ARE BORDERS INSIDE OR OUTSIDE?

Arturo Tozzi
Center for Nonlinear Science, Department of Physics, University of North Texas, Denton, Texas 76203, USA
1155 Union Circle, #311427, Denton, TX 76203-5017 USA
tozziarturo@libero.it
Arturo.Tozzi@unt.edu

When a boat disappears over the horizon, does a distant observer detect the last moment in which the boat is visible, or the first moment in which the boat is not visible? This apparently ludicrous way of reasoning, heritage of long-lasting medieval debates on decision limit problems, paves the way to sophisticated contemporary debates concerning the methodological core of mathematics, physics and biology. These ancient, logically-framed conundrums throw us into the realm of bounded objects with fuzzy edges, where our mind fails to provide responses to plain questions such as: given a closed curve with a boundary (say, a cellular membrane) how do you recognize what is internal and what is external? We show how the choice of an alternative instead of another is not arbitrary, rather points towards entirely different ontological, philosophical and physical commitments. This paves the way to novel interpretations and operational approaches to challenging issues such as black hole singularities, continuous time in quantum dynamics, chaotic nonlinear paths, logarithmic plots, demarcation of living beings. In the sceptical reign where judgements seem to be suspended forever, the contemporary scientist stands for a sort of God equipped with infinite power who is utterly free to dictate the rules of the experimental settings.

KEYWORDS: infinity; boundary; Jordan curve theorem; Bradwardine; Nicholas of Autrecourt.

Take an object with a border separating the internal from the external. We might ask: is the boundary inside or outside the object? At the very edge of the closed curve, does the last point of the internal occur, or the first point of the external? Medieval scholars confronted diverse types of these challenges that are currently termed “limit decision problems”. The ancient theologians/philosophers/scientists used the weapons available to them, i.e., Aristotelian analytic arguments and thought experiments, to appraise the limits and degrees of “powers” such as time, length, weight, distance, and so on (Offredus 1478; Nielsen 1982). Here I examine the subtleties of limit decision problems going through apparently silly questions such as: “the present time stands for: the current instant, or the last instant of the past, or the first instant of the future?” It is straightforward to claim that this intricate way of reasoning is perplexing for the baffled modern-day readers. Who cares about weird medieval enterprises such as the search for “*incipit*” and “*desinit*” (i.e., the “beginning” and the “end”)? What is the practical gist of absurd questions such as: “when we look at a boat disappearing over the horizon, does exist the last instant in which we see the boat, or the first instant in which we do not see the boat?” (O’Donnell, 1939). What these apparently idle remarks bring on the operational table of the pragmatic scientific method?
I will show in the sequel that these issues are not as bizarre as one might believe. The quite different solutions to this kind of difficulties were not delivered by medieval scholars just to satisfy a whim, rather to explain, assess and quantify different issues, including physical and biological ones. I will proceed as follows. At first, I will describe the practice of medieval limit decision problems from a general point of view. The ensuing specific chapters will treat physical and biological issues, portraying the medieval (and current) treatments of specific themes such as time, space, lengths and so on. Recounting how the ancient authors considered all options relative to these troublesome issues, my goal is tripartite. My first goal is to show to the baffled reader how the choice of an alternative instead of another is not arbitrary, rather it is points towards fully different ontological, philosophical, physical, and biological commitments. My second goal is to breathe new life into forgotten concepts tackled by Medieval scholars, turning them into precious gifts from the past. My third goal, the most important, is to demonstrate how these treasures from ancient times can be helpful (I would say: game changing) to tackle current mathematical, physical and biological issues under a novel methodological light.

DECISION LIMIT PROBLEMS: AN HISTORICAL LEITMOTIV

Ancient scholars, particularly Oxonians and Parisian middle-age authors, were fond of the subtilities and endeavours of the troublesome problem of limits. **Figure 1, upper part**, provides a general sketch of the challenges they faced. Take a line encompassing a segment bounded by two endpoints. When you evaluate the tiny spots remarkably close to the two endpoints, how do you define them? To make an example, what do you call the endpoint termed “beginning” in **Figure**

2A? Do you call it the last point before the beginning, of do you call it the first point after the beginning? The first move of medieval philosophers/scientists was to consider the problem of time passing, i.e., the *primo et ultimo istanti* (the first and the last instant), and *incipit* and *desinit* (the beginning and the end) (Shapiro and Shapiro, 1965; Heytesbury, 1979). Is this very moment the last instant after the past ones, or is it an instant of the present, or is it the first instant before the future ones? The issue is related to the Aristotelian dilemma: does the present instant cease to be when it is, or when it is not? (Trifogli 2017). Either this instant now is, and immediately afterwards will not be, or this instant now is not, and immediately beforehand it was. Successively, the temporal challenge of the *primo et ultimo istanti* was used to cope with other types of *maximum and minimum* magnitudes such as weights, or spatial distances (Walker, 2018). Here you are a typical set off: "an object cannot be seen from every distance, because there can be a distance so great that the object cannot be seen from it, and because there can be a minimum distance brought closed to the vision from which a visible thing cannot be seen".
Medieval scholars' next step was to relate the limit problem of spatial bodies to gnoseology and epistemology. What about the medium instant between the whole time in which the thing can be seen, and the whole time in which it cannot be seen? In this medium instant, can the thing be seen, or cannot (O'Donnell, 1939)? When a boat disappears over the horizon, what is the instant (o the minimum spatial range) of change between the whole state of visibility and that of non-visibility? The limit problem of spatial bodies became strictly correlated with human perceptions, because it involves what was called "intension and remission" of sensations: What is the smallest amount of matter that permits us to define an object? How much is an object divisible into particles so small that, once these particles are dispersed, the object no longer exercises influence on the senses (Autrecourt, 1939)? What does it mean that an object is too small, or too big, to be detected? The next move of the volitional scholars was to turn to the ranges of forces, such as pressure, or heat, and so on. Can an active capacity (force) be measured against a passive capacity (resistance)? Take a force: there must be a range in which the capacity can act or be acted upon, and another range in which it cannot act or be acted upon, and not both. If you can lift one hundred pounds, you can lift fifty pounds too. Conversely, if you can see a grain of millet from the distance of a mile, you can also see a skyscraper at the same distance. The force under evaluation, they reasoned, should only be able to take a value in the range on which it is measured from zero and the value which serves as its boundary. The last, but not the least, the case of the separation of two surfaces on the assumption of direct contact was frequently appraised as a hypothetical experimental setting invoked to deny the existence of natural and artificial vacuum (Grant, 1973).

To visualize the viable choices and relationships, we will use the informative sketches envisioned by Murdoch (1964). A black dot marks the left or right side of the segment's endpoint, according to the location where the scholar decided to pinpoint the event under examination. For example, **Figure 1, Internal points** says that the author is favouring the choice of internal points: in case of the human lifespan, he means that life goes from the first moment of existence (birth) to the last moment of existence (death). In turn, **Figure 1, External points** says that the author is favouring the choice of external points: he means that life goes from the last moment of non-existence before birth, to the first moment of non-existence after death. The choice depicted in **Figure 1, Left-side** was made by scholars who did not believe in the first instant of motion, rather did believe in the last instant of motion. In turn, **Figure 1, Right-side** was kept by scholars who believed in the first instant of motion but did not believe in the last instant of motion.
All the examples provided so far describe the dividing-point as mandatory located either before or after the segment's endpoint. The dividing-point was assigned by the most of Medieval scholars to one of the sides of the segment, and not to both, due to the principle of non-contradiction: "contradictory propositions cannot both be true at the same time and in the same sense". "If it is yes, it cannot be no": a point cannot be common to both the sides an endpoint. Nevertheless, a few strong-willed Medieval scholars realized that another intriguing possibility was left to their investigations: continuum and infinity (Bradwardine, 1955). Examining the treatment of space, time, and local motion provided by Aristotle's Physics, they grasped that terms such as "begins" and "ceases" have connections with the nature and properties of continua. They understood that the dividing endpoint point can be common to both the sides, so that "the last no" and "the first yes" are superimposed and inextricable (**Figure 1, Neutral**). The same holds for "the last yes" and "the first no". In the sequel, I will examine the medieval account of specific limit problems, such as the first temporal instant, the first moment is which a force is applied, and so on. I will use their lessons to assess contemporary challenges in scientific fields such as mathematics, physics and biology.

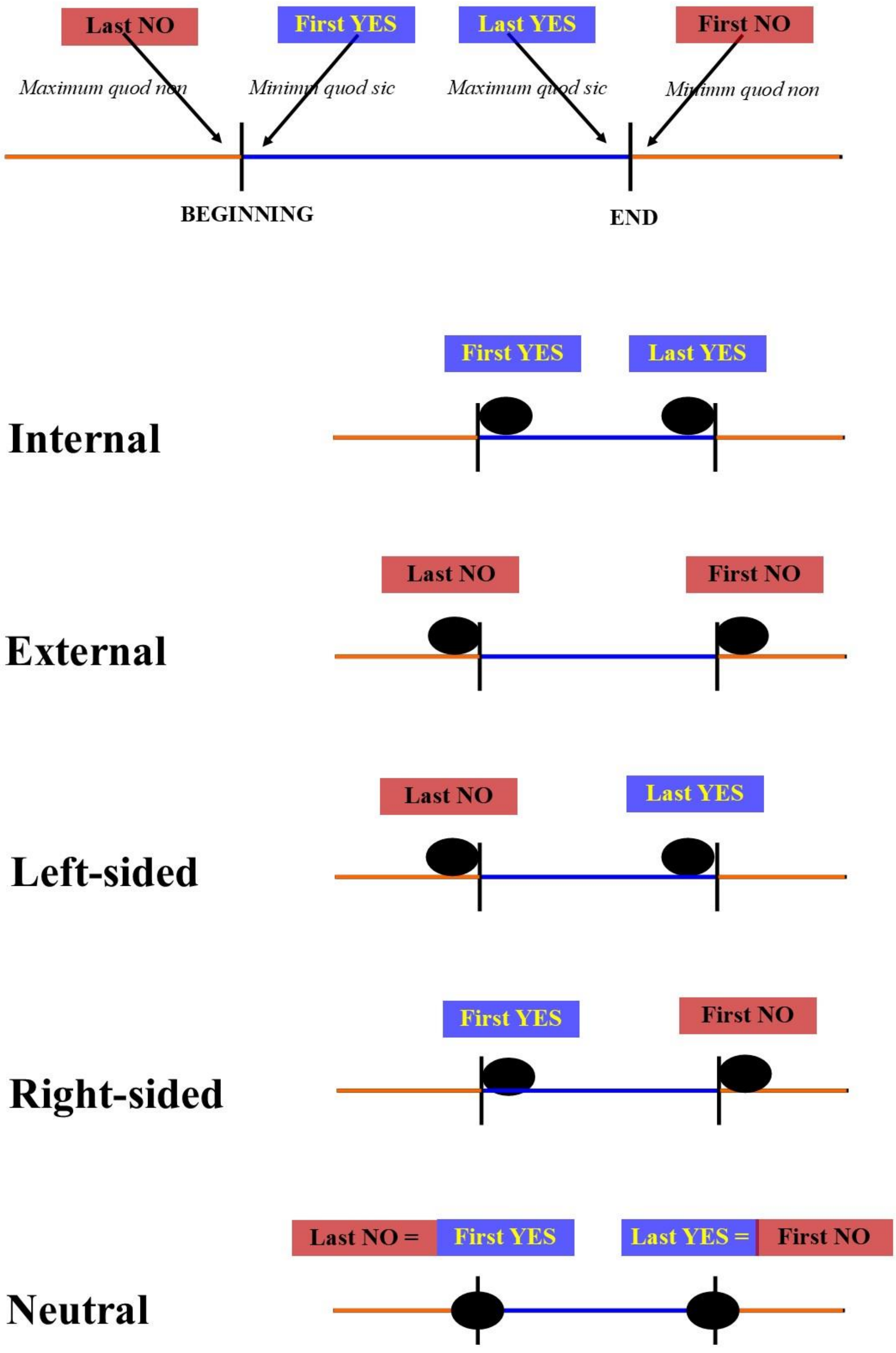

Figure 2. Different kinds of decision limit problems. The **Figure on the top** outlines the overall pattern described in this paper. When you mark with black dots the endpoints (beginning and end) of a given segment (blue line), you are allowed to perform different choices inside the tiny area close to the endpoints: you may choose to consider the last point before the line (termed "the last NO" in logical terms, and "*maximum quod non*" in medieval terminology), and so on. The **Figures below** illustrate the different allowed choices to assess decision limit problems. For sake of clarity, we termed the different assortments of choices with the arbitrary names of Internal, External, Left-sided, Right-sided, Neutral, based on the location of the points close to the segment's endpoints. Every scheme can be used with different meaning in dissimilar scientific contexts. See text for further details.

LESSONS FROM THE PAST 1: CURRENT INSIGHTS IN PHYSICAL ISSUES

Paradoxical properties of the temporal flow uncovered by Medieval scholars might be scientifically valuable. "Beginning" may be expounded by meaning that in the present instant a thing or a process is, and immediately prior to the present instant it was not (**Figure 1, Internal**) (Heytesbury, 1979). "Beginning" may be also expounded in the opposite way (**Figure 1, External**) (Trifogli 2017). In turn, people like Heytesbury (1984) favoured **Figure 1, Left-sided** when evaluating the "beginning" of time, and **Figure 1, Right-sided** when evaluating the "ceasing" of time. There was the logical necessity to ascribe the dividing point to one of the two sides as an attribute of time itself. Likewise, theological concerns pointed towards the endpoint as a discrete entity: points and lines must be immediate to one another and not superimposed, because God's knowledge of them must be clear and distinct (Sylla, 1998). Other scholars like Bradwardine took an opposite way, that is more interesting for our purposes: they declared that time flows continuously (Hannan and Villanova, 2016). If time is continuous, the present instant does not exist, and movements in the past denote movements in the present and the future too (**Figure 1, Neutral**).
This way of reasoning brings us to contemporary issues, suggesting the astounding possibility to treat time in quantum dynamics as a continuous parameter, rather than discrete. It is generally accepted that time is quantized, the smallest quantity of time being the Planck time (Wesson 1980). Nevertheless, the role of time in quantum dynamics is ambiguous: it is noteworthy that time plays a role neither in Wheeler-DeWitt equations, nor in the formulation of entangled states. Moreva et al. (2013) went so far as to state that the Universe is motionless for a hypothetical external observer: time exists just for observers inside the universe, because it is an emergent phenomenon arising from quantum entanglement. Any God-like observer outside the cosmos sees a static, unchanging Universe, just as the Wheeler-DeWitt equations predict. We propose, in touch with relativity that considers time as one of the four coordinates making the fabric of spacetime, to establish time in quantum dynamics in a peculiar way. Each physical system is associated with a (topologically) separable complex Hilbert space H with inner product $\langle \varphi|\psi \rangle$. We suggest to correlate the parameter time with the infinite-dimensional Hilbert space. If we use one of the countless coordinates of the quantum Hilbert space to locate time, we achieve a time that is continuous. But there is even more to it. In touch with Bradwardine, continuous time does not require to a single instant to be separated by the others, because the instant that separates two timeframes is common to both the segments. When Bradwardine (1955) states that "there is not a first instant of being and a first instant of nonbeing because they overlap" (**Figure 1, Neutral**), he makes a leap that will pave the way to infinity in philosophy, mathematics, biophysics… and theology. Once it becomes senseless to talk about past, present, and future, the argument of contingent futures is inadequate to tackle the theological issue of predestination and free will. God's atemporality is opposed to the temporal flow of the universe and humankind, so that statements concerning future states or objects can be verified just retrospectively (Hannan and Villanova, 2016). It is exactly what occurs during quantum measurements: the observer (the scientist, the researcher) cannot predict *a priori* the outcome of the experiment, because the quantum states are noncommutative and superimposed. Unless he finalizes the experiment and achieves decoherence, the researcher cannot be aware of the result: he is allowed, in touch with Bradwardine's analysis, to look at the results just retrospectively.

Defending Aristotle's contention that nature is a material plenum, medieval authors denied the existence of separate, continuously extended vacuum, either small or large (Grant, 1973). They proceeded as follows: if two plane surfaces, initially in uniform mutual contact with no air intervening, are separated in such manner that they remain continuously parallel, a vacuum is temporarily produced, until the newly-formed cavity is progressively filled by the expanding air surrounding the two surfaces. In opposition to this tradition, Blasius of Parma proposed an imaginative solution based on extrinsic boundaries (**Figure 1, External**) which denied the first moment of surfaces' separation (Grant, 1973; Pelacani, 2005). This unusual procedure prevents the occurrence of a vacuum: if the surfaces do not split, vacuum cannot occur. In touch with this line of argument, we turn to a present-day question. How can the event horizon of a black hole be demarcated? By the standpoint of an observer falling inside the black hole, the event horizon is not perceivable at all. In turn, the event horizon is clearly detectable by an observer at rest far from the black hole (Bekenstein, 1973; Gogberashvili, (2019). This subjective fuzziness in coping with boundaries in extreme physics leads us to an even more intricate problem: the singularity inside the black hole (Hawking 2005). What and where are the boundaries of black holes' singularities? If I choose to consider the border as located in the area external to the singularity (**Figure 1, External**), I achieve the conventional view of the black hole as information-sucker. In touch with Blasius de Parma's account of the vacuum, the choice of external limits allows us to exclude the possibility to detect any information located inside the black hole. However, medieval authors also suggest another possibility: what if I consider the black hole's boundaries according to the scheme depicted in **Figure 1, Internal**? In this case, we ponder the singularity in terms of the last internal point, with no relationships with the external. The black hole singularity becomes an entity independent of the surrounding space, possibly leading to wormholes. Studying the interior of rotating black holes coupled to a massless scalar field in asymptotically flat spacetime, Chesler (2019) noticed that the existence of a null singularity at the Cauchy horizon and a central spacelike singularity at radius r=0 is required. In touch with the choice described in **Figure 1, Internal**.

According to Albert of Saxony and Buridan, a moving body is continuously and successively in another place (Thijssen, 2009). Local motion stands for a separate entity different from the moving body and the crossed space, in the same way as the concept of whiteness may be thought despite the absence of a detectable white body. A question (not obvious for us at all) arose: do the celestial spheres move just locally with respect to its parts? If God decides to reverse the circular movement of the sky from West to East, then the local motion will lack an external point of reference. The Universe is not located in a surrounding container that might act as a reference frame, and yet it can move locally. In other words, is it feasible to detect the local motion of an object, when a reference frame in not available to my observation? The answer of medieval scholars such as Oresme was straightforward: yes, it is feasible, because the detection of movement is grounded in human perception. Oresme rejects the movement in relation to an external point of reference: the circular motion of the heaven depends just on the fact that it is another way than it was before (Thijssen, 2009). This "anti-reference" approach, conflicting with the tenets of Einstein's relativity, is amazingly close to recent claims in quantum mechanics. This theory has been reformulated as an approach that describes physical systems in terms of observer-dependent relational properties (Tozzi and Peters, 2019a). This means that the properties of the physical world are reliant on our observation (Rovelli 1996; Smerlak and Rovelli, 2007; Zizzi, 2018). Recent papers support this disconcerting statement, demonstrating that experimentally detected correlations in Bell tests strongly contradict the tenets of local realism (**The BIG Bell Test Collaboration, 2018**). In touch with Oresme's claim, the relational formulations of quantum mechanics strikingly state that features of quantum systems, such as superposition and entanglement, are demonstrated through the rules of counting the alternatives, without explicitly calling out the reference system (Yang 2018).

The choice of **Figure 1, left-sided** permits to physicists and biologists to shed fully novel light on ubiquitous behaviours that subtends countless natural and artificial phenomena: nonlinear and chaotic dynamics. Collective systems equipped with a large number of interacting and inter-dependent components are characterized by non-equilibrium dynamics, self-organized criticality that operates at the edge of chaos, metastable state of second-phase transitions, spontaneous avalanches, universal power laws, and so on (Bak et al, 2012; Perkins et al., 2014). Chaotic systems display dependence from initial conditions, positive Lyapunov exponents, attractors. These systems are apt to live close to a metastable state of second-order phase transitions, characterized by infinite correlation length, countless functional dimensions, spontaneous avalanches and universal power laws (de Arcangelis and Herrmann, 2010; Sherrill et al., 2020). To understand how the choice of **Figure 1, left-sided** fits in the discourse, take the bifurcation plot of logistic maps (**Figure 2A**), a behaviour that is typical of systems at the edge of criticality. The logistic map is a one-dimensional nonlinear difference equation used in dynamic systems theory (Jordan and Smith, 2007; Richardson et al., 2014). It is visualized on a bifurcation diagram as a function of a fixed phase parameter located on the axis x. When the phase parameter increases, a growing number of doubling bifurcations progressively takes place, until a chaotic state is reached (Smith, 2013). What are the boundaries of a single bifurcation? These boundaries cannot occur after the beginning of a bifurcation, because the split gives rise to two, instead of one, lines. Also, the boundaries cannot occur after the end of a bifurcation for the same reason. Therefore, to assess the length of a bifurcation we need to calculate the first point before the beginning of the bifurcation (the last NO in **Figure 2A**) and the last point before the end of the bifurcation (the last YES in **Figure 2A**): this matches the scheme illustrated in **Figure 1, left-sided**.

Similar discrepancies might occur when scientists assess two different graphs: the exponential and the linear plot (**Figure 2B**). While the beginning of both the plots can be described with "the first YES", the end of the two lines displays a different behaviour. The end of the linear line must be described in terms of "the last YES", while the end of the exponential curve must be described in terms of "the first NO" after the plateau. Indeed, the exponential curve converges towards a point external to the highest value in the axis y, but never touches it. Therefore, we can state that the linear dynamics adhere to the scheme in **Figure 3, Internal**, while the exponential dynamics adhere to the scheme in **Figure 3, Right side**.

The *Liber sex principiorum* (the book of the six principles), dated about 1150, describes the Aristotelian cosmic spheres in terms of a structure that closely resembles a four-dimensional torus, i.e., a donut-like structure that has been used to describe the nervous dynamics of the brain activity (Tozzi, 2019). The anonymous author (Porretano, 2009) states that "controversy arises concerning the border of the sphere (i.e, the *Primum Mobile*, the outermost moving sphere in the geocentric model of the universe). Nothing is outside the sphere; however, a locus cannot be given in such a sphere, because the sphere itself is the locus that surrounds; then, once admitted that the sphere is local, it must also be admitted that something does exist outside the sphere, in which the site of the border is located; however, nothing exists outside the sphere border, therefore the sphere is nowhere. Indeed, nothing does exist outside itself; in turn, no place is the place of itself ; once admitted that the sphere is local, it must be also admitted that the extreme border of the sky must be else than itself; however, nothing is beyond itself and, therefore, the border is nowhere. It must be admitted that talking about these arguments is somewhat weird, obscure and against the experience of our senses." If you look at the movements of a hypersphere, you appreciate a strikingly analogy: the same point becomes alternatively external or internal in a rotating hypersphere (**Figure 2C**). A video is available here: "A stereographic projection of a Clifford torus performing a simple rotation" https://en.wikipedia.org/wiki/Clifford_torus#/media/File:Clifford-torus.gif. This means that, in a few cases of formally developed mathematical concepts, it is almost impossible to decide whether a point is internal or external. In other words, it is not feasible to provide a distinction between the cases depicted in **Figure 3, Internal** and **Figure 3, External**. This brings us to the troublesome dilemma of the boundaries, that will be tackled in the next Section.

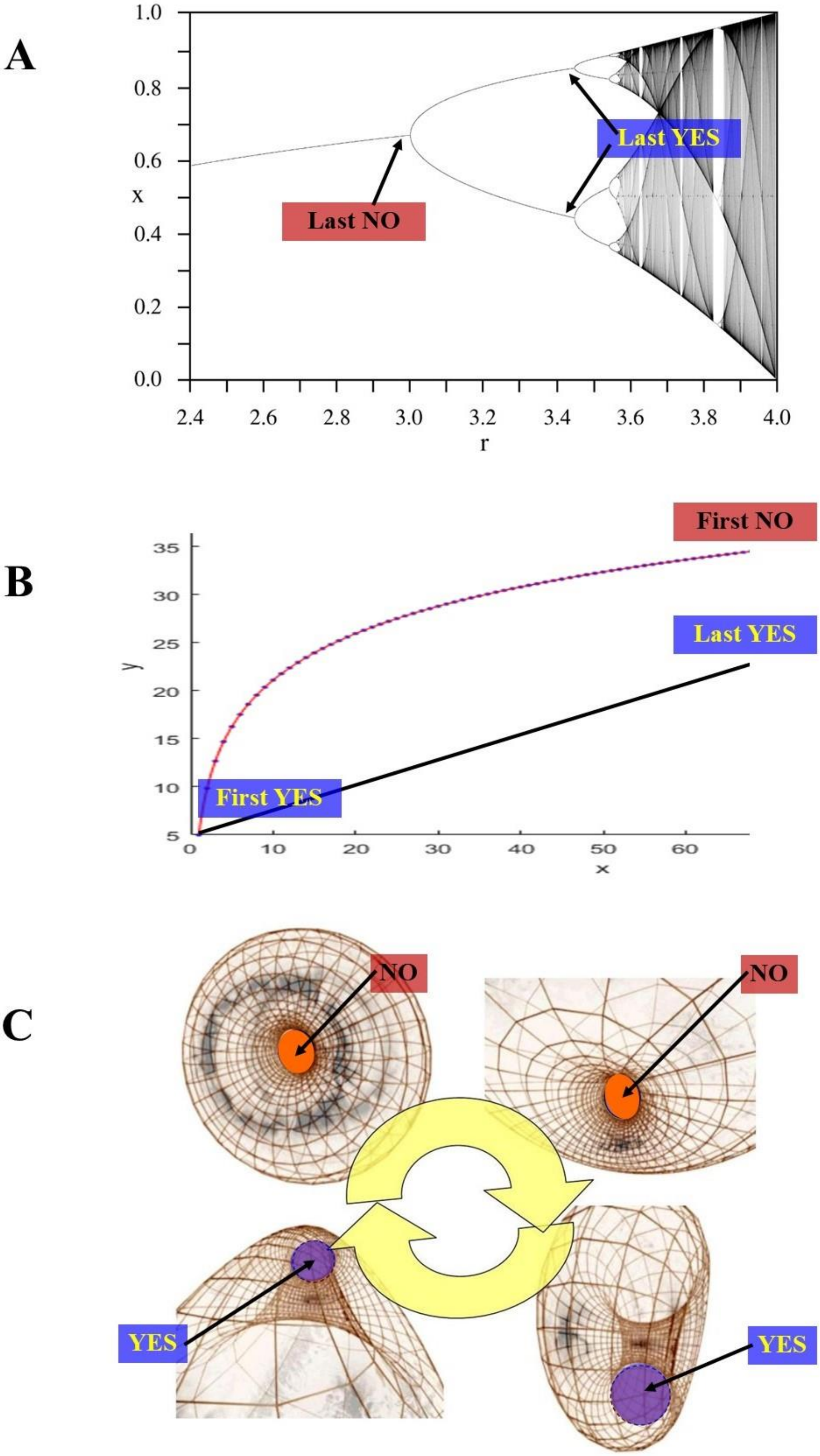

Figure 2. Different possibilities to cope with internal and external boundaries. **Figure 2A** illustrates a logistic plot, **Figure 2B** an exponential (the highest line) and a linear graph. **Figure 2C** depicts the rotations of a four-dimensional Clifford torus: during the movements, the same point (red and blue dot) is alternatively internal and external to the donut-like structure.

LESSONS FROM THE PAST 2: CURRENT INSIGHTS IN BIOLOGICAL ISSUES

Augustine identified the Being with the Good and the Evil with the Not-Being. Hints of this conception are still visible in the current notions of set theory and topology, where the attention is drawn towards the closed, compact, continuous, bounded manifolds, rather than towards fuzzier and untreatable concepts such as open lines and manifolds without boundaries. Despite this reassuring commitment towards simpler concepts, the troubles are not over: if the closed curves are intersections pertinent of both internal and external, how can I make a distinction between inside and outside? If the boundaries can be located both inside and outside, what does make me sure that the interior does not extend throughout the exterior? Or, in turn, why couldn't the external fill the internal? When considering physical and biological features, the issue is related to the indeterminacy of object delimitation and the difficulty to quantify its boundaries. For example, in the case of a porous membrane of a living cell, where does the cytoplasm finish and where the surrounding environment start? Are membrane pores and channels part of the interior of the cell, or of the surrounding environment? These seemingly negligible concerns may be crucial in several fields. Take, e.g., metrology: when you are evaluating a microscopic object, weight measurement might be affected by the surrounding electronic cloud (Tozzi and Papo, 2019). In vertebrates, the central and peripheral nervous system are segregated at the cellular level **(**Suter and Jaworski, 2019), because physical barriers and combinations of attractive and repulsive cues control cell behaviors at the dividing line. However, especially during nervous system development, neurons and signals negotiate the boundary zone: a small number of select cells traverse the boundary and connect the two components **(**Suter and Jaworski, 2019). Boundaries can be regulated through cellular micro-components too: to provide an example, the capability of proteins and nucleic acids to undergo liquid–liquid phase separation is a vital molecular principle that permits cells to quickly and reversibly compartmentalize their components into non–membrane-bound sub-compartments (Hondele et al., 2019; Frottin et al., 2019).

The distinction between internal and external limits provides a subtle, intriguing difference that can be fully appreciated if we consider biological entities. If a scientist favours the approach in **Figure 1, Internal** he is providing internal limits to living beings. Once equipped with internal boundaries, living beings become homeostatic, self-sustaining entities provided with an ontological status that sharply separates them from external environment. Living cell become confined structures split from the surrounding environment. In turn, if a scientist favours the approach in **Figure 1, External** he is suggesting that external limits to living beings occur. The choice of external limits lean towards the scientific/philosophical view of dynamic systems theory (Friston, 2010): moving objects, events, cellular automata and living beings are structures embodied/embedded in the environment, with which they establish relational interactions (Ceruti and Damiano, 2018).
In chemistry, stochastic activation rate is conventionally defined by the mean first passage time (MFPT) of a Brownian particle to a small target. Likewise, in biological gradient/descent apparatuses of Bayesian machines, the stochastic activation rate is set on the mean of the first passage time around the goal, i.e., the local minimum (Constant et al., 2019). Schuss et al. (2017) proposed a different approach in cellular biology, which sets the time scale of activation not anymore to MFPT, rather to the "mean time of the first among many arrivals" of particles at the target. According to this approach, termed Statistics of the extreme, fast activation processes occurring in living cells require a biological activation time that is not anymore necessarily exponential. For example, the fertilization rate does not depend anymore on the number and redundancy of the spermatozoa, rather it depends on the time required by the first spermatozoon to reach the egg (**Figure 1, Internal**). This means that, in the field of optimal random search, just one arrive counts, while the other arrivals are not useful. This also means that an incomplete knowledge of the event does not preclude the possibility to predict the outcome. Here the Lévy's zero–one law comes into play: if we are learning gradually all the information that determines the outcome of an event, then we will become gradually certain what the outcome will be. Certain types of events will either almost surely happen or almost surely not happen, because their probability is either zero or one: e.g., in an infinite sequence of coin-tosses, a sequence of 100 consecutive heads must necessarily occur. Therefore, even if you do not have knowledge of all the collected data of system dynamics (e.g., you cannot know the trajectories of all the spermatozoa), you expect that the results always tend towards either zero, or one. In other words, your statistical knowledge of all the intermediate steps does not matter: a spermatozoon will always either fertilize, or not fertilize, the egg.

The present-day solution of the boundary problem is topological. Given a spherical object, are the boundary points located inside or outside the sphere? In operational terms, when I calculate the area of the interior, must I also include the tiny surface covered by the boundary? The Jordan Curve Theorem (JCT) says that any planar continuous simple closed curve separates the plane in two disjoint regions, i.e., an "interior" region bounded by the curve and an "exterior" region containing all the nearby and far away unbounded points (Jordan, 1893). The boundary is the common element of the bounded and the unbounded components. Leaving apart technicalities, we can plainly state that in topology every point can be considered either internal, or external. The border can be indifferently approached from the inside and the outside of an object. Despite this mathematical designation of the boundary problem, practical difficulties in decision limit problems concerning physical/biological systems are not solved at all (Koczkodaj et al., 2017; Tozzi and Peters, 2019): what is a set, and more importantly, how to provide a boundary to a set? Paraphrasing ancient scholars, we may ask: in topology, does the principle of non-contradiction hold? If the boundary points described by the Jordan theorem are both internal/not internal, the principle of non-contradiction (and also of identity!) breaks down. The problem is also more

distressing, if we consider fractal structures such as the topological Cantor set, which is totally disconnected and is equipped with points that at the same time spread apart and are packed together.
How to get out of this wood? There is a chance and is a good one: to get rid of… points. Here the theory of descriptive nearness comes into play, allowing us to deal with subsets that share some common properties even if they are not spatially close (Di Concilio et al., 2018). Set descriptions arise from the connection between relations on an object space and relations on the corresponding feature space, so that they do not require the use of points as marks to perform operations. Other types of point-free geometries have been used for different purposes, such as: to compute relative stabilities among different experimental structures of organic molecules (Noé et al., 2019), to address the problem of detecting the curvature of a free probing particle inside a magnetic field (Bonalda et al., 2019), to identify the topological holes that continuously appear and disappear in a series of movie frames (Tozzi and Peters, 2020). In case of uncertainty due to lack of information and absence of sharp boundaries, we can delimit the interior through a ciclic chain-wise set of pixels, instead of a conventional boundary line (Don et al., 2020). In these sophisticated approaches, the points are removed and replaced by complex structures that consider topological vortices and Betti numbers. Therefore, the requirement of points and boundaries for the description of systems and phenomena fades away, as indirectly suggested many centuries ago.

CONCLUSION: DIFFERENT CHOICES, DIFFERENT MEANINGS

We showed how medieval approaches to decision limit problems can be actualized, providing fresh insights in ongoing scientific issues. We conclude that in mathematics, physics and biology, the border between the external and the internal is conventionally stipulated for practical and operational purposes. The feasibility to detect, identify and characterize a boundary depends on the technology and equipment available to the researchers. Since the choice of the point's location is arbitrary, the verbs "to start" and "to end" become just relative terms. The attribution of the division to one or another segment is just a logical, not real necessity: in modern terms, we may call it a metalinguistic necessity.
In the 14th Century, Buridan wondered why constant forces have not been detected in nature if they are required for motion (Grant and Nelson, 1962). He noticed that, although these rules (we could say in modern terms: scientific laws) are conditional and true, they are almost never found to produce their effects in the real world. Nevertheless, if the ideal conditions stated in the rules are observed, everything would occur just as the rules assert: this means that the rules are not useless or fictitious. The inference for Buridan is straightforward: although these conditions are not fulfilled by natural powers, it is nevertheless possible to them to be fulfilled by divine power, i.e., by *Potentia Dei infinita*. God sometimes forgets about His *Potentia Dei Ordinata* that rules our physical world and guarantees the regularity of the natural laws. Rather, He sometimes opts for actions out of the ordinary paths of events, such as miracles. Paraphrasing these ancient and forgotten scholars, we could nowadays state that our scientists make the same methodological approach when performing their experiments: mimicking the *Potentia Dei infinita*, they carefully shape a strict experimental setting, thoroughly determining the quantitative parameters according to their own will. Therefore, the experimental results stand for the data achieved by a Divinity (the scientist) who artificially sculpts the environment where the predicted event under examination must take place. Buridan provided an effort to upholding the tenet of the Christian faith of God's omnipotence, while clinging to defensible epistemological position that guarantees the application of human reason. The case of the modern scientist is far from different. According to the concept of *Potentia Dei infinita*, only by means of secondary cause is God (the scientist) able to provoke effects. Scientists are under some obligation, and this serves to eliminating their arbitrary interference in the normal course of events.
The distinction between God's ordained (laws of nature) and absolute powers (strict experimental setting) safeguards the contention that scientists must exercise restraint to guarantee natural certitude, in the meantime preserving the normal course of events. Even if God annihilates an object that the observer is watching, the observer continues to perceive the object. It is exactly what happens when scientists examine quantum systems, perturbing and annihilating them through instrumental measures. Still, there is a difference between the infinite powerful God and the current scientists. One of the limits of medieval God's infinite power was that He must be constrained by the same logical principles that He created by Himself. God Himself cannot escape logic. In turn, scientific investigation is an empirical enterprise that is not automatically framed on logical premises. Scientist are not necessarily exceedingly reasonable and coherent. The same Isaac Newton, who haughtily stated: "*hypotheses non fingo"* ("I do not contrive hypotheses"), spent half of his life looking for historical forecasts in Johannes' Apocalypse, that he thought was a prophetical holy book. Scientists may have exotic intuitions that could, against all odds, be subsequently confirmed by ensuing experiments. Illogicity is sometimes the key for serendipitous discoveries. This means that illogical premises could be scientifically fruitful, if, and only if, scientists are capable to test and confirm them in a painstaking experimental setting.

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