The information volume of uncertain information: (2) Fuzzy membership function

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Abstract

In fuzzy set theory, the fuzzy membership function describes the membership degree of certain elements in the universe of discourse. Besides, Deng entropy is a important tool to measure the uncertainty of an uncertain set, and it has been wildly applied in many fields. In this paper, firstly, we propose a method to measure the uncertainty of a fuzzy MF based on Deng entropy. Next, we define the information volume of the fuzzy MF. By continuously separating the BPA of the element whose cardinal is larger than 1 until convergence, the information volume of the fuzzy sets can be calculated. When the hesitancy degree of a fuzzy MF is 0, information volume of the fuzzy membership function is identical to the Shannon entropy. In addition, several examples and figures are expound to illustrated the proposed method and definition.

Keywords: information volume, fuzzy sets, membership function, Shannon entropy, Deng entropy.

1. Introduction

In the past decades, plenty of theories have been developed for expressing and dealing with the uncertainty in the uncertain environment, for instance,

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probability theory [1], fuzzy set theory [2], Dempster-Shafer evidence theory [3, 4], and D numbers [5].

Fuzzy set theory has been widely applied in many fields, like uncertainty measurements [6] and data fusion[7]. However, there are still some issues to be solved. Among them, how to measure the uncertainty in fuzzy set theory has attracted much attention.

Since firstly derived from thermodynamics, different kinds of entropy have been proposed, such as Shannon entropy [8], Tsallis entropy [9], and nonadditive entropy [10]. Entropy is one of the methods for measuring uncertainty, which can be extended to measure the uncertainty degree in fuzzy set. For example, Kai Yao and Hua Ke propose a entropy operator for membership function[11].

Recently, a new entropy, called Deng entropy [12], is presented for measuring the uncertainty in evidence theory. Deng entropy is the generalization of Shannon entropy. Compared with traditional methods, Deng entropy is more reasonable, and it takes both discord and non-specificity into account. Because of these efficiency, Deng entropy has various applications, such as data fusion [13] and decision making [14]. Moreover, based on the maximum entropy principle, the maximum form of Deng entropy, named as the maximum Deng entropy [15], is proposed, whose properties are analyzed in [16].

In the first paper of this series [17], the information volume of mass function has been defined. In addition, the mass function which can yield the maximum Deng entropy is called the Deng distribution.

This is the second paper of this series. In this paper, firstly, we propose a method to measure the uncertainty of a fuzzy membership function by two steps. Step 1 is to convert the MF into the associated BPA, and step 2 is to calculate the uncertainty of the associated BPA based on Deng entropy. And then, based on the information volume defined in [17], we define the information volume of the fuzzy membership function through continuously separating the associated BPA of the element whose cardinal is larger than 1 until the Deng entropy converges. The value of the information volume of the fuzzy sets can be calculated.

To summarize, the contributions of this paper are as follows:

(1) A method to measure the uncertainty of a fuzzy membership function is present.

(2) The definition for the information volume of the fuzzy membership function is proposed.

(3) When the hesitancy degree of a fuzzy MF equals to zero, the information volume of fuzzy MF is identical to the Shannon entropy.

(4) Several examples and figures are expound to illustrated the proposed method and definition.

The rest of this paper is organized as follows. In section **2**, some preliminaries are briefly reviewed. In section **3**, based on Deng entropy, a method for measuring the uncertainty of a fuzzy membership function is proposed. In section **4**, we define the information volume of the fuzzy membership function. In section **5**, numerical examples are expounded to illustrated the proposed method and definition. In section **6**, we have a brief conclusion.

2. Preliminaries

Several preliminaries are briefly introduced in this section, including basic probability assignment, Deng entropy, the maximum Deng entropy, fuzzy sets and intuitionistic fuzzy sets.

2.1. Basic probability assignment

Dempster-Shafer evidence theory[3, 4] can be used to deal with uncertainty. Besides, evidence theory satisfies the weaker conditions than the probability theory, which provides it with the ability to express uncertain information directly. Some basic conceptions of evidence theory are given as follows:

Definition 2.1: *Frame of discernment and its power set*

Let Θ, called the frame of discernment, denote an exhaustive nonempty set of hypotheses, where the elements are mutually exclusive. Let the set Θ have N

elements, which can be expressed as:

$$\Theta = \{\theta_1, \theta_2, \theta_3, \cdots, \theta_N\} \tag{1}$$

The power set of Θ*, denoted as* 2^Θ*, contains all possible subsets of* Θ *and has* 2^N *elements, and* 2^Θ *is represented by*

$$\begin{aligned} 2^\Theta &= \{A_1, A_2, A_3, \cdots, A_{2^N}\} \\ &= \{\ \emptyset, \{\theta_1\}, \{\theta_2\}, \cdots, \{\theta_N\}, \{\theta_1, \theta_2\}, \\ &\qquad \{\theta_1, \theta_3\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \Theta\ \} \end{aligned} \tag{2}$$

where the element A_i *is called the focal element of* Θ*, if* A_i *is nonempty.*

Definition 2.2: *Basic probability assignment (BPA)*

A BPA is a mass function mapping m *from* 2^Θ *to* $[0, 1]$*, and it is defined as follows:*

$$m : 2^\Theta \rightarrow [0, 1] \tag{3}$$

which is constrained by the following conditions:

$$\sum_{A \in 2^\Theta} m(A) = 1 \tag{4}$$

$$m(\emptyset) = 0 \tag{5}$$

2.2. Deng entropy

In information theory, entropy can be used to measure the uncertainty of a system. Recently, a novel entropy, named as Deng entropy [12], is proposed to measure the uncertainty in evidence theory.

Definition 2.5: *Deng entropy*

Deng entropy is defined as:

$$H_{DE}(m) = -\sum_{A \in 2^{\Theta}} m(A) \log(\frac{m(A)}{2^{|A|} - 1}) \tag{6}$$

where $|A|$ is the cardinal of a certain focal element A.

Deng entropy is the generalization of Shannon entropy. When every focal element is singleton, Deng entropy degenerates into Shannon entropy.

Through a simple transformation, Eq.(6) can be rewritten as follows:

$$H_{DE}(m) = \sum_{A \in 2^{\Theta}} \log(2^{|A|} - 1) - \sum_{A \in 2^{\Theta}} m(A) \log m(A) \tag{7}$$

where $\sum_{A \in 2^{\Theta}} \log(2^{|A|} - 1)$ and $-\sum_{A \in 2^{\Theta}} m(A) \log m(A)$ are measurements of nonspecificity and discord, respectively. As a result, Deng entropy is a composite measurement of nonspecificity and discord, which means that it is a tool for measuring total uncertainty.

2.3. The maximum Deng entropy

Assume A is the focal element of a certain frame of discernment Θ and $m(A)$ is the BPA for A. According to [15], the analytic solution of the maximum Deng entropy and the conditions of BPA distribution is as follows:

Theorem 2.1: *The analytic solution of the maximum Deng Entropy and its BPA distribution*

If and only if $m(A) = \frac{(2^{|A|}-1)}{\sum_{A \in 2^{\Theta}} (2^{|A|}-1)}$, Deng entropy reaches its maximum value, and the analytic solution of the maximum Deng entropy is

$$H_{MDE}(m) = \log \sum_{A \in 2^{\Theta}} (2^{|A|} - 1) \tag{8}$$

2.4. Fuzzy sets

Let E be a universe of discourse. A fuzzy set A based on E can be characterized by the set of pairs which is defined as [2]:

Definition 2.6: *Fuzzy sets*

$$A = \{\langle x, \mu_A(x)\rangle \,|x \in E\} \tag{9}$$

where $\mu_A : E \to [0, 1]$ *is the membership function (MF) of A.* $\mu_A(x)$ *describes the membership degree of each element* x *to the fuzzy set A. The closer* $\mu_A(x)$ *is to* 1*, the more likely* x *belongs to* A*.*

2.5. Intuitionistic fuzzy sets

Given a universe of discourse E, a intuitionistic fuzzy set (IFS) A is defined as follows [18]:

Definition 2.7: *Intuitionistic fuzzy sets (IFS)*

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \,|x \in E\} \tag{10}$$

where $\mu_A : E \to [0, 1]$ *and* $\gamma_A : E \to [0, 1]$ *are the membership function (MF) and the non-membership function (non-MF) of A, respectively.* $\mu_A(x)$ *describes the membership degree of* x *to the set A, and* $\gamma_A(x)$ *describes the non-membership degree of* x *to the set A. For every* $x \in E$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$*. The hesitancy degree is defined as:*

$$\eta_A(x) = 1 - \mu_A(x) - \gamma_A(x) \tag{11}$$

which represents the hesitancy degree of each element $x \in E$*.*

3. The uncertainty of a fuzzy membership function

In the past decades, plenty of theories have been developed for expressing and dealing with the uncertainty in the uncertain environment, for instance, probability theory, fuzzy set theory , Dempster-Shafer evidence theory, and so on. However, there are still some issues to be solved. Among them, how to measure the uncertainty of fuzzy environment has attracted much attention.

In this section, firstly, we present a method to measure the uncertainty of MF. Then, some numerical examples and discussions are expounded for better understanding of the proposed conceptions.

3.1. Proposed method for measuring the uncertainty of a fuzzy MF

In the course of information processing in fuzzy environment, it is important to identify the uncertainty of the data. Taking intuitionistic fuzzy sets (IFS) for example, the uncertainty of MF $\langle x_1, 0.9, 0.1\rangle$ is larger than that of $\langle x_2, 0.4, 0.4\rangle$. As a result, in this subsection, an uncertainty measurement for fuzzy membership function is proposed. The concrete steps are listed as follows:

step 1: Input a fuzzy membership function. Then, convert the fuzzy membership function into BPA distributions. Let μ_A be MF, γ_A be non-MF, η_A be hesitancy degree and x be the element of universe E. Let $U = \{\theta_1, \theta_2\}$ be the frame of discernment while $X = \{\theta_1\}$ and $Y = \{\theta_2\}$ be singletons.

(1) For the MF of classical fuzzy sets, the associated BPA is that:

$$\begin{cases} m(X) = \mu_A\left(x\right), \\ m(U) = 1 - \mu_A\left(x\right). \end{cases} \tag{12}$$

(2) For the MF of intuitionistic fuzzy sets, the associated BPA is that:

$$\begin{cases} m(X) = \mu_A\left(x\right), \\ m(Y) = \gamma_A\left(x\right), \\ m(U) = \eta_A\left(x\right) = 1 - \mu_A\left(x\right) - \gamma_A\left(x\right). \end{cases} \tag{13}$$

It should be noted that, the associated BPA of classical fuzzy MF is a special case of that of intuitionistic fuzzy MF. Namely, when $\gamma_A\left(x\right) = 0$, the associated BPA of intuitionistic fuzzy MF degenerates into that of classical fuzzy MF.

step 2: Calculate the uncertainty of the BPA based on Deng entropy [12]. Then, output the uncertainty of the fuzzy MF $H_{MF}(m)$.

(1) The uncertainty of the BPA associated with classical fuzzy MF can be calculated as:

$$\begin{aligned} H_{MF}(m) &= -m(X)\log\frac{m(X)}{2^1-1} - m(U)\log\frac{m(U)}{2^2-1} \\ &= -m(X)\log m(X) - m(U)\log\frac{m(U)}{3} \end{aligned} \tag{14}$$

(2) The uncertainty of the BPA associated with intuitionistic fuzzy MF can be calculated as:

$$\begin{aligned} H_{MF}(m) &= -m(X)\log\frac{m(X)}{2^1-1} - m(Y)\log\frac{m(Y)}{2^1-1} - m(U)\log\frac{m(U)}{2^2-1} \\ &= -m(X)\log m(X) - m(Y)\log m(Y) - m(U)\log\frac{m(U)}{3} \end{aligned} \tag{15}$$

3.2. Numerical examples and discussions

In this subsection, some numerical examples are shown to illustrate above conceptions. In the following examples, let the universe discourse be E, the frame of discernment be $U = \{\theta_1, \theta_2\}$, $X = \{\theta_1\}$ and $Y = \{\theta_2\}$ be singletons. The base of the logarithmic function is 2.

Example 3.1: Consider that a classical fuzzy MF is that $\langle x_1, 0.9\rangle$ where $x_1 \in E$. The procedure of calculating the uncertainty of this MF is as follows:

step 1: Firstly, convert this MF into BPA distribution:

$$\begin{cases} m(X) = 0.9, \\ m(U) = 1 - 0.9 = 0.1. \end{cases} \tag{16}$$

step 2: Then, use Deng entropy to calculate the uncertainty of the associated BPA distribution.

$$H_{MF}(m) = -0.9\log_2(0.9) - 0.1\log_2(\frac{0.1}{3}) = 0.627492 \tag{17}$$

As a result, based on Deng entropy, the uncertainty of this MF is 0.627492.

Next, let us consider another example.

Example 3.2: Let a intuitionistic fuzzy MF be $\langle x_2, 0.6, 0.2\rangle$ where $x_2 \in E$. The procedure of calculating the uncertainty of this MF is that:

step 1: Firstly, this MF can be converted into BPA distribution:

$$\begin{cases} m(X) = 0.6, \\ m(Y) = 0.2, \\ m(U) = 1 - 0.6 - 0.2 = 0.2. \end{cases} \tag{18}$$

step 2: Then, based on Deng entropy, calculate the uncertainty of the associated BPA distribution.

$$H_{MF}(m) = -0.6\log_2(0.6) - 0.2\log_2(0.2) - 0.2\log_2(\frac{0.2}{3}) = 1.68794 \tag{19}$$

Hence, the uncertainty of this MF is 1.68794. This result is larger than the result of **Example 3.1**, which means that $\langle x_2, 0.6, 0.2\rangle$ is more uncertain than $\langle x_1, 0.9\rangle$.

Example 3.3: Let a intuitionistic fuzzy MF be $\langle x_3, 0.2, 0.2\rangle$ where $x_3 \in E$. The procedure of calculating the uncertainty of this MF is as follows:

step 1: Firstly, convert this MF into BPA:

$$\begin{cases} m(X) = 0.2, \\ m(Y) = 0.2, \\ m(U) = 1 - 0.2 - 0.2 = 0.6. \end{cases} \tag{20}$$

step 2: Then, using Deng entropy, we can calculate the uncertainty of the associated BPA.

$$H_{MF}(m) = -0.2\log_2(0.2) - 0.2\log_2(0.2) - 0.6\log_2(\frac{0.6}{3}) = 2.32193 \tag{21}$$

So, the uncertainty of this MF is 2.32193. Actually, this BPA is the BPA distribution of the maximum Deng entropy [15] when the cardinal of the frame of discernment is 2, which is also called Deng distribution [17]. Besides, 2.32193 is the associated maximum value of Deng entropy, named as the maximum Deng entropy [15]. This example shows that $\langle x_3, 0.2, 0.2\rangle$ is the most uncertain case of intuitionistic fuzzy MF based on Deng entropy.

4. The information volume of the fuzzy membership function

4.1. Introduction of this section

In **Example 3.3**, we discuss about the most uncertain case of fuzzy MF and the value of the maximum Deng entropy. In information theory, the information volume of probability is the maximum Shannon entropy. When the base of the logarithmic function is 2, can we simply consider 2.32193, the value of the maximum Deng entropy, as the information volume of a fuzzy MF? The answer is NO. Consider following example:

Example 4.1: Let a intuitionistic fuzzy MF be $\left\langle x_4, \frac{1}{5}, \frac{1}{5}\right\rangle$ where $x_4 \in E$. Based on the steps in section **3.1**, the uncertainty of this MF is 2.32193. However, if we change the calculating steps as follows, the result will be larger than 2.32193.

step 1: Firstly, convert this MF into BPA:

$$\begin{cases} m(X_0) = \frac{1}{5}, \\ m(Y_0) = \frac{1}{5}, \\ m(U_0) = 1 - \frac{1}{5} - \frac{1}{5} = \frac{3}{5}. \end{cases} \tag{22}$$

step 2: Next, focus on the element whose cardinal is larger than 1, and separate its BPA based on this proportion: $\frac{1}{5} : \frac{1}{5} : \frac{3}{5}$. Namely, we focus on $m(U_0)$, and separate this BPA based on $\frac{1}{5}:\frac{1}{5}:\frac{3}{5}$. The result is that:

$$\begin{cases} m(X_0) = \frac{1}{5}, \\ m(Y_0) = \frac{1}{5}, \\ m(X_1) = \frac{1}{5} \times \frac{3}{5} = \frac{3}{25}, \\ m(Y_1) = \frac{1}{5} \times \frac{3}{5} = \frac{3}{25}, \\ m(U_1) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}. \end{cases} \tag{23}$$

where $m(X_1)$, $m(Y_1)$ and $m(U_1)$ can be seen as derivatives of $m(U_0)$. For better understanding, the calculating procedure of step 2 is illustrated in Figure 1.

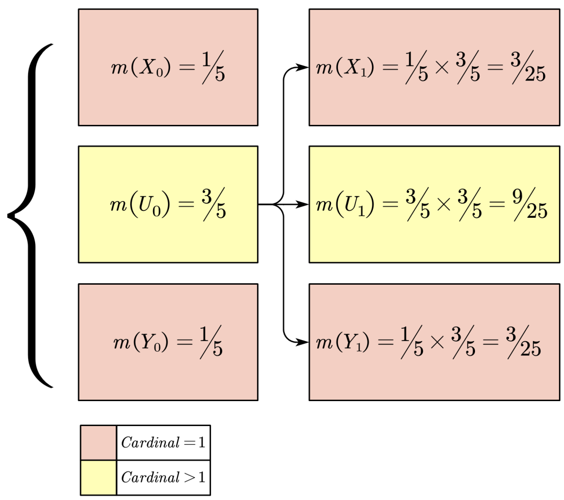

Figure 1: The procedure of step 2

step 3: Finally, based on Deng entropy, calculate the uncertainty of the new

BPA distribution.

$$\begin{aligned} H_{MF}(m) =& -\frac{1}{5}\log_2(\frac{1}{5}) - \frac{1}{5}\log_2(\frac{1}{5}) - \frac{3}{25}\log_2(\frac{3}{25}) \\ & -\frac{3}{25}\log_2(\frac{3}{25}) - \frac{9}{25}\log_2(\frac{\frac{9}{25}}{3}) \\ =& 2.76411 \end{aligned} \tag{24}$$

Based on this calculating steps, the uncertainty of fuzzy MF is 2.76411, which is larger than 2.32193. This example shows that, the uncertainty of fuzzy MF increases when the BPA is separated.

In the first paper of this series [17], the information volume of mass function is defined based on continuously separating the mass function. Inspired by the idea of continuous separation, we can also continuously divide the associated BPA of fuzzy MF until $H_{MF}(m)$ converges to a certain value, which is actually the information volume of the fuzzy membership function.

Hence, in the rest of this section, firstly, we define the information volume of the fuzzy membership function. Then, one example is shown for better understanding of the definition, and some brief discussions are followed after the example.

4.2. The definition of the information volume of the fuzzy membership function

Definition 2.8: *The information volume of the fuzzy membership function*

Let the universe discourse be E, the frame of discernment be $U = \{\theta_1, \theta_2\}$, $X = \{\theta_1\}$ and $Y = \{\theta_2\}$ be singletons. Let the fuzzy MF be $\langle x, \mu_A, \gamma_A\rangle$ where $x \in E$. If the fuzzy MF is based on classical fuzzy sets, y equals to 0. Use index i to denote the times of the loop. The information volume of a fuzzy membership function can be calculated by following steps:

step 1: Input the fuzzy MF of the fuzzy sets. Then, convert this fuzzy MF into associated BPA:

$$
\begin{cases}
m(X_0) = \mu_A, \\
m(Y_0) = \gamma_A, \\
m(U_0) = \eta_A = 1 - \mu_A - \gamma_A.
\end{cases}
\tag{25}
$$

where $i = 0$ means the initial time of the loop.

step 2: Next, continuously separate the associated BPA of the element whose cardinal is larger than 1 until convergence. Concretely, repeat the loop from step 2-1 to step 2-3 until Deng entropy is convergent.

step 2-1: Focus on the element whose cardinal is larger than 1. Since U_i is the only element that $|U_i| > 1$, we only focus on U_i. And then, separate the mass function of U_i based on the proportion of Deng distribution [17]:

$$
m_D(A_i) = \frac{(2^{|A_i|} - 1)}{\sum_{A_i \in 2^{U_i}} (2^{|A_i|} - 1)}
\tag{26}
$$

where A_i can be X_i, Y_i or U_i.

Since $U = \{\theta_1, \theta_2\}$, the proportion is that $\frac{1}{5} : \frac{1}{5} : \frac{3}{5}$. The ith times of separation divide $m(U_{i-1})$ and yield following new BPAs: $m(X_i)$, $m(Y_i)$, $m(U_i)$ which are derived from $m(U_{i-1})$. In addition, they satisfy these equations:

$$
m(X_i) + m(Y_i) + m(U_i) = m(U_{i-1})
\tag{27}
$$

$$
m(X_i) : m(Y_i) : m(U_i) = \frac{1}{5} : \frac{1}{5} : \frac{3}{5}
\tag{28}
$$

step 2-2: Based on Deng entropy, calculate the uncertainty of the new BPA distribution. The result is denoted as $H_{MFi}(m)$.

step 2-3: Calculate $\Delta_i = H_{MFi}(m) - H_{MFi-1}(m)$. When Δ_i satisfies following condition, jump out of this loop.

$$
\Delta_i = H_{MFi}(m) - H_{MFi-1}(m) < \varepsilon
\tag{29}
$$

where ε is the allowable error.

step 3: Output $H_{IV-MF}(m) = H_{MFi}(m)$, which is the information volume of fuzzy membership function.

4.3. The maximum information volume of fuzzy membership function

Theorem 4.1: *The maximum information volume of fuzzy membership function*

If and only if the associated BPA of fuzzy MF is Deng distribution [17], the information volume of fuzzy MF achieve its maximum value, which is called the maximum information volume of fuzzy membership function $H_{MIV-MF}(m)$.

4.4. Numerical examples and discussions

For better understanding of the proposed definition, some examples are expounded. In the following examples, the base of the logarithmic function is 2, and the allowable error is 0.001.

Example 4.2: Let the universe discourse be E, the frame of discernment be $U = \{\theta_1, \theta_2\}$. The base of the logarithmic function of Deng entropy is 2. Let a intuitionistic fuzzy MF be $A = \{\langle x_5, \frac{1}{2}, \frac{1}{2}\rangle | x_5 \in E\}$.

Firstly, convert this fuzzy MF into associated BPA: $m(X_0) = m(Y_0) = \frac{1}{2}$, $m(U_0) = 0$.

Because there is no focal element whose cardinal is larger than 1, the step 2-1 can be skipped for all the times of the loop. Then, in step 2-2, use Deng entropy to calculate the uncertainty of this mass function:

$$H_{MFi}(m) = -\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{2}\log_2(\frac{1}{2}) = 1 \tag{30}$$

After going through the loop again, the new $H_{MFi}(m)$ is also 1 since step 2-1 is always skipped. As a result, we escape from the loop and get the information volume of this mass function $H_{IV-MF}(m) = 1$.

Actually, this associated BPA of fuzzy MF is the probability distribution $P_1 = P_2 = \frac{1}{2}$. Hence, when the hesitancy degree of a fuzzy MF is $\eta_A = 1 - \mu_A - \gamma_A = 0$, the value of $H_{IV-MF}(m)$ is identical to the Shannon entropy.

Example 4.3: Let the universe discourse be E, the frame of discernment be $U = \{\theta_1, \theta_2\}$. The base of the logarithmic function of Deng entropy is 2. Let a intuitionistic fuzzy MF be $A = \{\langle x_6, \frac{1}{5}, \frac{1}{5}\rangle | x_6 \in E\}$.

Based on the definition of the information volume of fuzzy sets, the calculating procedure is illustrated in Figure 2.

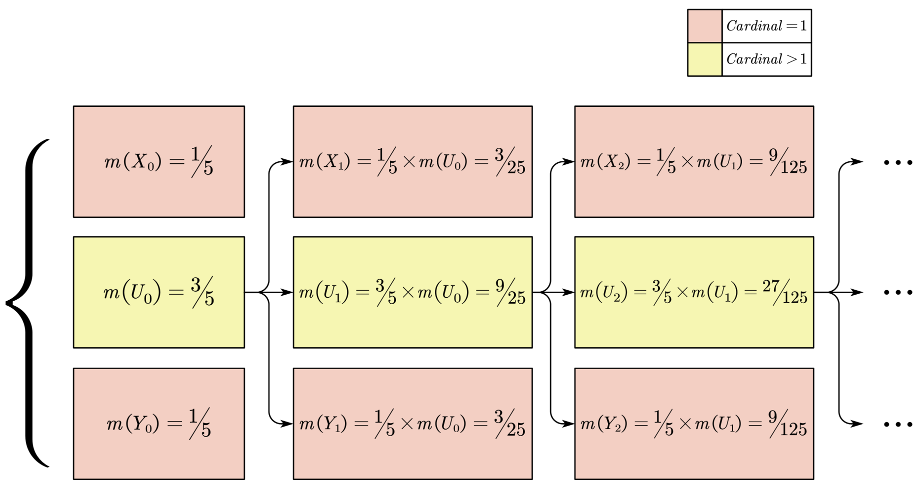

Figure 2: The procedure of from step 2-1 to step 2-3

Then, the convergence procedure of $H_{MFi}(m)$ is listed in Table 1.

Table 1: The convergence procedure of $H_{MFi}(m)$

i	$H_{MFi}(m)$	i	$H_{MFi}(m)$
1	2.321928	8	3.396431
2	2.764107	9	3.408809
3	3.029415	10	3.416236
4	3.188600	11	3.420692
5	3.284110	12	3.423366
6	3.341417	13	3.424970
7	3.375801	14	3.425933

According to Table 1, when we continuously separate the associated BPA of the element whose cardinal is larger than 1, the Δ_i becomes smaller and smaller. When $i = 14$, $H_{MFi}(m) - H_{MFi-1}(m) < 0.001$, which means that

$H_i(m)$ finally converges to 3.425933. Hence the information volume of this fuzzy set is $H_{IV-MF}(m) = 3.425933$.

This associated BPA of fuzzy MF is the same as Deng distribution. Under this kind of fuzzy MF, $H_{IV-MF}(m) = 3.425933$ is actually the maximum information volume of fuzzy membership function $H_{MIV-MF}(m)$.

5. Conclusion

In this paper, firstly, we propose a method to measure the uncertainty of a fuzzy MF through two steps. Step 1 is to convert the MF into BPA, and step 2 is to calculate the uncertainty of the associated BPA based on Deng entropy. Then, we define the information volume of the fuzzy MF. By continuously separating the BPA of the element whose cardinal is larger than 1 until convergence, the value of the information volume of the fuzzy MF can be calculated. In addition, several examples and figures are expound to illustrated the proposed method and definition. An interesting point is that, when the hesitancy degree of a fuzzy MF is 0, information volume of the fuzzy membership function is identical to the Shannon entropy.

Acknowledgment

The work is partially supported by National Natural Science Foundation of China (Grant No. 61973332).

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