Observations on the Pair Correlation of the Riemann Zeros

Brian Scannell June 2020
brian.scannell@ntlworld.com

Abstract

We look at the pair correlation of the Riemann zeros at larger ranges. The evolution of this curve computed at different ranges is shown. The ripply shape of this curve is examined using spectral analysis. There seems to be a main spectral peak and possibly some other structures. The percentage RMS in the peak is shown. The results are compared to the spectrum of both noise and Montgomery's pair correlation conjecture function $1-\mathrm{sinc}^2 x$. The pair correlation of unnormalised zeros shows dips at the zero positions.

1. Background

Riemann [1] hypothesised that the non-trivial zeros of the zeta function had real part one half. The terms of art represent these complex zeros as $1/2+i\gamma_i$ where γ_i is called the height of the i^{th} zero. The gap between two adjacent zeros normalised to have a unity average spacing is $\delta_i=(\gamma_{i+1}-\gamma_i)n_i$ where the normalisation $n_i=\dfrac{\ln(\gamma_i/2\pi)}{2\pi}$ accounts for the increased spacing of the zeros at increased height.

The Matlab [2] output of the pair correlation for the first 100,000 zeros along with Montgomery's conjecture is shown in Figure 1. The largest x which here is 3 is named the range.

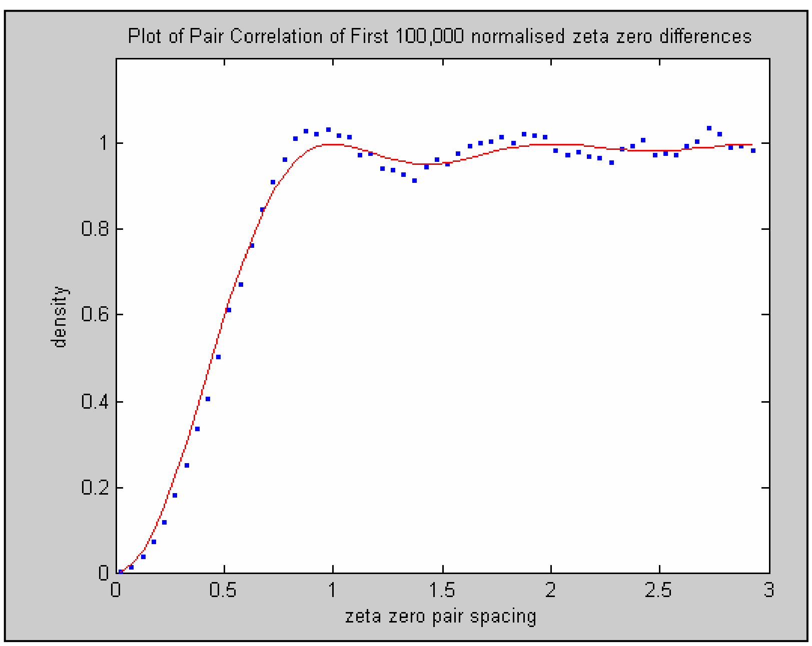

Figure 1 Pair correlation function for the first 10^5 zeros (blue dots) along with Montgomery's $1-\mathrm{sinc}^2 x$ (red)

Normally the pair correlation function is shown with range up to 3. Here we extend the range to 1000 and examine its structure.

brian.scannell@ntlworld.com
June 2020

2 The Riemann Zeros Data

The zeros data is downloaded from:

- Odlysko [3].
  - The first 100,000 zeros
  - The first 2e6 zeros
- LMFBD [4]
  - 6,009,630 zeros at height 403,646,000
  - 7,000,000 zeros at height 30,607,946,000
  - They are in stored in binary format and converted to text by Matlab.

3 Pair Correlation of First 100,000 Zeros

Method of Computation

The process is shown in the flow diagram computing the pair correlation over a range of x=10.

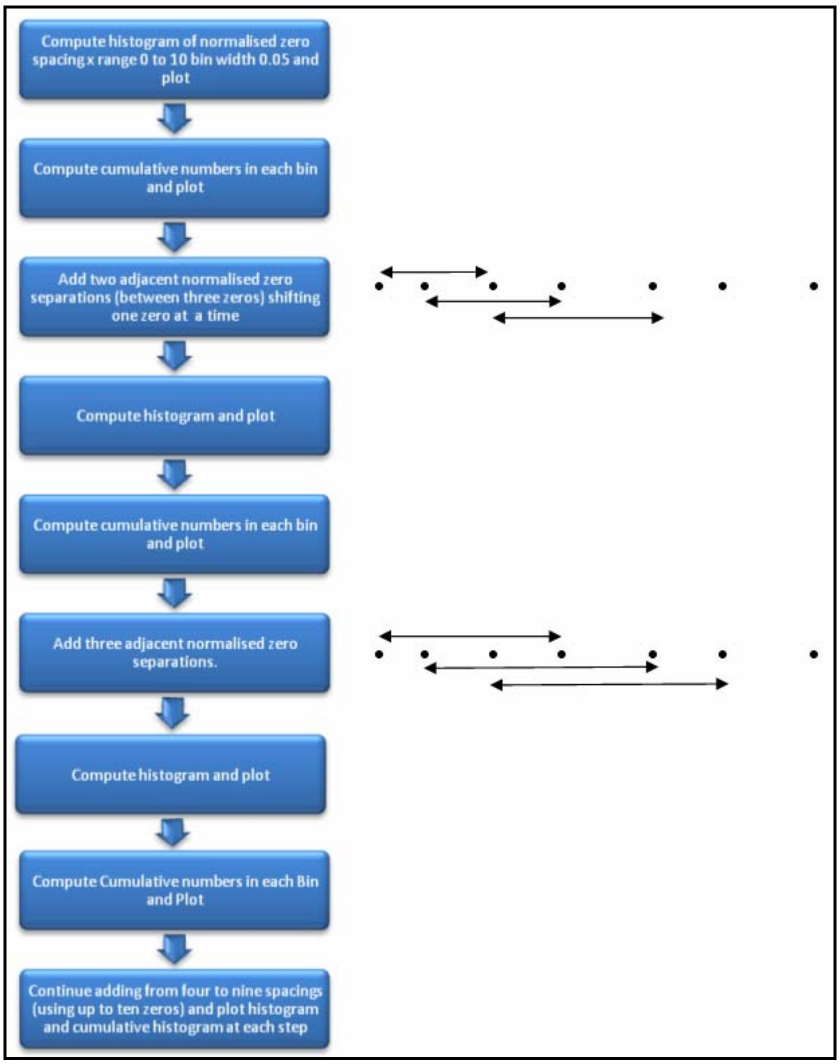

Figure 2 Process for computing pair correlation for a range about 10

brian.scannell@ntlworld.com
June 2020

The probability density is obtained by normalising the cumulative values in each bin by (the total number of spacings * bin width).

The figures below show the evolution of the pair correlation curve over an x range of 10 using 2, 4 and 9 adjacent zeros. The x axis is labeled Pair Correlation Function x, but it could also be called range or maximum number of zeta zero spacings used.

Due to the relatively small number of zeros the curve is quite peaky.

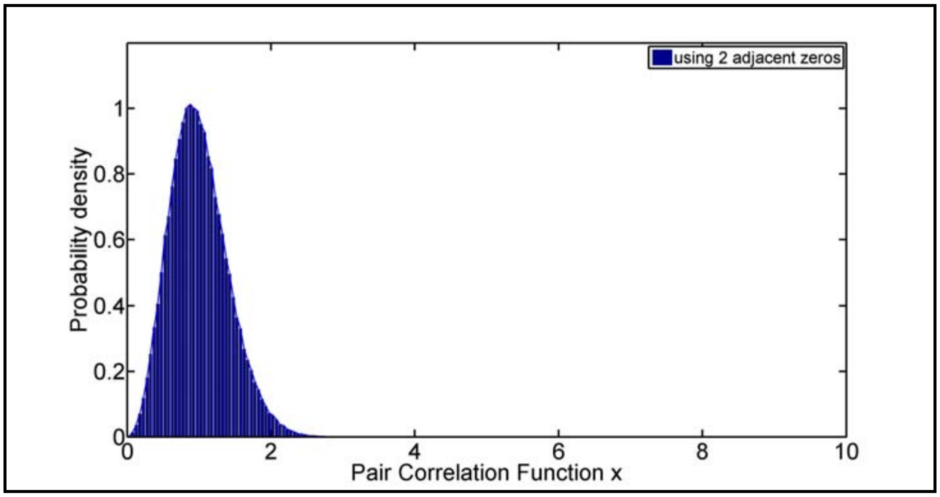

Figure 3 Probability density of pair correlation function for adjacent normalised zero differences (that is using two zeros) for the first 1e5 zeros

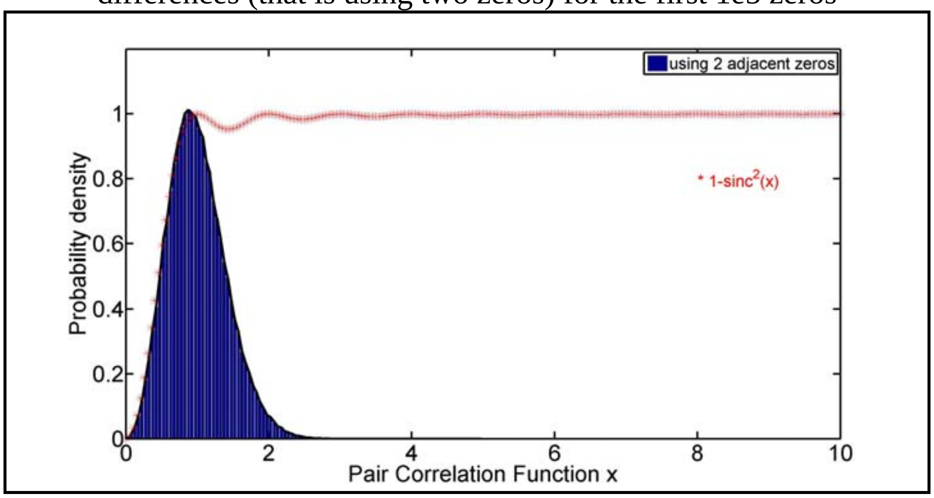

Figure 4 Cumulative probability density of pair correlation function for adjacent normalised zero differences for the first 1e5 zeros

brian.scannell@ntlworld.com
June 2020

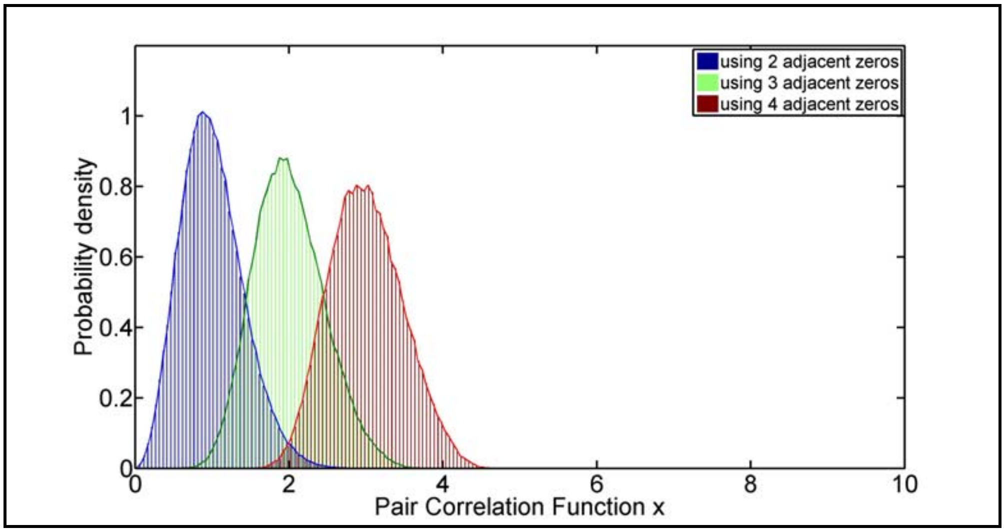

Figure 5 Probability density of normalised spacing between two, three and four adjacent zeros for first 1e5 zeros

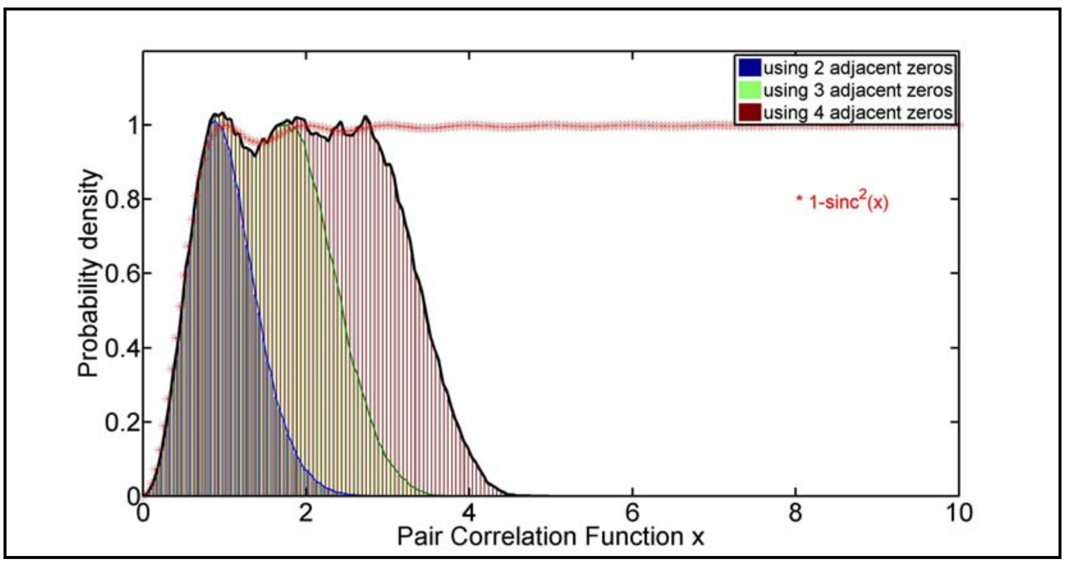

Figure 6 Cumulative probability density of normalised spacing between two, three and four adjacent zeros for first 1e5 zeros

brian.scannell@ntlworld.com

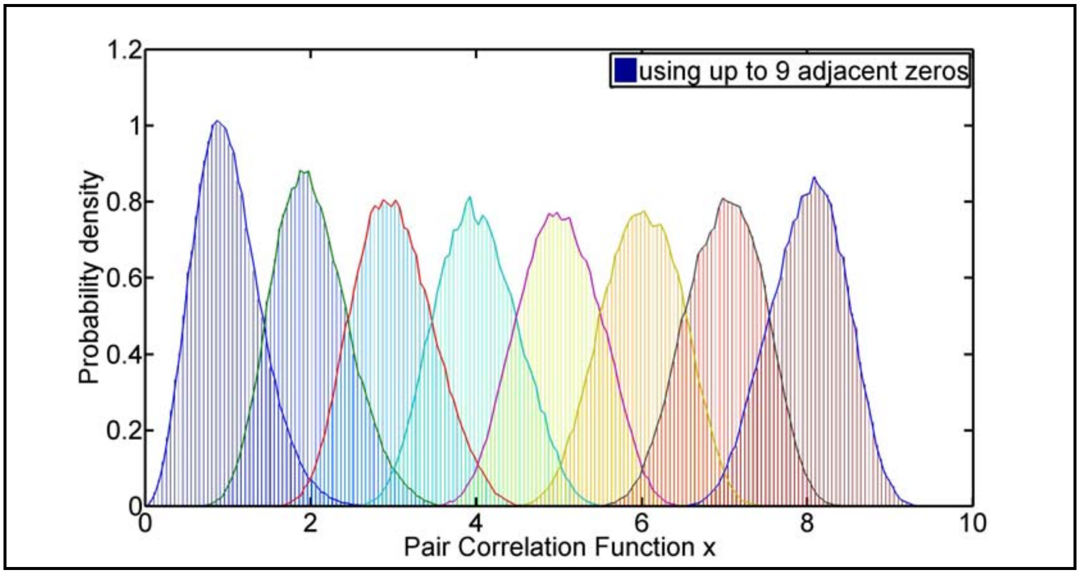

Figure 7 Probability density of normalised spacing using from two to nine adjacent zeros for the first 1e5 zeros

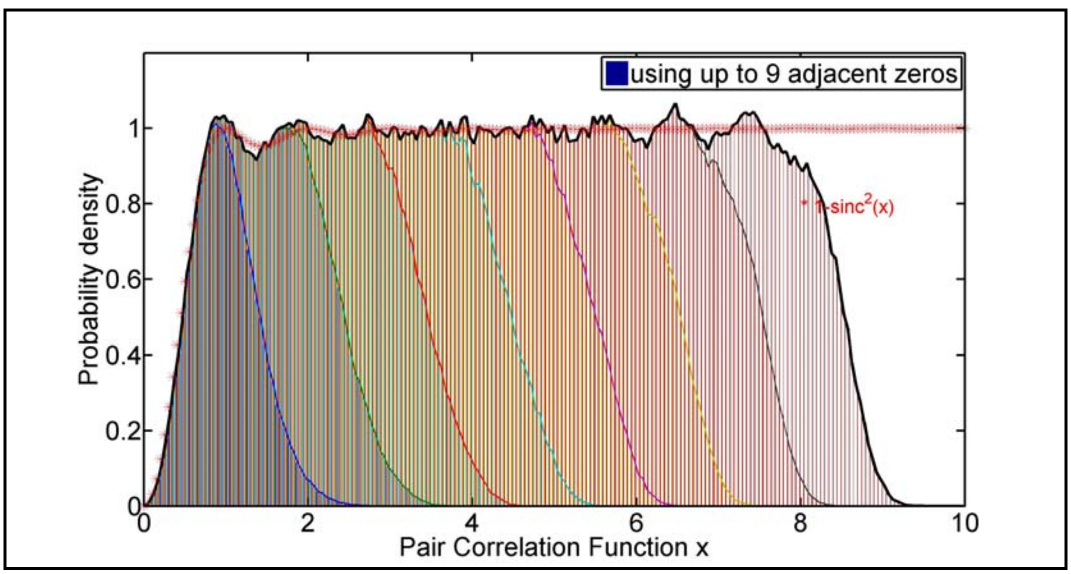

Figure 8 Cumulative probability density of normalised spacing using from two to nine adjacent zeros for the first 1e5 zeros

brian.scannell@ntlworld.com
June 2020

4 Pair Correlation of First 2e6 Zeros

Here we just show the final pair correlation curve for a range of 20 using the first two million normalised zero spacings. There seems to be some ripple which will be investigated.

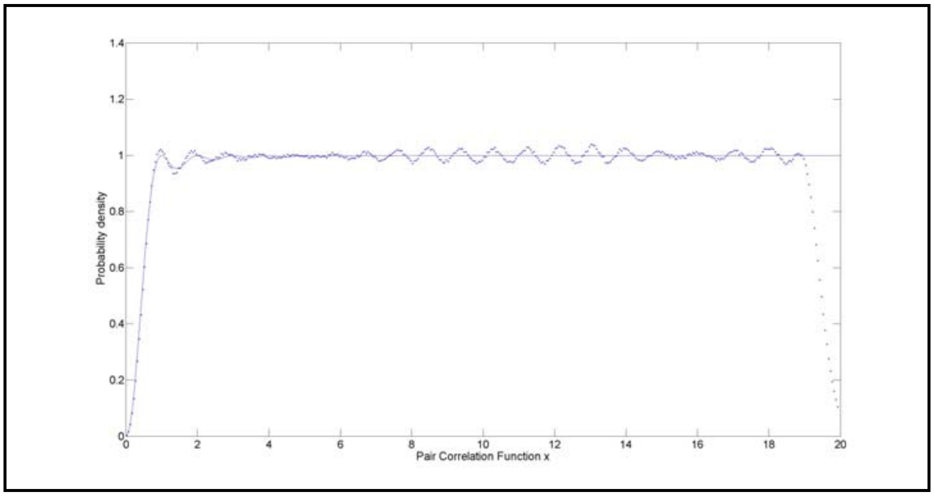

Figure 9 Pair correlation (cumulative probability density) of normalised spacing using up to nineteen adjacent zeros for the first 2e6 zeros

5 Pair Correlation of 6,009,630 zeros at height 403,646,000

This is the pair correlation higher up.

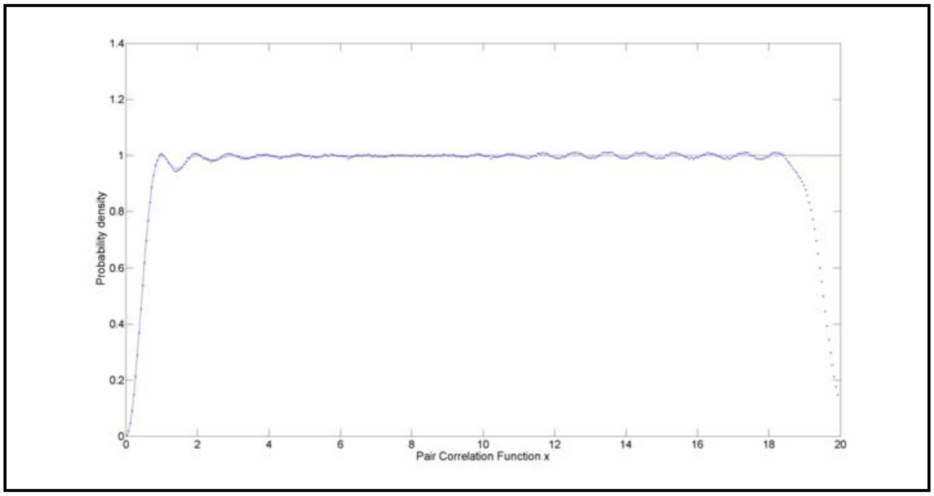

Figure 10 Pair correlation of normalised spacing using up to nineteen adjacent zeros for the 6e6 zeros at height 403e6

brian.scannell@ntlworld.com
June 2020

6 Pair Correlation of 7,000,000 zeros at height 30,607,946,000

This is the pair correlation at the largest height for which I can find published zeros.

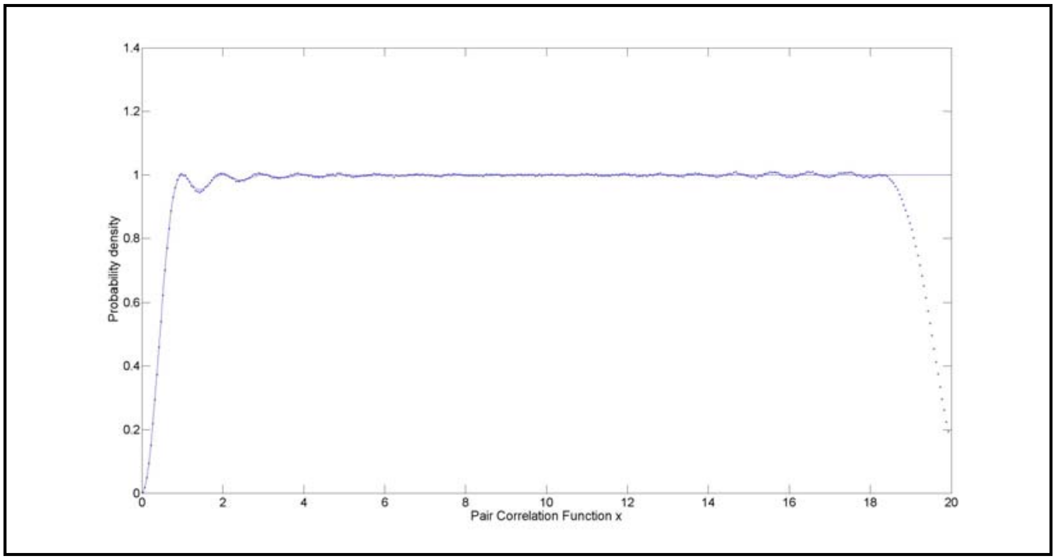

Figure 11 Pair correlation of normalised spacing using up to nineteen adjacent zeros for the 7e6 zeros at height 30e9

Below the range is extended to 100. There seems to be some repeating structure.

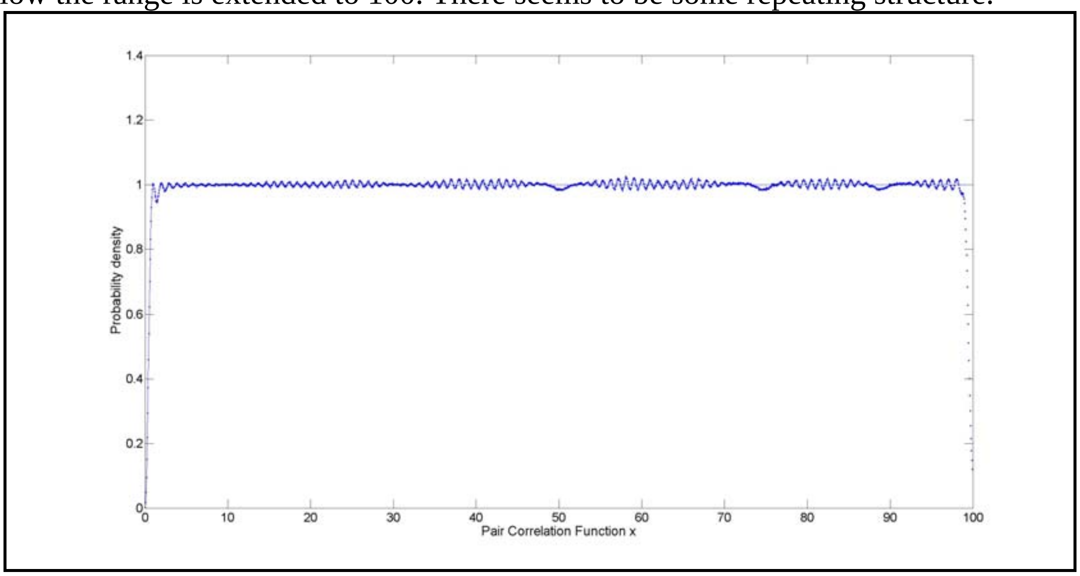

Figure 12 Pair correlation of normalised spacing using up to ninety-nine adjacent zeros for the 7e6 zeros at height 30e9

Below the x range is extended further to 1000. Again some structure is apparent. The next section investigates this pair correlation structure using a range of 1000 for all heights.

brian.scannell@ntlworld.com

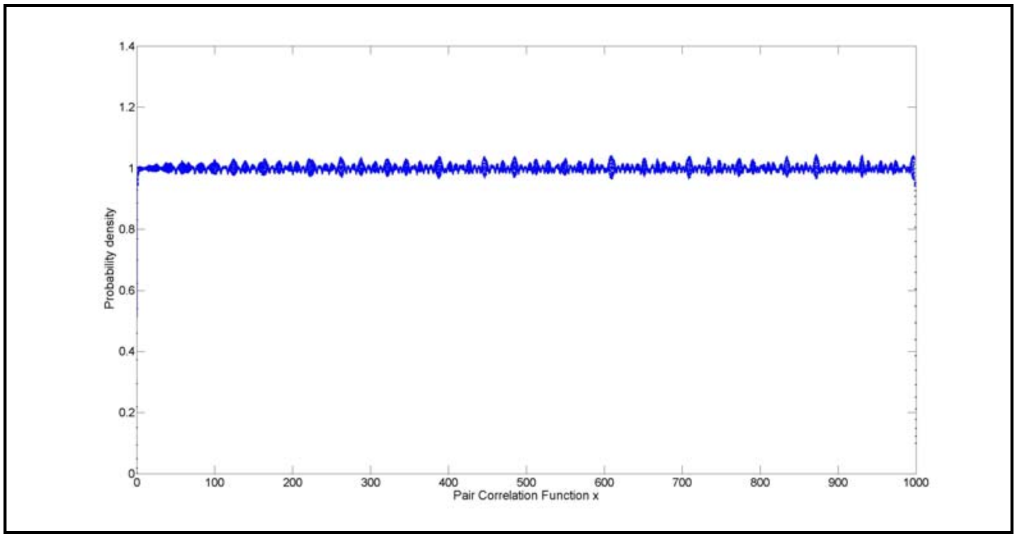

Figure 13 Pair Correlation of normalised spacing using up to nine hundred and ninety-nine adjacent zeros for the 7e6 zeros at height 30e9

7 Spectral Density of the Pair Correlation Curve: First 100,000 Zeros

Notation:
Looking for example at Figure 9 there seems to be a ripple in the pair correlation of about 1 unit. I will use a familiar notation and say this has a wavelength of about 1 unit. Similarly its frequency is 1/wavelength also equal to about 1 unit.

To make this more quantitative the spectral density of the pair correlation curve is computed. The DC term is subtracted by analysing only the flattish portion of the curve from x=1.875 to 998.125. This upper value avoids the computational 'end-effect' of the curve dropping off.

A property of the spectral density is that the area under the curve (its integral) is the variance of the flattish part of the pair correlation curve that was used to generate it.

The computation of the pair correlation over a range of 1000 uses up to around a cumulative 1000 zero spacings shifted across the list of zeros and with a histogram bin-width of 0.05. This governs the frequency range of the plots from between 0.001 (1/1000) to 20 (1/0.05).

7.1 Spectral Density vs. Frequency

The spectral density of the pair correlation for the first 100,000 zeros shows a peak around 1 unit as expected by-eye.

The peak is quite noisy with a range between 0.8 and 1.2.

brian.scannell@ntlworld.com

There seems to be some lower frequency structure between 0.07 to 0.3 or higher.

There is no indication of any peaks at frequency 2 which represents a half integer variation in pair correlation x. This may have some relevance to the Alternative Hypothesis [5].

The spikes at the top frequency are presumably a computational end-effect.

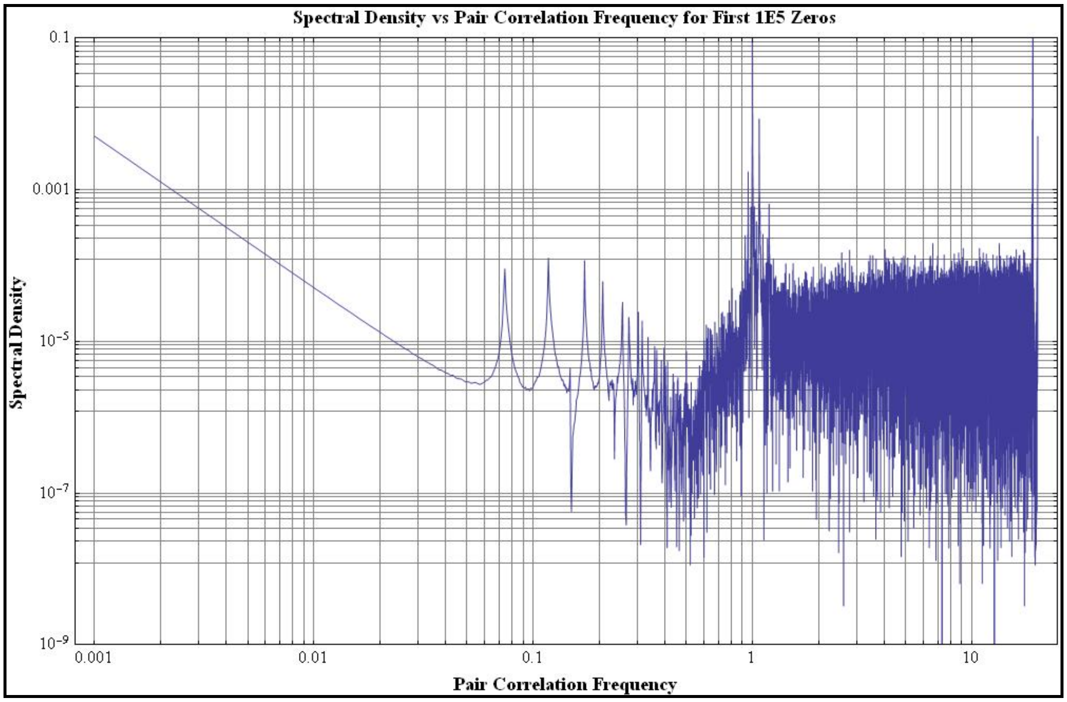

Figure 14 Spectral density vs. frequency of the pair correlation for a range of x equal to 1000 for the first 1e5 zeros

7.2 Spectral Density vs. Wavelength

The equivalent spectral density versus wavelength shows the equivalent peak around 1.

There is a -2 slope due to the 1/wavelength^2 scaling in this format.

brian.scannell@ntlworld.com
June 2020

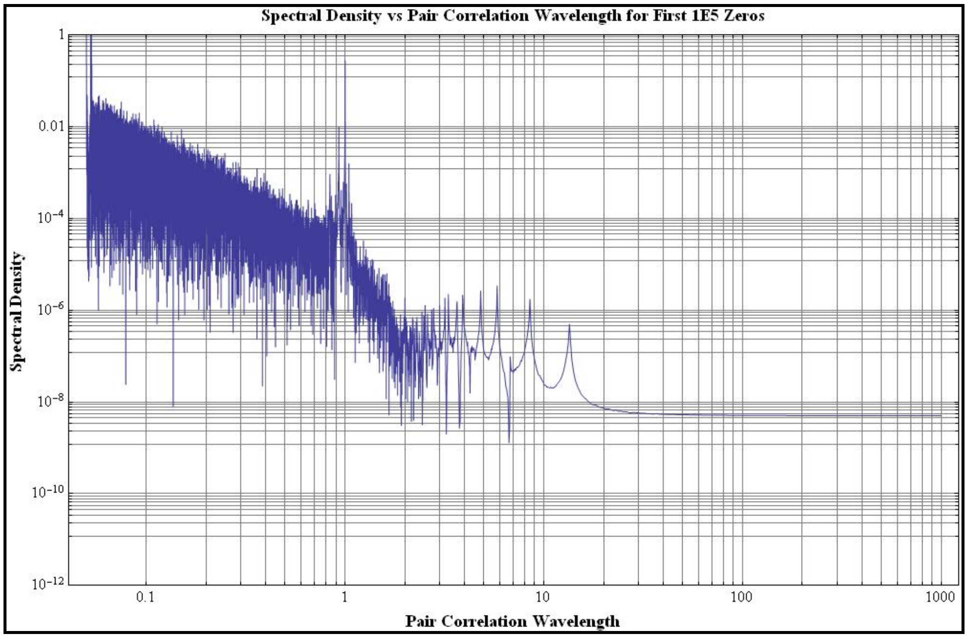

Figure 15 Spectral density vs. wavelength of the pair correlation for a range of x equal to 1000 for the first 1e5 zeros

8 Comparison of Spectral Density of the Pair Correlation Curves

Here we compare the spectral densities for the zeros at different heights: the first 1e5, the first 2e6, 6e6 at height 403e6 and 7e6 at height 30e9.

8.1 Spectral Density vs. Frequency

The main feature is the consistent spike at around 1 for all heights.

There is a downward level shift at larger heights reflecting a lower variance in the pair correlation.

brian.scannell@ntlworld.com
June 2020

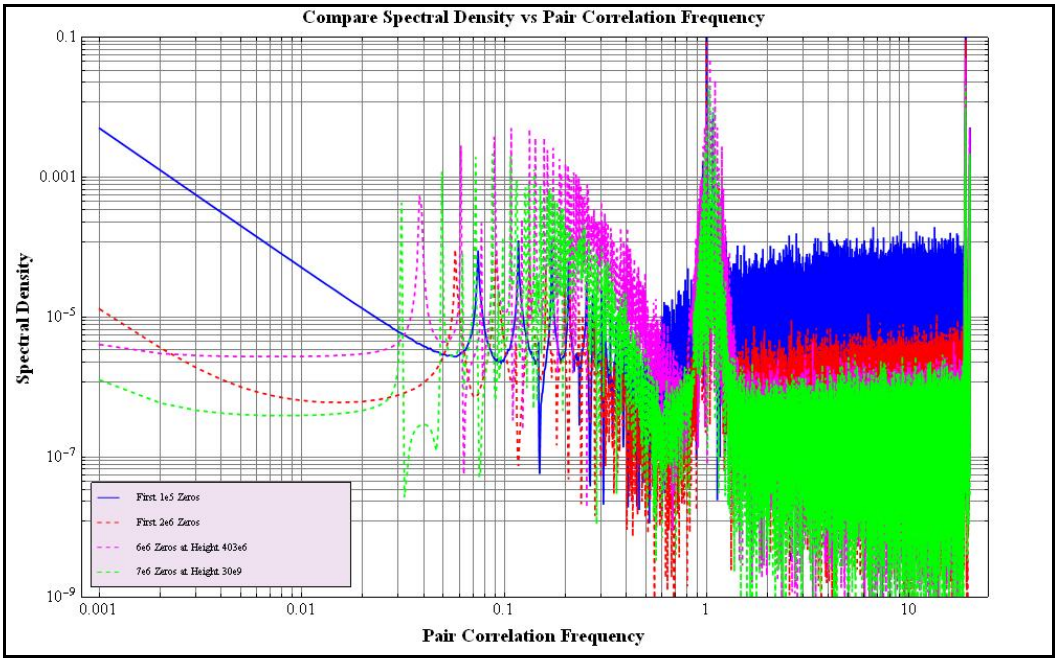

Figure 16 Comparing at different heights the spectral density vs. frequency of the pair correlation for a range of x equal to 1000

8.2 Spectral Density vs. Wavelength

The same comments apply to the wavelength format.

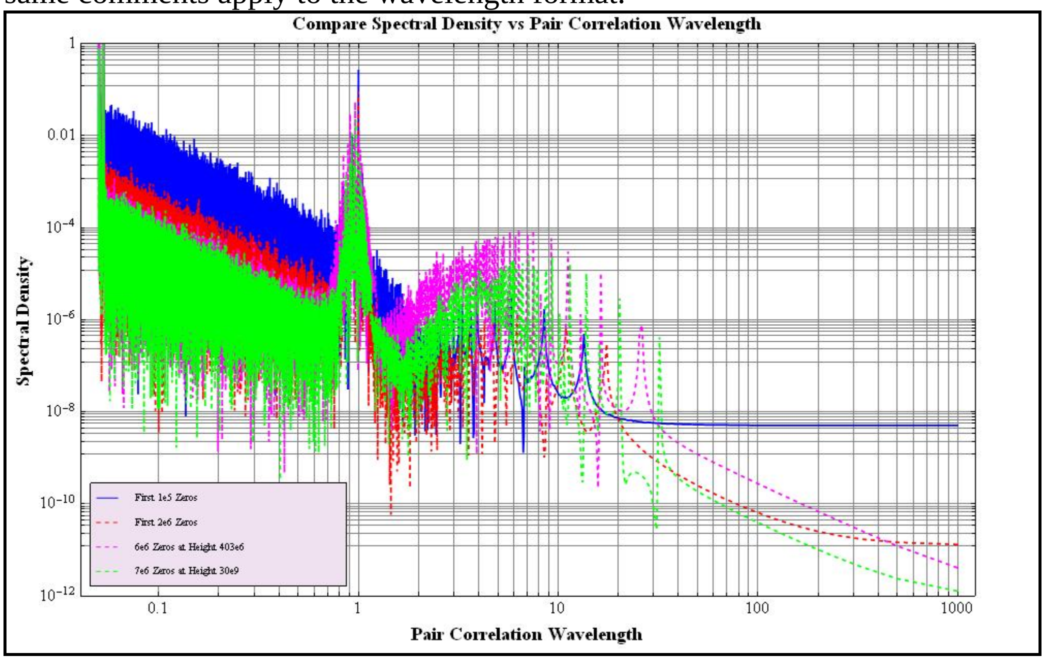

Figure 17 Comparing at different heights the spectral density vs. wavelength of the pair correlation for a range of x equal to 1000

brian.scannell@ntlworld.com

9 Comparison of Spectral Density of Noise

To assess the amount of noise in the pair correlation, the spectral density of 7e6 zeros at height 30e9 is compared to that of uniform noise with the same RMS computed over the same range.

9.1 Spectral Density vs. Frequency

The main features pointed out previously do seem to be real artifacts and not noise.

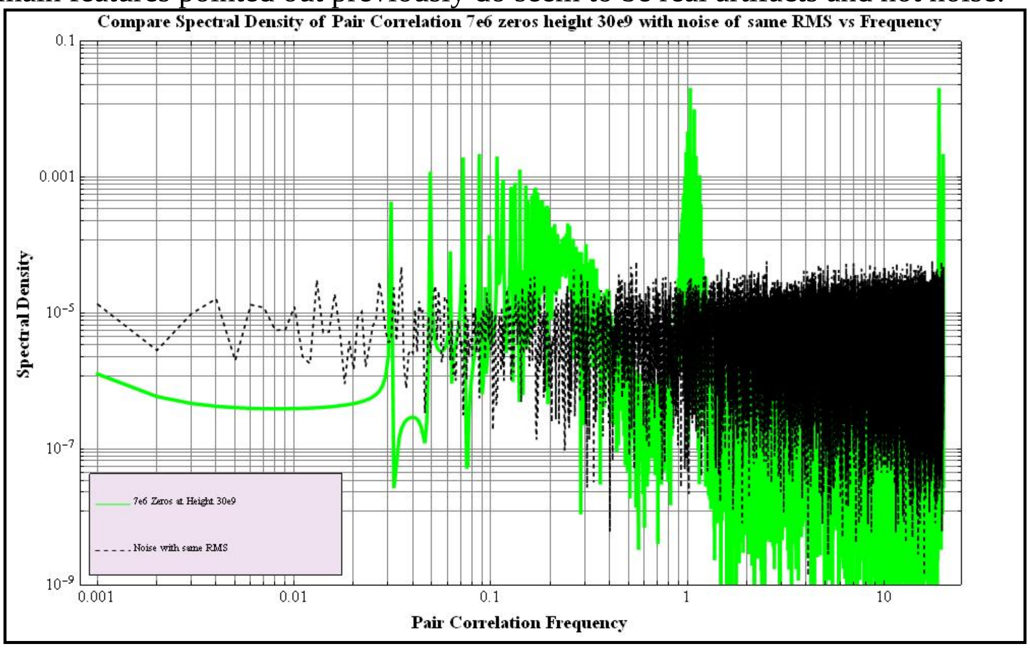

Figure 18 Comparing noise of the same RMS with the spectral density vs. frequency of the pair correlation for a range of x equal to 1000 for 7e6 zeros at height 30e9

brian.scannell@ntlworld.com
June 2020

9.2 Spectral Density vs. Wavelength

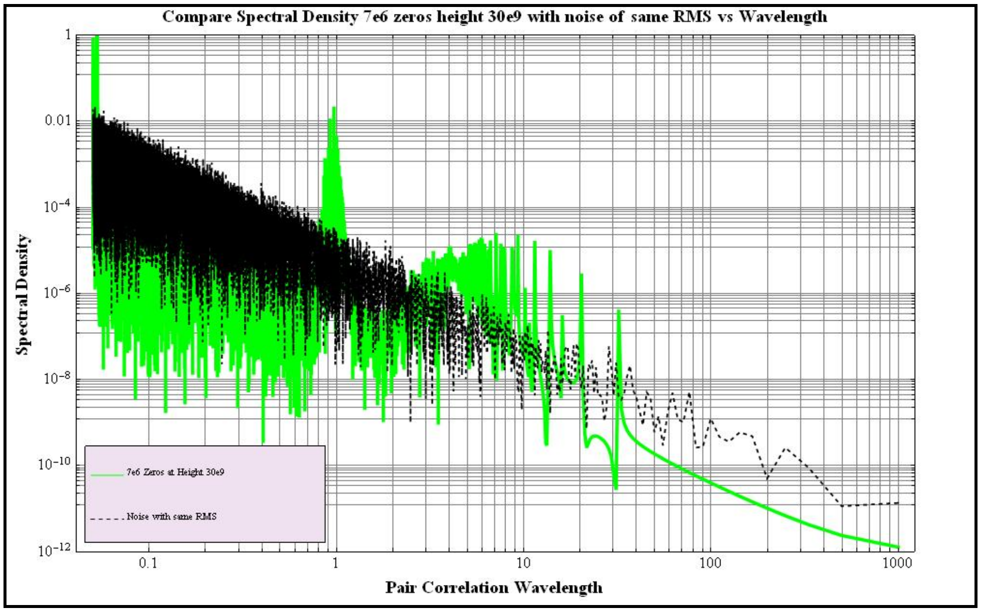

Figure 19 Comparing noise of the same RMS with the spectral density vs. wavelength of the pair correlation for a range of x equal to 1000 for 7e6 zeros at height 30e9

10 Spectral Density of 1-sinc^{2}x

Here we look at the spectral density of Montgomery's 1- sinc^{2}x over the same range (i.e. just the flattish part) and bin width.

10.1 Spectral Density vs. Frequency

There does not seem to be any obvious structure and not a peak at one which is seen in the data using 1e5 to 7e6 zeros.

Figure 20 Spectral density vs. frequency of 1- sinc^{2}x

brian.scannell@ntlworld.com

10.2 Spectral Density vs. Wavelength

Similar comments apply for this format.

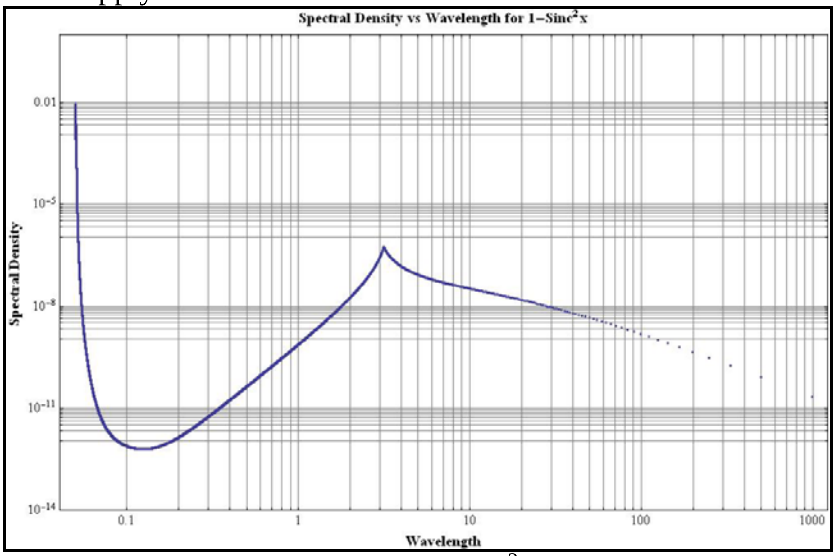

Figure 21 Spectral density vs. wavelength of 1- sinc^2x

11 Comparison of RMS in Main Peak of Spectral Density

The bar chart shows, with height, the RMS of the pair correlation curves used above. The RMS is computed from the square root of the integral of the spectral density. The largest variation is obtained using the smallest number of zeros. It is unclear if there is a height variation.

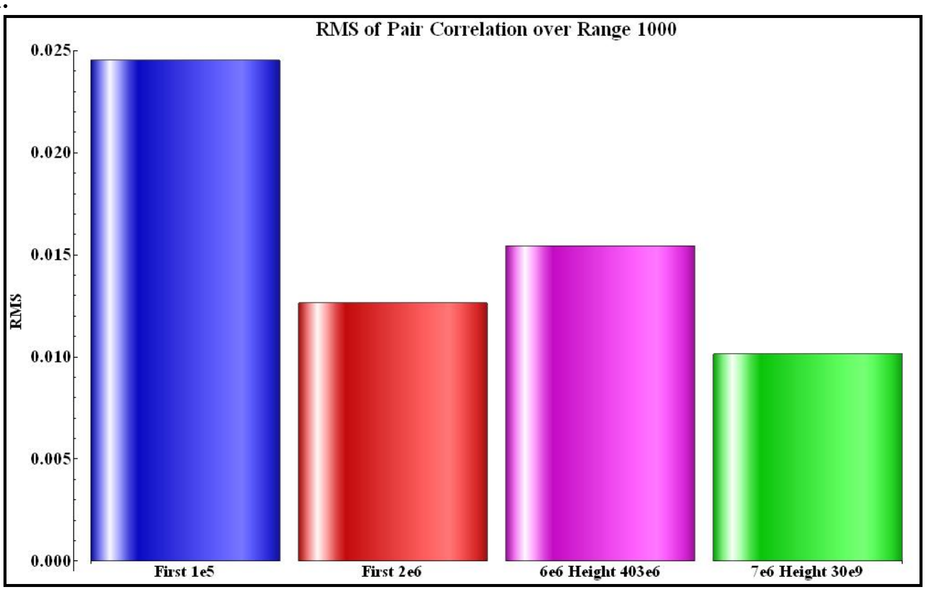

Figure 22 RMS of pair correlation curve at different heights

brian.scannell@ntlworld.com

Below shows the percentage RMS in the main spectral peak relative to the total RMS above. The RMS in the main peak is the integral of spectral curve over the frequency range 0.8 to 1.2.

There seems to be a more or less constant value of 80% at different heights.

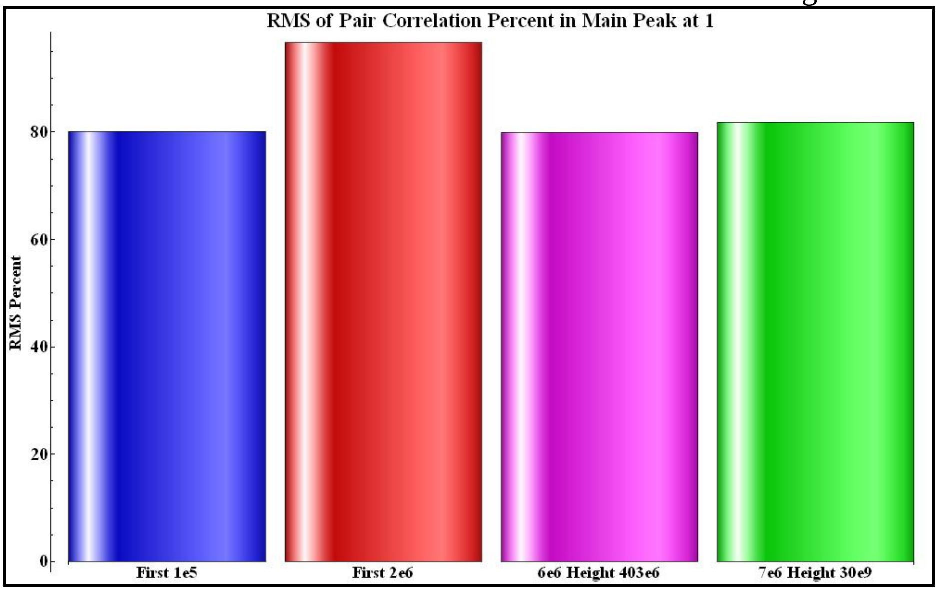

Figure 23 Percentage RMS in main peak at different heights

12 Pair Correlation of Unnormalised (Raw) Zeros

The figure below shows the pair correlation of the unnormalised zeros at different heights. In this format the Riemann zeros are shown as the prominent dips. The depth of the dips gets less pronounced at increased heights presumably in line with the Ratios Theorem [6].

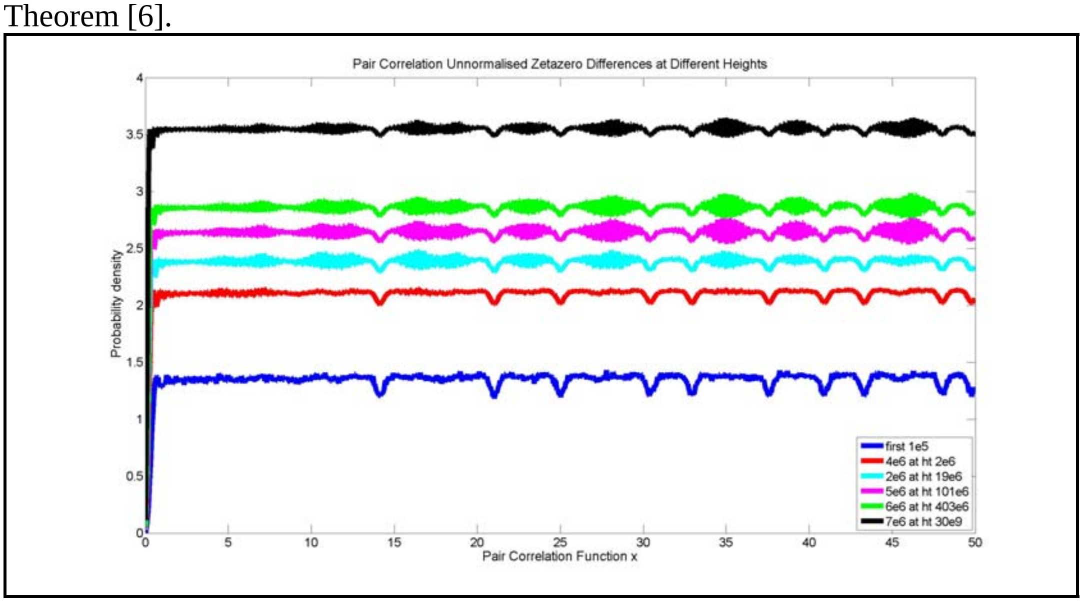

Figure 24 Pair correlation of the unnormalised zeros at different heights. The dips are the Riemann zeros.

brian.scannell@ntlworld.com

Conclusions

There is structure in the pair correlation of the Riemann zeros at all heights for the number of zeros considered.

Spectral analysis of these curves shows a main peak at around 1 identified as an easily observed waviness in the pair correlation curve of unity wavelength. There may be some larger wavelength structures. These structures do not seem to be attributable to noise.

There is no spectral peak at 1 for the Montgomery function 1-sinc^2x. It would be interesting to see if the peak at unity observed using millions of zeros here dissolved upon using billions of zeros in line with his theorem.

The RMS variation in the main peak seems constant with height.

The Riemann zeros are seen in the pair correlation of the raw zeros. The dips showing the position of the zeros gets less pronounced at increasing height perhaps in line with the ratios theorem.

Reference

1) Bernhard Riemann. On the number of prime numbers less than a given quantity. University of Berlin Nov 1859.
2) The Matlab codes to compute the pair correlation and to convert the downloaded zeros from binary is available upon request. brian.scannell@ntlworld.com
3) http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html
4) https://www.lmfdb.org/zeros/zeta/?limit=100&N=10000000000
5) e.g. C. Baluyot, On the pair correlation conjecture and the alternative hypothesis, J. Number Theory 169 (2016), 183–226
6) Snaith, NC. (2010). Riemann Zeros and Random Matrix Theory. *Milan Journal of Mathematics*, *78*(1), 135-152. https://doi.org/10.1007/s00032-010-0114-7 also citing E.B. Bogomolny and J.P. Keating, Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation, *Phys. Rev. Lett.*, 77(8):1472{1475, 1996.

brian.scannell@ntlworld.com
June 2020