x_j) = \text{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\text{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\text{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \neq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.4. *We have*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $1 \geq m > 0$.

Proposition 2.2. *Let* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *with* $x_i \neq 0$ *for each* $1 \leq i \leq n$ *and* $x_i \neq x_j$ *for* $i \neq j$, *then we have*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m\log\left(1+\frac{n-1}{\text{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\leq m\sum_{k=0}^{n-1}\frac{1}{\text{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

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Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

3. The ball induced by compression

In this section we introduce the notion of the ball induced by a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of a given scale. We launch more formally the following language.

Definition 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and $x_i \neq 0$ for all $1 \leq i \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of scale $1 \geq m > 0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality.

Remark 3.2. Next we prove that smaller balls induced by points should essentially be covered by the bigger balls in which they are embedded. We state and prove this statement in the following result.

In the geometry of balls induced under compression of scale $m > 0$, we assume implicitly that

$$0 < m \leq 1.$$

For simplicity we will on occasion choose to write the ball induced by the point $\vec{x} = (x_1, x_2, \ldots, x_n)$ under compression as

$$\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}[\vec{x}].$$

We adopt this notation to save enough work space in many circumstances. We first prove a preparatory result in the following sequel. We find the following estimates for the compression gap useful.

Proposition 3.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, if $m = m(n) = o(1)$ as $n \longrightarrow \infty$, then we have the estimate

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

for $\vec{x} \in \mathbb{R}^n$ with $x_i \geq 1$ for each $1 \leq i \leq n$.

Proposition 3.1 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

with $m := m(n) = o(1)$ as $n \longrightarrow \infty$ if and only if $||\vec{x}|| \lesssim ||\vec{y}||$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $x_i \geq 1$ for all $1 \leq i \leq n$. This important transference principle will be mostly put to use in obtaining our results. In particular, we note that in the latter case, we can write the asymptotic

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \sim \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] = ||\vec{x}||^2.$$

Lemma 3.3 (Compression estimate). *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_i \geq 1$ for all $1 \leq i \leq n$ with $x_i \neq x_j$ $(i \neq j)$. If $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\sup(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\sup(x_j^2)}\right)^{-1} - 2mn.$$

Theorem 3.4. *Let $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$ with $z_i \geq 1$ for all $1 \leq i \leq n$ and $m := m(n) = o(1)$ as $n \longrightarrow \infty$. Then $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}|| < ||\vec{y}||$ if and only if*

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

with $||\vec{y} - \vec{z}|| < \epsilon$ for some $\epsilon > 0$.

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$ and $z_i \geq 1$ for all $1 \leq i \leq n$ such that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}|| \lesssim ||\vec{z}||$, which is absurd. In this case, we can take $\epsilon := \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}]$. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 3.1 that $||\vec{z}|| \lesssim ||\vec{y}||$. Under the requirement $||\vec{y} - \vec{z}|| < \epsilon$ for some $\epsilon > 0$, we obtain the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| + \epsilon \\ &= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] + \epsilon\end{aligned}$$

with $m = m(n) = o(1)$ as $n \longrightarrow \infty$. By choosing $\epsilon > 0$ sufficiently small, we deduce that $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. $\square$

In the geometry of balls under compression, we will assume that n is sufficiently large for $\mathbb{R}^n$. In this regime, we will always take the scale of compression $m := m(n) = o(1)$ as $n \longrightarrow \infty$.

Theorem 3.5. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$ with $x_i\geq 1$ for each $1\leq i\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ for $||\vec{y}-\vec{x}||<\delta$ for $\delta>0$ sufficiently small, then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for $m:=m(n)=o(1)$ as $n\longrightarrow\infty$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ for $||\vec{y}-\vec{x}||<\delta$, then it follows from Theorem 3.4 that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$ with $||\vec{y}-\vec{x}||<\delta$ for $\delta>0$ sufficiently small. Suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}||<||\vec{y}||$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{z}-\vec{y}||<\epsilon$ for $\epsilon>0$ sufficiently small. It is not very difficult to see that this point does exist. Notice that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\mathbb{V}_m[\vec{y}]]}[\mathbb{V}_m[\vec{y}]]$$

so that under the regime where the two balls overlap then either $\vec{y}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ or $\mathbb{V}_m[\vec{y}]\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ since these points are symmetric to the center of ball. However in the latter case, we choose the point $\vec{z}$ such that to $||\mathbb{V}_m[\vec{y}]||<||\vec{z}||$. We can assume without loss of generality that $\vec{y}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ so that we choose the point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}||<||\vec{y}||$ such that $||\vec{z}-\vec{y}||<\epsilon$ for $\epsilon>0$ sufficiently small, then $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 3.4 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$$

with $||\vec{z}-\vec{x}||<\epsilon+\delta$. It follows from Theorem 3.4 that $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ since ϵ,δ are taken to be sufficiently small. This is inconsistent with $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The case where the balls do not overlap is easier and can be treated in the same manner. This completes the proof. ☐

Remark 3.6. Theorem 3.5 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

3.1. **Interior points and the limit points of balls induced under compression.** In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

Definition 3.7. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then a point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ is an interior point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for most $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$. An interior point $\vec{z}$ is then said to be a limit point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for all $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$

Remark 3.8. Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

Theorem 3.9. *Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ with $y_i \geq 1$ for all $1 \leq i \leq n$. Then the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains an interior point and a limit point.*

Proof. Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ with $x_i \geq 1$ for all $1 \leq i \leq n$ and suppose on the contrary that $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains no limit point. Then pick

$$\vec{z}_1 \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

with $||\vec{z}_1|| < ||\vec{x}||$ such that $||\vec{z}_1 - \vec{x}|| < \epsilon$ for $\epsilon > 0$ sufficiently small. Then by Theorem 3.5 and Theorem 3.4, it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1] \lesssim \mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Again pick $\vec{z}_2 \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$ with $||\vec{z}_2|| < ||\vec{z}_1||$ such that $||\vec{z}_2 - \vec{z}_1|| < \delta$ for $\delta > 0$ sufficiently small. Then by employing Theorem 3.5 and Theorem 3.4, we have

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]}[\vec{z}_2] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2] \lesssim \mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]$. By continuing the argument in this manner we obtain the infinite descending sequence of the gap of compression

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}] \gtrsim \mathcal{G}\circ\mathbb{V}_m[\vec{z}_1] \gtrsim \mathcal{G}\circ\mathbb{V}_m[\vec{z}_2] \gtrsim \cdots \gtrsim \mathcal{G}\circ\mathbb{V}_m[\vec{z}_n] \gtrsim \cdots$$

thereby ending the proof of the theorem. □

Proposition 3.2. *The point $\vec{x} = (x_1, x_2, \ldots, x_n)$ with $x_i = 1$ for each $1 \leq i \leq n$ is the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$ for any $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i > 1$ for each $1 \leq i \leq n$.*

Proof. Applying the compression $\mathbb{V}_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ on the point $\vec{x} = (x_1, x_2, \ldots, x_n)$ with $x_i = 1$ for each $1 \leq i \leq n$, we obtain $\mathbb{V}_1[\vec{x}] = (1, 1, \ldots, 1)$ so that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}] = 0$ and the corresponding ball induced under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ contains only the point $\vec{x}$. It follows by Definition 3.9 the point $\vec{x}$ must be the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$. It follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

for any $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i > 1$ for all $1 \leq i \leq n$. For if the contrary

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

holds for some $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i > 1$ for each $1 \leq i \leq n$, then there must exists some point $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$. Since $\vec{x}$ is the only point in the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$, it follows that

$$\vec{x} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

which is inconsistent with the fact that $\vec{x}$ is the limit point of the ball. □

3.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 3.10. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 3.11. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 3.12. *Let $\vec{x}\in\mathbb{R}^n$ with $x_i\neq x_j$ $(i\neq j)$ such that $x_i\geq 1$ for all $1\leq i\leq n$ and set $m:=m(n)=o(1)$ as $n\longrightarrow\infty$. The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ such that $||\vec{y}-\vec{x}||<\epsilon$ for $\epsilon>0$ sufficiently small is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ such that $||\vec{y}-\vec{x}||<\epsilon$ for $\epsilon>0$ sufficiently small be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Without loss of generality, we can choose some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{z}||<||\vec{x}||$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

for $||\vec{z}-\vec{x}||<\delta$ for $\delta>0$ sufficiently small. Applying Theorem 3.4, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

This already contradicts the equality $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. The latter equality of compression gaps follows from the requirement that the balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the two indistinguishable balls and so must satisfy the equality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&=\left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\&=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. $\square$

Next we obtain an equivalent notion of the area of the circle induced by points under compression in the plane $\mathbb{R}^2$ in the following result.

Proposition 3.3. *Let $\vec{x} \in \mathbb{R}^2$ with $x_i \neq 0$ for each $1 \leq i \leq 2$. Then the area of the circle induced by point $\vec{x}$ under compression of scale m, denote by $\mathbb{V}_m[\vec{x}]$ is given by*

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

Proof. This follows from the mere definition of the area of a circle and noting that the radius r of the circle induced by the point $\vec{x} \in \mathbb{R}^2$ under compression is given by

$$r = \frac{\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}{2}.$$

□

4. The lower bound

Theorem 4.1. *Let $\Delta(s)$ denotes the minimal area of the triangle formed by s points in the unit disc. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

Proof. First let $s \geq 4$ and let $1 \geq m := m(s) > 0$ be fixed. Pick arbitrarily a point $(x_1, x_2) = \vec{x} \in \mathbb{R}^2$ with $x_j > 1$ for $1 \leq j \leq 2$ so that $x_1 \neq x_2$ and set $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < 1$. This ensures the circle induced under compression is contained in some unit disc. Next we apply the compression of scale $1 \geq m > 0$, given by $\mathbb{V}_m[\vec{x}]$ and construct the circle induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])}{2}$. On this circle locate $(s-3)$ admissible points so that the chord joining each pair of adjacent $(s-1)$ admissible points including $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$ are equidistant. Let us now join each of the $(s-1)$ admissible point considered to the center of the circle given by

$$\vec{y} := \frac{1}{2}\bigg(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}\bigg).$$

Invoking Proposition 3.3, the area of the circle induced under compression is given by

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

We join all pairs of adjacent admissible points considered by a chord and produce $(s-1)$ triangles of equal area. We note that we can use the area of each sector formed from this construction to approximate the area of each of the triangles inscribed in the sector as we increase the number of such admissible points on the circle. It follows that the area of each sector formed must be the same and given by

$$\mathcal{A} := \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4 \times (s-1)}$$
$$\gg \frac{2\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{1}{\sup(x_j^2)}\right)^{-1} - 4m}{4 \times s}.$$

The lower bound follows by taking

$$m := \frac{\log^2 s}{4s} < 1 \quad \text{and} \quad \mathrm{Inf}(x_j) := 1 + \frac{\log s}{\sqrt{s}}$$

since points $\vec{x} = (x_1, x_2)$ can only have a compression gap $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < 1$ if $x_1 = 1 + \delta$ and $x_2 = 1 + \epsilon$ for any small $\delta, \epsilon > 0$. □

Albeit Heilbronn's triangle problem is a max $-$ min problem, the area of each triangle espoused in the construction is the same, to which the underlying condition has little relevance in this particular framework.

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