Generalized sedeonic equations of hydrodynamics

V. L. Mironov[1,2] and S. V. Mironov[1]

[1]Institute for Physics of Microstructures, Russian Academy of Sciences, 603950, Nizhny Novgorod, GSP-105, Russia
[2]Lobachevsky State University of Nizhny Novgorod, 603950, Nizhny Novgorod, Gagarin Avenue, 23, Russia,

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Abstract

We discuss a generalization of the equations of hydrodynamics based on space-time algebra of sedeons. It is shown that the fluid dynamics can be described by sedeonic second-order wave equation for scalar and vector potentials. The generalized sedeonic Navier-Stokes equations for a viscous fluid and vortex flows are also discussed. The main peculiarities of the proposed approach are illustrated on the equations describing the propagation of sound waves.

1 Introduction

The analogy between the equations of hydrodynamics and electrodynamics is actively discussed for a long time. Apparently first, some similarity between vortex dynamics of fluid and electromagnetic phenomena induction was pointed out by H. Helmholtz in [1]. Subsequently, several attempts were made to describe the fluid dynamics by vector fields (similar to electric and magnetic fields) satisfying some Maxwell-like equations [2]-[10]. However a common drawback of the approach used in these works is that the equation for the vortex component of the fluid motion is obtained simply by taking the "curl" operator from the Euler equation for velocity and therefore it is not independent. In particular, in [4] the linearized equations for a free isentropic compressible fluid are reduced to the following form:

$$\begin{aligned}
&c^2[\nabla \times \mathbf{H}] - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J},\\
&[\nabla \times \mathbf{E}] + \frac{\partial \mathbf{H}}{\partial t} = 0,\\
&(\nabla \cdot \mathbf{E}) = g,\\
&(\nabla \cdot \mathbf{H}) = 0,
\end{aligned} \tag{1}$$

where vector fields $\mathbf{E}$ and $\mathbf{H}$ are defined by the following expressions:

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{v}}{\partial t} - \nabla h, \\ \mathbf{H} &= [\nabla \times \mathbf{v}], \end{aligned} \tag{2}$$

and the field sources

$$\begin{aligned} g &= -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) - \triangle h, \\ \mathbf{J} &= \frac{\partial^2 \mathbf{v}}{\partial t^2} + \frac{\partial}{\partial t}(\nabla h) + c^2[\nabla \times [\nabla \times \mathbf{v}]]. \end{aligned} \tag{3}$$

Here $\mathbf{v}$ is the velocity of the fluid, h is the enthalpy per unit mass, c is the speed of sound.

By the form, the system (1) coincides with Maxwell's equations, however, these equations do not have any predictive power, since the field sources are determined through the quantities $\mathbf{v}$ and h, which themselves must be found from the equations. In addition, by substituting the definition of fields (2) and sources (3) into equation (1), we obtain the identity. A similar situation is observed in the works of other authors.

During the past decades the essential progress is observed in the reformulation of the equations for electromagnetic field and fluid motion based on the different algebras of hypercomplex numbers such as quaternions [11]-[14] and octonions [15-18], which take into account the symmetry of physical values with respect to operation spatial inversion. A natural generalization of this approach is the inclusion of time reversal symmetry in an algebraic structure, which requires consideration of extended sixteen-component algebras such as sedenions [19], [20].

Recently, we proposed an associative algebra of sixteen-component sedeons, which takes into account the properties of physical quantities with respect to space-time inversion and implements a scalar-vector representation of the Poincare group [21]. This formalism has been successfully applied to describe classical and quantum fields [21]-[24]. In the present paper we discuss the application of sedeonic algebra to the generalization of the equations describing dynamics of viscous fluid.

2 Algebra of space-time sedeons

The algebra of sedeons encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars (A) and absolute vectors ($\mathbf{A}$) are not transformed under spatial and time inversion.

2. Time scalars ($B_{\mathbf{t}}$) and time vectors ($\mathbf{B_t}$) change sign under time inversion and are not transformed under spatial inversion.

3. Space scalars ($C_{\mathbf{r}}$) and space vectors ($\mathbf{C_r}$) are changed under spatial inversion and are not transformed under time inversion.

4. Space-time scalars ($D_{\mathbf{tr}}$) and space-time vectors ($\mathbf{D_{tr}}$) change sign under spatial and time inversion.

The indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{S}}$, which is defined by the following expression:

$$\tilde{\mathbf{S}} = A + \mathbf{A} + B_{\mathbf{t}} + \mathbf{B_t} + C_{\mathbf{r}} + \mathbf{C_r} + D_{\mathbf{tr}} + \mathbf{D_{tr}}. \tag{4}$$

The system of sedeons is based on the Macfarlane's quaternion algebra [25]. Any vector is presented in the basis of unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ as

$$\mathbf{A} = A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}, \tag{5}$$

with the following rules of multiplication

$$\mathbf{a_n a_m} = \delta_{\mathbf{nm}} + i\varepsilon_{\mathbf{nmk}}\mathbf{a_k}, \tag{6}$$

where $\delta_{\mathbf{nm}}$ is Kronecker delta, $\varepsilon_{\mathbf{nmk}}$ is Levi-Civita symbol ($\mathbf{n}, \mathbf{m}, \mathbf{k} \in \{1, 2, 3\}$) and i is imaginary unit ($i^2 = -1$). The rules of multiplication and commutation for unit vectors are also summarized in Table 1.

The main advantage of this approach is the Clifford's product of vectors

$$\mathbf{AB} = (\mathbf{A} \cdot \mathbf{B}) + i[\mathbf{A} \times \mathbf{B}], \tag{7}$$

that allows to write the equations in very compact form.

The space-time properties of physical values can be taken into account using an additional basis $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$, where $\mathbf{e_t}$ is the time scalar unit; $\mathbf{e_r}$ is the spatial scalar unit; $\mathbf{e_{tr}}$ is the space-time scalar unit. The rules of multiplication and commutation for space-time units are presented in Table 2. The space-time units $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$ commute with vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$

$$\mathbf{a_n e}_\alpha = \mathbf{e}_\alpha \mathbf{a_n}, \tag{8}$$

$$\alpha \in (\mathbf{t}, \mathbf{r}, \mathbf{tr}), \quad \mathbf{n} \in (\mathbf{1}, \mathbf{2}, \mathbf{3}).$$

In general algebra of sedeons is the tensor product of two algebras of Macfarlane quaternions [25] $\{\mathbf{a_n}\}$ and $\{\mathbf{e}_\alpha\}$. It is associative algebra, which is isomorphic to the algebra of (4×4) Dirac matrices [26].

Using the space-time basis we can rewrite the sedeon (4) in terms of absolute scalars and absolute vectors as follows:

$$\tilde{\mathbf{S}} = A + \mathbf{A} + \mathbf{e_t}B + \mathbf{e_t}\mathbf{B} + \mathbf{e_r}C + \mathbf{e_r}\mathbf{C} + \mathbf{e_{tr}}D + \mathbf{e_{tr}}\mathbf{D}. \tag{9}$$

Thus the sedeon $\tilde{\mathbf{S}}$ is a compound space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

Table 1: The rules of multiplication for absolute unit vectors

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2: The rules of multiplication for space-time units

	$\mathbf{e_t}$	$\mathbf{e_r}$	$\mathbf{e_{tr}}$
$\mathbf{e_t}$	1	$i\mathbf{e_{tr}}$	$-i\mathbf{e_r}$
$\mathbf{e_r}$	$-i\mathbf{e_{tr}}$	1	$i\mathbf{e_t}$
$\mathbf{e_{tr}}$	$i\mathbf{e_r}$	$-i\mathbf{e_t}$	1

3 Symmetric form of equations for ideal fluid

The dynamics of an ideal vortex-less fluid is described by the following well known system of equations

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{q}, \\
&\frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho + \rho(\nabla\cdot\mathbf{v}) = \mathbf{0}, \\
&[\nabla\times\mathbf{v}] = 0,
\end{aligned} \tag{10}$$

where $\mathbf{v}$ is local flow velocity of fluid, ρ is a density, p is a pressure, $\mathbf{q}$ is a force for unit mass [27]. This system includes the Euler equation, the continuity equation and the condition of the absence of vortices.

The solutions of equations (10) depend on the specific type of fluid motion. In the cases of barotropic, isothermal and isentropic motion under the additional assumption of constant velocity of sound the system (10) becomes simplified. Below we show that in all these specific cases the equations (10) can be rewritten in a universal symmetric form, which allows the natural generalization on the basis of the sedeonic approach.

3.1 The barotropic fluid motion

In the case of a simple model of barotropic fluid, the the pressure depends only on density, so the state equation takes the following form:

$$p = p(\rho). \tag{11}$$

We assume that the speed of sound in the medium is constant:

$$c_B^2 = \frac{\partial p}{\partial \rho} = const. \tag{12}$$

Then equations (10) take the form:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{c_B^2}{\rho}\nabla \rho = \mathbf{q}, \\
&\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}(\mathbf{v} \cdot \nabla)\rho + (\nabla \cdot \mathbf{v}) = \mathbf{0}, \\
&[\nabla \times \mathbf{v}] = 0.
\end{aligned} \tag{13}$$

Let us introduce new notation:

$$\begin{aligned}
u_B &= c_B \ln(\rho), \\
\mathbf{f}_B &= \frac{1}{c_B}\mathbf{q},
\end{aligned} \tag{14}$$

then the system of equations for ideal fluid becomes symmetric:

$$\begin{aligned}
&\frac{1}{c_B}\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\right)\mathbf{v} + \nabla u_B = \mathbf{f}_B, \\
&\frac{1}{c_B}\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\right)u_B + (\nabla \cdot \mathbf{v}) = 0, \\
&[\nabla \times \mathbf{v}] = 0.
\end{aligned} \tag{15}$$

3.2 The isothermal fluid motion

For the isothermal fluid we assume the constant speed of sound

$$c_T^2 = \left(\frac{\partial p}{\partial \rho}\right)_T = const, \tag{16}$$

and use the thermodynamic relation for the Gibbs potential

$$dz = -sdT + \frac{1}{\rho}dp, \tag{17}$$

where z is the Gibbs potential, s is the entropy referred to the unit mass, T is the temperature. In the case of $T = const$ we have:

$$dz = \frac{1}{\rho}dp = \frac{c_T^2}{\rho}d\rho, \tag{18}$$

and therefore

$$\begin{aligned}
&\frac{1}{\rho}\nabla p = \nabla z, \\
&\frac{\partial \rho}{\partial t} = \frac{\rho}{c_T^2}\frac{\partial z}{\partial t}, \\
&\nabla \rho = \frac{\rho}{c_T^2}\nabla z = 0.
\end{aligned} \tag{19}$$

Then equations (10) take the form:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla z = \mathbf{q},\\
&\frac{1}{c_T^2}\frac{\partial z}{\partial t} + (\mathbf{v}\cdot\nabla)z + (\nabla\cdot\mathbf{v}) = \mathbf{0},\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{20}$$

Introducing the notations:

$$\begin{aligned}
u_T &= \frac{1}{c_T} z,\\
\mathbf{f}_T &= \frac{1}{c_T}\mathbf{q},
\end{aligned} \tag{21}$$

we derive the system of equations for ideal fluid, which again becomes symmetric:

$$\begin{aligned}
&\frac{1}{c_T}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u_T = \mathbf{f}_T,\\
&\frac{1}{c_T}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u_T + (\nabla\cdot\mathbf{v}) = 0,\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{22}$$

3.3 The isentropic fluid motion

We denote the speed of sound in the case of isentropic motion as

$$c_S^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = const. \tag{23}$$

Let us use the thermodynamic relation for enthalpy

$$dh = T ds + \frac{1}{\rho} dp. \tag{24}$$

Here h is the enthalpy referred to the unit mass. In the case of $s = const$ we have:

$$dh = \frac{1}{\rho} dp = \frac{c_S^2}{\rho} d\rho, \tag{25}$$

and therefore

$$\begin{aligned}
&\frac{1}{\rho}\nabla p = \nabla h,\\
&\frac{\partial \rho}{\partial t} = \frac{\rho}{c_S^2}\frac{\partial h}{\partial t},\\
&\nabla\rho = \frac{\rho}{c_S^2 T}\nabla h = 0.
\end{aligned} \tag{26}$$

Then equations (10) take the form:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla h = \mathbf{q},\\
&\frac{1}{c_S^2}\frac{\partial h}{\partial t} + (\mathbf{v}\cdot\nabla)h + (\nabla\cdot\mathbf{v}) = \mathbf{0},\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{27}$$

Let us introduce new notation:

$$\begin{aligned}
u_S &= \frac{1}{c_S}h,\\
\mathbf{f}_S &= \frac{1}{c_S}\mathbf{q},
\end{aligned} \tag{28}$$

then the system of equations for ideal fluid takes the following symmetric form:

$$\begin{aligned}
&\frac{1}{c_S}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u_S = \mathbf{f}_S,\\
&\frac{1}{c_S}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u_S + (\nabla\cdot\mathbf{v}) = 0,\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{29}$$

4 The sedeonic equation for a vortex-free flow

Using the algebra of sedeons, equations (15), (22) and (29) can be represented as a single generalized first-order wave equation in the following form:

$$\left(i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right) - \mathbf{e_r}\nabla\right)(\mathbf{e_{tr}}\mathbf{v} - u) = \mathbf{f}, \tag{30}$$

where the set

$$\{c, u, \mathbf{f}\} \in \{\{c_B, u_B, \mathbf{f}_B\}, \{c_T, u_T, \mathbf{f}_T\}, \{c_S, u_S, \mathbf{f}_S\}\} \tag{31}$$

depending on the type of fluid motion. Indeed, after the action of the operator on the left side of equation (30), we have

$$\begin{aligned}
&\mathbf{e_r}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + i\mathbf{e_t}(\nabla\cdot\mathbf{v}) + i\mathbf{e_t}[\nabla\times\mathbf{v}]\\
&+i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u + \mathbf{e_r}\nabla u = \mathbf{e_r}\mathbf{f}.
\end{aligned} \tag{32}$$

Separating the quantities with different space-time properties, we obtain the following system of equations:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\mathbf{v}+\nabla u=\mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)u+(\nabla\cdot\mathbf{v})=0,\\
&[\nabla\times\mathbf{v}]=0.
\end{aligned}\tag{33}$$

As can be seen, equations (33) coincide with equations (15), (22) and (29).

Using analogy with electrodynamics, a generalized equation describing the dynamics of a fluid can be represented in the form of a sedeonic wave equation for potentials. Let us introduce scalar φ and vector $\mathbf{A}$ potentials according to the following relations:

$$\begin{aligned}
&u=\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\varphi+\nabla\cdot\mathbf{A},\\
&\mathbf{v}=-\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\mathbf{A}-\nabla\varphi,\\
&[\nabla\times\mathbf{A}]=0,
\end{aligned}\tag{34}$$

and denote the operator

$$\widehat{\nabla}=\left\{i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)-\mathbf{e_r}\nabla\right\},\tag{35}$$

then equations (33) are equivalent to the following second-order wave equation:

$$\widehat{\nabla}\widehat{\nabla}\left(i\mathbf{e_t}\varphi+\mathbf{e_r}\mathbf{A}\right)=\mathbf{e_r}\mathbf{f}.\tag{36}$$

Indeed after the first operator action we have

$$\widehat{\nabla}\left(i\mathbf{e_t}\varphi+\mathbf{e_r}A\right)=\mathbf{e_{tr}}\mathbf{v}-u\tag{37}$$

and equation (36) is rewritten as

$$\widehat{\nabla}\left(\mathbf{e_{tr}}\mathbf{v}-u\right)=\mathbf{e_r}\mathbf{f},\tag{38}$$

that coinsides with equation (30).

5 Sedeonic equations for vortex flow

The equation (30) can be generalized for the vortex motion. Let us introduce the vector $\mathbf{w}$ as

$$\mathbf{w}=[\nabla\times\mathbf{A}].\tag{39}$$

Here $\mathbf{w}(\mathbf{r},t)$ is vector field of vortex lines [1] in the fluid

$$\mathbf{w} = c\,2\boldsymbol{\Theta}, \tag{40}$$

where $\boldsymbol{\Theta}$ is the vector of angle of rotation for vortex line. It connected with speed of vortex line rotation $\boldsymbol{\omega}$ [1] as

$$\frac{1}{c}\frac{d\mathbf{w}}{dt} = 2\boldsymbol{\omega}. \tag{41}$$

In this case the relations for potentials are changed as following:

$$\begin{aligned} u &= \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\varphi + \nabla\cdot\mathbf{A}, \\ \mathbf{v} &= -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{A} - \nabla\varphi, \\ \mathbf{w} &= [\nabla\times\mathbf{A}]. \end{aligned} \tag{42}$$

Then we have

$$\widehat{\nabla}\left(i\mathbf{e_t}\varphi + \mathbf{e_r}A\right) = -u + \mathbf{e_{tr}}\mathbf{v} + i\mathbf{w} \tag{43}$$

and the generalized wave equation (30) is rewritten as

$$\widehat{\nabla}\left(-u + \mathbf{e_{tr}}\mathbf{v} + i\mathbf{w}\right) = \mathbf{e_r}\mathbf{f}. \tag{44}$$

This equation is equivalent to the following system:

$$\begin{aligned} &\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u + (\nabla\cdot\mathbf{v}) = 0, \\ &\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = \mathbf{f}, \\ &\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{w} - [\nabla\times\mathbf{v}] = 0, \\ &(\nabla\cdot\mathbf{w}) = 0. \end{aligned} \tag{45}$$

Here the third equation is well known relation between velocity of vortex line rotation $\boldsymbol{\omega}$ and vorticity of linear velocity [1]:

$$2\boldsymbol{\omega} = [\nabla\times\mathbf{v}]. \tag{46}$$

6 Sedeonic equations for viscous fluid

The equation (30) can be generalized for the description of viscous fluid. The viscosity can be taken into account by modifying the operator (35) as

$$\widehat{\nabla}_\nu = \left\{ i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right) - \mathbf{e}_2\nabla \right\}, \tag{47}$$

where ν is the coefficient of kinematic viscosity [28]. Then generalized equation for the viscous vortex flow is

$$\widehat{\nabla}_{\nu}\left(-u+\mathbf{e}_3\mathbf{v}+i\mathbf{w}\right)=\mathbf{e}_2\mathbf{f}. \tag{48}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)-\nu\triangle\right)u+(\nabla\cdot\mathbf{v})=0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)-\nu\triangle\right)\mathbf{v}+\nabla u+[\nabla\times\mathbf{w}]=\mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)-\nu\triangle\right)\mathbf{w}-[\nabla\times\mathbf{v}]=0,\\
&(\nabla\cdot\mathbf{w})=0.
\end{aligned} \tag{49}$$

The sedeonic relation (48) is the generalized Navier-Stokes equation for viscous vortex flow.

The system (49) can be also represented as the second-order wave equation for potentials

$$\widehat{\nabla}_{\nu}\widehat{\nabla}_{\nu}\left(i\mathbf{e}_1\varphi+\mathbf{e}_2\mathbf{A}\right)=\mathbf{e}_2\mathbf{f}, \tag{50}$$

where the relations between parameters $u,\mathbf{v},\mathbf{w}$ and potentials $\varphi,\mathbf{A}$ have the following form:

$$\begin{aligned}
&u=\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)-\nu\triangle\right)\varphi+\nabla\cdot\mathbf{A},\\
&\mathbf{v}=-\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)-\nu\triangle\right)\mathbf{A}-\nabla\varphi,\\
&\mathbf{w}=[\nabla\times\mathbf{A}].
\end{aligned} \tag{51}$$

Besides, the linearized system (49) can be written also for the time derivative of enthalpy $\dot{u}$, linear acceleration $\mathbf{a}$ and speed of rotation $\mathbf{\Omega}$ as

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t}-\nu\triangle\right)\dot{u}+(\nabla\cdot\mathbf{a})=0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}-\nu\triangle\right)\mathbf{a}+\nabla\dot{u}+[\nabla\times\mathbf{\Omega}]=\frac{\partial\mathbf{f}}{\partial\mathbf{t}},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}-\nu\triangle\right)\mathbf{\Omega}-[\nabla\times\mathbf{a}]=0,\\
&(\nabla\cdot\mathbf{\Omega})=0,
\end{aligned} \tag{52}$$

where $\mathbf{\Omega}=c\boldsymbol{\omega}$. The third equation in (52) describes the vortex diffusion. The term $[\nabla\times\mathbf{\Omega}]$ describes the circulation of $\mathbf{\Omega}$ and in particular it is responsible for the generation of toroidal vortex under a force impulse.

7 Sound waves in ideal fluid

Let us consider the sound waves in ideal fluid. In this case we can neglect the convective derivative and sedeonic wave equation (36) is equivalent to the following system

$$\begin{aligned} &\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)\varphi = 0, \\ &\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)\mathbf{A} = \mathbf{f}, \end{aligned} \tag{53}$$

which is similar to the wave equations for electromagnetic field. The parameters $u, \mathbf{v}, \mathbf{w}$ are expressed through the potentials as

$$\begin{aligned} &u = \frac{1}{c}\frac{\partial\varphi}{\partial t} + (\nabla \cdot \mathbf{A}), \\ &\mathbf{v} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi, \\ &\mathbf{w} = [\nabla \times \mathbf{A}], \end{aligned} \tag{54}$$

and the system (45) is rewritten as

$$\begin{aligned} &\frac{1}{c}\frac{\partial u}{\partial t} + (\nabla \cdot \mathbf{v}) = 0, \\ &\frac{1}{c}\frac{\partial \mathbf{v}}{\partial t} + \nabla u + [\nabla \times \mathbf{w}] = \mathbf{f}, \\ &\frac{1}{c}\frac{\partial \mathbf{w}}{\partial t} - [\nabla \times \mathbf{v}] = 0, \\ &(\nabla \cdot \mathbf{w}) = 0. \end{aligned} \tag{55}$$

If we neglect the changes of enthalpy ($u = 0$), which is equivalent to the condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + (\nabla \cdot \mathbf{A}) = 0 \tag{56}$$

similar to Lorentz gauge, then the system (55) is reduced to the Maxwell-like equations

$$\begin{aligned} &\frac{1}{c}\frac{\partial \mathbf{v}}{\partial t} + [\nabla \times \mathbf{w}] = \mathbf{f}, \\ &\frac{1}{c}\frac{\partial \mathbf{w}}{\partial t} - [\nabla \times \mathbf{v}] = 0, \\ &(\nabla \cdot \mathbf{v}) = 0, \\ &(\nabla \cdot \mathbf{w}) = 0. \end{aligned} \tag{57}$$

Multiplying the first two equations by $\mathbf{v}$ and $\mathbf{w}$, respectively, and adding, we obtain

$$\frac{1}{2c}\frac{\partial}{\partial t}(\mathbf{v}^2 + \mathbf{w}^2) + (\nabla \cdot [\mathbf{v} \times \mathbf{w}]) = (\mathbf{v} \cdot \mathbf{f}). \tag{58}$$

This expression is an analogue of the Poynting relation for sound waves in the uncompressible fluid.

Besides, differentiating (57) with respect to time, we obtain the equations relating the angular velocity of the vortex $\mathbf{\Omega}$ with the linear acceleration $\mathbf{a}$ of the fluid in the sound wave

$$\begin{aligned} &\frac{1}{c}\frac{\partial \mathbf{a}}{\partial t} + [\nabla \times \mathbf{\Omega}] = \frac{\partial \mathbf{f}}{\partial t}, \\ &\frac{1}{c}\frac{\partial \mathbf{\Omega}}{\partial t} - [\nabla \times \mathbf{a}] = 0, \\ &(\nabla \cdot \mathbf{a}) = 0, \\ &(\nabla \cdot \mathbf{\Omega}) = 0. \end{aligned} \tag{59}$$

8 Sound waves in viscous fluid

Let us consider the free sound waves in viscous fluid. In this case neglecting the convective derivative the equation (48) is rewritten as

$$\left\{ i\mathbf{e}_1 \frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) - \mathbf{e}_2\nabla \right\}(-u + \mathbf{e}_3\mathbf{v} + i\mathbf{w}) = 0. \tag{60}$$

This equation is equivalent to the following system:

$$\begin{aligned} &\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) u + (\nabla \cdot \mathbf{v}) = 0, \\ &\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) \mathbf{v} + \nabla u + [\nabla \times \mathbf{w}] = 0, \\ &\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) \mathbf{w} - [\nabla \times \mathbf{v}] = 0, \\ &(\nabla \cdot \mathbf{w}) = 0. \end{aligned} \tag{61}$$

Multiplying the first three equations by u, $\mathbf{v}$ and $\mathbf{w}$, respectively, and adding, we obtain

$$\begin{aligned} &\frac{1}{2c}\frac{\partial}{\partial t}(u^2 + \mathbf{v}^2 + \mathbf{w}^2) + (\nabla \cdot (u\mathbf{v} + [\mathbf{v} \times \mathbf{w}])) \\ &-\frac{1}{c}\nu\left(u\triangle u + (\mathbf{v} \cdot \triangle\mathbf{v}) + (\mathbf{w} \cdot \triangle\mathbf{w})\right) = (\mathbf{v} \cdot \mathbf{f}). \end{aligned} \tag{62}$$

This expression is an analogue of the Poynting relation for sound waves in viscous fluid.

The equation (60) has the plane wave solution. Let us find the solutions in the following form:

$$\begin{aligned} u &= u_0 \exp(i\omega t - i(\mathbf{k} \cdot \mathbf{r})), \\ \mathbf{v} &= \mathbf{v}_0 \exp(i\omega t - i(\mathbf{k} \cdot \mathbf{r})), \\ \mathbf{w} &= \mathbf{w}_0 \exp(i\omega t - i(\mathbf{k} \cdot \mathbf{r})), \end{aligned} \tag{63}$$

where $u_0, \mathbf{v}_0, \mathbf{w}_0$ are amplitudes that are independent of coordinates and time, ω is frequency, $\mathbf{k}$ is wave vector. The dispersion relation for the equation (60) is

$$\omega^2 - i2\nu k^2\omega - \nu^2 k^4 - c^2 k^2 = 0, \tag{64}$$

and consequently

$$\omega = \pm ck + i\nu k^2. \tag{65}$$

Here $k = |\mathbf{k}|$. Substituting (63) into the system (61) we have

$$\begin{aligned}
&u_0 = (\mathbf{n}\cdot\mathbf{v}_0),\\
&\mathbf{v}_0 = u_0\mathbf{n} + [\mathbf{n}\times\mathbf{w}_0],\\
&\mathbf{w}_0 = -[\mathbf{n}\times\mathbf{v}_0],\\
&(\mathbf{n}\cdot\mathbf{w}_0) = 0,
\end{aligned} \tag{66}$$

where $\mathbf{n} = \mathbf{k}/k$. In case of vortex-less motion ($\mathbf{w}_0 = 0$) we have

$$\begin{aligned}
&u_0 = (\mathbf{n}\cdot\mathbf{v}_0),\\
&\mathbf{v}_0 = u_0\mathbf{n},
\end{aligned} \tag{67}$$

and in sound wave the vector $\mathbf{v}_0$ is parallel to the vector $\mathbf{k}$. In case of incompressible fluid ($u_0 = 0$) the system (66) is reduced to

$$\begin{aligned}
&(\mathbf{n}\cdot\mathbf{v}_0) = 0,\\
&\mathbf{v}_0 = [\mathbf{n}\times\mathbf{w}_0],\\
&\mathbf{w}_0 = -[\mathbf{n}\times\mathbf{v}_0],\\
&(\mathbf{n}\cdot\mathbf{w}_0) = 0,
\end{aligned} \tag{68}$$

and we have transverse sound wave, where $\mathbf{v}_0 \perp \mathbf{w}_0$.

9 Conclusion

Thus, we proposed the generalized equations of hydrodynamics based on space-time algebra of sedeons. It has been shown that the fluid dynamics can be described by sedeonic second-order wave equation for scalar and vector potentials. In simple model the viscosity was taken into account by modifying the differential operator using $\nu\triangle$ term. The equations for vortex flow have been derived by introducing angle vector $\mathbf{w}$ in sedeonic wave equation. As a result, we have obtained the generalized Navier-Stokes equation in sedeonic form (50) and in equivalent form as the system (49). In system (49) the first equation describes the convection-diffusion [29, 30] of enthalpy. Essentially, it is a continuity equation taking into account the process of self-diffusion in viscous fluid.

The second equation is the diffusion of linear momentum. The term $[\nabla \times \mathbf{w}]$ in this equation is responsible for the vortex line distortion. The third equation describes the vortex diffusion. In case of viscous-less fluid it is well known relation between angle rotation of vortex line and vorticity of linear velocity [1].

As an example we considered the linearized equations for sound waves. It is seen from (66) that in sound wave the vector of vortex lines $\mathbf{w}_0$ always perpendicular to the wave vector $\mathbf{k}$, while vector $\mathbf{v}_0$ has some angle with wave vector $\mathbf{k}$. In case of incompressible fluid the oscillations of fluid are described by Maxwell-like equations (57) and we have transverse sound waves (68) with $\mathbf{w}_0 \times \mathbf{v}_0 = \mathbf{n}$. In this wave, fronts with oppositely directed speed $\mathbf{v}$ alternate with vortex planes with opposite vorticity of $\mathbf{w}$.

The proposed sedeonic equations of hydrodynamics may become the convenient theoretical platform for the further analysis of the complex vortex dynamics and turbulent flows.

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