A basic approach to the perfect extensions of spaces

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Abstract. In this paper we generalize the notion of *perfect compactification* of a Tychonoff space to a generic extension of any space by introducing the concept of *perfect pair*. This allow us to simplify the treatment in a basic way and in a more general setting. Some [S_1], [S_2], and [D]'s results are improved and new characterizations for perfect (Hausdorff) extensions of spaces are obtained.

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1. Introduction

The notion of *perfect compactification* of a Tychonoff space was introduced and studied by E.G. Skljarenko since 1961 ([S_1], [S_2]) by using proximal techniques. In [D], B. Diamond gave some additional characterizations of perfectness for compactifications of Tychonoff spaces by using proximities, too.

The aim of this paper is to generalize the notion of perfectness from a Hausdorff compactification of a Tychonoff space to a generic extension of any space by introducing the notion of *perfect pair*. This definition allow us to simplify the treatment in a basic way (without using proximities) and in a more general setting, removing any additional hypothesis about the space.

Thus we are able to improve some Skljarenko and Diamond's results contained in [S_1], [S_2], [D] and to establish new characterizations for perfect (Hausdorff) extensions of spaces.

2. Notation and preliminaries

The word "space" will mean "topological space" on which, unless otherwise specified, no separation axiom is assumed.

If X is a space, $\tau(X)$ will denote the set of open sets of X while $\sigma(X)$ will denote the set of closed sets of X.

Terms and undefined concepts are used as in [E] and [PW].

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Definition. Let Y be a generic extension of a space X and U be an open set of X. We define the *maximal extension of U in Y* and we will denote it by $\langle U\rangle_Y$ (or $\langle U\rangle$ for short) by setting $\langle U\rangle_Y = \bigcup\{V \in \tau(Y) : V \cap X = U\}$.

The main properties of the operator $\langle\cdot\rangle : \tau(X) \to \tau(Y)$ are summarized in the following:

Lemma 2.1. *For every extension Y of X and every pair of open set U, V of X, the following holds:*

(1) $\langle U\rangle = Y\backslash cl_Y(X\backslash U)$;
(2) $U \subseteq V \implies \langle U\rangle \subseteq \langle V\rangle$;
(3) *if $Z \subseteq Y$ is another extension of X, then* $\langle U\rangle_Z = \langle U\rangle_Y \cap Z$;
(4) $\langle U \cap V\rangle = \langle U\rangle \cap \langle V\rangle$;
(5) $\langle U\rangle \subseteq cl_Y(U)$;
(6) $cl_Y(\langle U\rangle) = cl_Y(U)$;
(7) *U is dense in $\langle U\rangle$*;
(8) $bd_Y(\langle U\rangle)\backslash bd_X(U) \subseteq Y\backslash X$;
(9) $bd_X(U) \subseteq bd_Y(\langle U\rangle)$;
(10) $cl_Y(bd_X(U)) \subseteq bd_Y(\langle U\rangle)$.

Lemma 2.2. *Let Y be an extension of X, $U \in \tau(X)$ and $C \in \sigma(Y)$ such that $C \subseteq X$, then:*

(1) $\langle U\rangle = \langle U\backslash C\rangle \cup (U \cap C)$;
(2) $\langle U\backslash C\rangle = \langle U\rangle\backslash C$.

Proof: (1) Since $C = C \cap X \in \sigma(X)$, by 2.1.(4) and 2.1.(1), $\langle U\backslash C\rangle = \langle U\rangle \cap \langle X\backslash C\rangle = \langle U\rangle \cap (Y\backslash C)$ and so $\langle U\backslash C\rangle \cap (Y\backslash X) = \langle U\rangle \cap (Y\backslash X)$. Hence, $\langle U\rangle = (\langle U\rangle \cap (Y\backslash X)) \cup (\langle U\rangle \cap X) = (\langle U\backslash C\rangle \cap (Y\backslash X)) \cup U = (\langle U\backslash C\rangle \cap (Y\backslash X)) \cup ((U\backslash C) \cup (U \cap C)) = (\langle U\backslash C\rangle \cap (Y\backslash X)) \cup ((\langle U\backslash C\rangle \cap X) \cup (U \cap C)) = \langle U\backslash C\rangle \cup (U \cap C)$.

(2) It follows directly from (1) as the sets $\langle U\backslash C\rangle$ and $U \cap C$ are disjoint. □

Lemma 2.3 [D]**.** *If Y is an extension of X and $U, V \in \tau(X)$, then* $\langle U \cup V\rangle\backslash(\langle U\rangle \cup \langle V\rangle) \subseteq cl_Y(U) \cap cl_Y(V) \cap (Y\backslash X)$.

Lemma 2.4. *Let Y be an extension of X, $U \in \tau(X)$ and $V \in \tau(Y)$, then* $\langle U \cap V\rangle_V = \langle U\rangle_Y \cap V$.

Proof: Obviously V is an extension of $V \cap X$ and $U \cap V \in \tau(V) \subseteq \tau(Y)$ implies $\langle U \cap V\rangle_V \subseteq \langle U \cap V\rangle_Y \subseteq \langle U\rangle_Y$ by 2.1.(2). Thus $\langle U \cap V\rangle_V \subseteq \langle U\rangle_Y \cap V$. On the other hand, $U \in \tau(X)$ implies $U \cap V \in \tau(U) \subseteq \tau(X)$. So, being $\langle U\rangle_Y \cap V \in \tau(V)$ and $(\langle U\rangle_Y \cap V) \cap X = U \cap V$, it follows that $\langle U\rangle_Y \cap V \subseteq \langle U \cap V\rangle_V$. This proves the equality. □

Corollary 2.5. *Let Y be an extension of X, $\mathcal{V}$ be an open cover of Y and $U \in \tau(X)$, then* $\langle U\rangle_Y = \bigcup_{V\in\mathcal{V}}\langle U \cap V\rangle_V$.

3. Perfect extensions of arbitrary spaces and their characterizations

Definitions [S$_1$]. Let Y be an extension of a space X.

(i) If U is an open set of X, we say that Y is a *perfect extension of* X *with respect to* U if $cl_Y(bd_X(U)) = bd_Y(\langle U \rangle)$.

(ii) We say that Y is a *perfect extension of* X if it is a perfect extension of X with respect to every open set of X.

Now, we introduce some new definitions closely connected with the previous ones.

Definitions. Let Y be an extension of X, $U \in \tau(X)$ and $x \in Y \backslash X$.

(i) We say that the *pair* (x, U) is *perfect* if $x \in cl_Y(bd_X(U))$ provided $x \in bd_Y(\langle U \rangle)$.

(ii) We say that Y is a *perfect extension of* X *relatively to* U if for every $y \in Y \backslash X$ the pair (y, U) is perfect.

(iii) We say that Y is a *perfect extension of* X *relatively to* x if for every $W \in \tau(X)$ the pair (x, W) is perfect.

Remark. It is clear that Y is a perfect extension of X iff all the pairs (x, U) (with $x \in Y \backslash X$ and $U \in \tau(X)$) are perfect iff Y is a perfect extension of X relatively to any open set of X (any point of the remainder $Y \backslash X$).

Moreover, we give the following definitions.

Definition. Let Y be an extension of X, $U \in \tau(X)$ and $x \in Y \backslash X$. We say that $Y \backslash X$ *cuts* X *at* x *relatively to* U if there exists some O neighbourhood of x in Y and some V open set of X such that $O \cap X = (O \cap U) \cup V$, $(O \cap U) \cap V = \emptyset$ and $x \in cl_Y(O \cap U) \cap cl_Y(V)$.

Note. Obviously in the previous definition it results $U \cap V = \emptyset$.

Definition [S$_1$]. Let X be a space, $F \subseteq X$ and $U, V \in \tau(X)$. We say that F *separates* X *in* U *and* V if $U \cap V = \emptyset$ and $X \backslash F = U \cup V$.

Note. It is clear that in the last definition, F is a closed set of X.

Definition. Let X be a space, $A, C \subseteq X$ and $U, V \in \tau(X)$. We say that the set A C*-separates* X *in* U *and* V if $U \cap V = \emptyset$ and $X \backslash (A \cup C) = U \cup V$.

First we give the following characterization for a perfect pair.

Proposition 3.1. *Let Y be an extension of X, $U \in \tau(X)$ and $x \in Y \backslash X$. The following are equivalent:*

(1) *the pair (x, U) is perfect;*

(2) *$Y \backslash X$ does not cut X at x relatively to U;*

(3) *there is no neighbourhood O of x in Y such that $O \cap X = (O \cap U) \cup (O \cap (X \backslash cl_X(U)))$ and $x \in cl_Y(O \cap U) \cap cl_Y(O \cap (X \backslash cl_X(U)))$;*

(4) *for every $V \in \tau(X)$ such that $U \cap V = \emptyset$, $x \notin \langle U \cup V \rangle \backslash (\langle U \rangle \cup \langle V \rangle)$;*

(5) $x \notin \langle U \cup (X \setminus cl_X(U))\rangle \setminus (\langle U\rangle \cup \langle X \setminus cl_X(U)\rangle)$;

(6) *for every* $V \in \tau(X)$ *such that* $U \cap V = \emptyset$, $x \in cl_Y(X \setminus (U \cup V)) \cup \langle U\rangle \cup \langle V\rangle$;

(7) $x \in cl_Y(bd_X(U)) \cup \langle U\rangle \cup \langle X \setminus cl_X(U)\rangle$;

(8) *for every* $F \in \sigma(Y)$ *such that* $F \subseteq X$, *the pair* $(x, U \setminus F)$ *is perfect;*

(9) *for every* $F \in \sigma(Y)$ *such that* $F \subseteq X$, $Y \setminus X$ *does not cut* X *at* x *relatively to* $U \setminus F$;

(10) *for every* $V \in \tau(X)$ *such that* $cl_Y(U \cap V) \subseteq X$, $x \notin \langle U \cup V\rangle \setminus (\langle U\rangle \cup \langle V\rangle)$;

(11) *for every* $F \in \sigma(X)$ *and* $C \in \sigma(Y)$ *such that* $C \subseteq X$ *and* F *C-separates* X *in* U *and* V, *then* $x \in cl_Y(F) \cup C \cup \langle U\rangle \cup \langle V\rangle$;

(12) *for every* $F \in \sigma(X)$ *which separates* X *in* U *and* V, $x \in cl_Y(F) \cup \langle U\rangle \cup \langle V\rangle$;

(13) *for every* $C \in \sigma(Y)$ *and* $V \in \tau(X)$ *such that* $C \subseteq X$ *and* $(U \cup C) \cap V = \emptyset$, *then* $x \in cl_Y(X \setminus ((U \setminus C) \cup V)) \cup \langle U \setminus C\rangle \cup \langle V\rangle$.

PROOF: First of all, let us observe that the implications (2)⇒(3), (4)⇒(5), (8)⇒(1), (9)⇒(2), (10)⇒(4), (11)⇒(12) and (13)⇒(6) are trivial.

(1)⇒(2) Suppose that the pair (x, U) is perfect and let us observe that if $x \in \langle U\rangle \cup (Y \setminus cl_Y(\langle U\rangle))$, $Y \setminus X$ does not cut X at x relatively to U. In fact, if — by contradiction — there is some O neighbourhood of x in Y and some $V \in \tau(X)$ such that $O \cap X = (O \cap U) \cup V$, $(O \cap U) \cap V = \emptyset$ and $x \in cl_Y(O \cap U) \cap cl_Y(V)$, it follows that $U \cap V = \emptyset$ and by 2.1.(4), $\langle U\rangle \cap \langle V\rangle = \emptyset$. Hence, $\langle U\rangle \cap cl_Y(\langle V\rangle) = \emptyset$ where $x \in cl_Y(V) = cl_Y(\langle V\rangle)$ by 2.1.(6). Thus, $x \notin \langle U\rangle$ and if $x \in Y \setminus cl_Y(\langle U\rangle)$ by 2.1.(2) and (6), we obtain $x \in cl_Y(O \cap U) \subseteq cl_Y(U) = cl_Y(\langle U\rangle)$ which is a contradiction.

So, we have only to consider the case $x \in bd_Y(\langle U\rangle)$. Since the pair (x, U) is perfect, $x \in cl_Y(bd_X(U))$ and if — by contradiction — $Y \setminus X$ cuts X at x relatively to U, i.e. if there is some O neighbourhood of x in Y and some $V \in \tau(X)$ such that $O \cap X = (O \cap U) \cup V$, $(O \cap U) \cap V = \emptyset$ and $x \in cl_Y(O \cap U) \cap cl_Y(V)$, it follows that $O \cap bd_X(U) = O \cap X \cap bd_X(U) = ((O \cap U) \cup V) \cap bd_X(U) \subseteq (U \cup V) \cap bd_X(U) = V \cap bd_X(U) \subseteq V \cap cl_X(U) = \emptyset$ and so $x \notin cl_Y(bd_X(U))$. A contradiction which proves that $Y \setminus X$ does not cut X at x relatively to U.

(3)⇒(4) Let $V \in \tau(X)$ such that $U \cap V = \emptyset$. If, by contradiction, $x \in \langle U \cup V\rangle \setminus (\langle U\rangle \cup \langle V\rangle)$, by 2.3., $x \in cl_Y(U) \cap cl_Y(V)$. Now, from $U \cap V = \emptyset$ follows $V \subseteq X \setminus cl_X(U) = V'$ with $V' \in \tau(X)$ and so $O = \langle U \cup V'\rangle$ is a neighbourhood of x in Y such that $O \cap X = U \cup V'$, $O \cap U = U$, $O \cap V' = V'$ and $O \cap X = (O \cap U) \cup (O \cap (X \setminus cl_X(U)))$. Further, $x \in cl_Y(U) = cl_Y(O \cap U)$ and $x \in cl_Y(V) \subseteq cl_Y(V') = cl_Y(O \cap V') = cl_Y(O \cap (X \setminus cl_X(U)))$ imply $x \in cl_Y(O \cap U) \cap cl_Y(O \cap (X \setminus cl_X(U)))$ which is a contradiction to (3).

(5)⇒(6) Suppose that $x \notin \langle U \cup (X \setminus cl_X(U))\rangle \setminus (\langle U\rangle \cup \langle X \setminus cl_X(U)\rangle)$ and — by contradiction — that there exists some $V \in \tau(X)$ such that $U \cap V = \emptyset$ and $x \notin cl_Y(X \setminus (U \cup V)) \cup \langle U\rangle \cup \langle V\rangle$. So, from $x \notin cl_Y(X \setminus (U \cup V))$ follows that there is some W neighboourhood of x in Y such that $W \cap cl_Y(X \setminus (U \cup V)) = \emptyset$. Hence, $(W \cap X) \setminus (U \cup V) = \emptyset$ implies $W \cap X \subseteq U \cup V$. So, by definition of maximal extension and 2.1.(2), we obtain $x \in W \subseteq \langle W \cap X\rangle \subseteq \langle U \cup V\rangle$. Further, from $U \cap V = \emptyset$ follows $V \subseteq X \setminus cl_X(U)$ and again, by 2.1.(2), $x \in$

$\langle U \cup (X\backslash cl_X(U))\rangle$. Since $x \in \langle U \cup V\rangle\backslash(\langle U\rangle \cup \langle V\rangle)$, by 2.3. and 2.1.(6), we have that $x \in cl_Y(U) \cap cl_Y(V) = cl_Y(\langle U\rangle) \cap cl_Y(\langle V\rangle)$. On the other hand, by 2.1(4), $U \cap V = \emptyset$ implies $\langle U\rangle \cap \langle V\rangle = \emptyset$ and $\langle U\rangle \cap cl_Y(\langle V\rangle) = \emptyset$. So, $x \notin \langle U\rangle$. Moreover, from $U \cap (X\backslash cl_X(U)) = \emptyset$ we obtain $\langle U\rangle \cap \langle X\backslash cl_X(U)\rangle = \emptyset$ and by 2.1.(4) follows $cl_Y(\langle U\rangle) \cap \langle X\backslash cl_X(U)\rangle = \emptyset$ and $x \notin \langle X\backslash cl_X(U)\rangle$. Thus $x \in \langle U \cup (X\backslash cl_X(U))\rangle\backslash (\langle U\rangle \cup \langle X\backslash cl_X(U)\rangle)$. A contradiction to (5).

(6)⇒(7) It suffices to put $V = X\backslash cl_X(U)$ and observe that $bd_X(U) = X\backslash(U \cup V)$.

(7)⇒(1) Let $x \in bd_Y(\langle U\rangle)$. Obviously $x \notin \langle U\rangle$. Furthermore, being $U \cap (X\backslash cl_X(U)) = \emptyset$, by 2.1.(4) we obtain $\langle U\rangle \cap \langle X\backslash cl_X(U)\rangle = \emptyset$ and $bd_Y(\langle U\rangle) \cap (X\backslash cl_X(U)) = \emptyset$ which implies that $x \notin \langle X\backslash cl_X(U)\rangle$. So, as from (7), $x \in cl_Y(bd_X(U)) \cup \langle U\rangle \cup \langle X\backslash cl_X(U)\rangle$, it follows that $x \in cl_Y(bd_X(U))$ and this proves that the pair (x, U) is perfect.

(1)⇒(8) Suppose (x, U) be perfect and let $F \in \sigma(Y)$ such that $F \subseteq X$. Obviously $x \notin F$, $F = F \cap X \in \sigma(X)$ and $U\backslash F \in \tau(X)$. So, if $x \in bd_Y(\langle U\backslash F\rangle)$, by 2.2.(2), $x \in bd_Y(\langle U\rangle)\backslash F$ and this leads to $x \in bd_Y(\langle U\rangle)$. By perfectness of (x, U), $x \in cl_Y(bd_X(U))$ and being clearly $bd_X(U) \subseteq F \cup bd_X(U\backslash F)$, it follows that $x \in F \cup cl_Y(bd_X(U\backslash F))$ which implies $x \in cl_Y(bd_X(U\backslash F))$ and proves that the pair $(x, U\backslash F)$ is perfect.

(2)⇒(9) Suppose that $Y\backslash X$ does not cut X at x relatively to U and let $F \in \sigma(Y)$ such that $F \subseteq X$. If, by contradiction, $Y\backslash X$ cuts X at x relatively to $U\backslash F$, i.e. if there exists some O neighbourhood of x in Y and some $V \in \tau(Y)$ such that $O \cap X = (O \cap (U\backslash F)) \cup V$, it is clear that $(U\backslash F) \cap V = \emptyset$. Now, $O' = O\backslash F$ is a neighbourhood of x in Y and $V' = V\backslash F$ is an open set of Y such that $O' \cap X = (O\backslash F) \cap X = (O \cap X)\backslash F = ((O \cap (U\backslash F)) \cup V)\backslash F = (((O\backslash F) \cap U) \cup V)\backslash F = ((O' \cap U) \cup V)\backslash F = (O' \cap U) \cup (V\backslash F) = (O' \cap U) \cup V'$. Since $x \in cl_Y(V)$ and $x \notin F \in \sigma(Y)$, $x \in cl_Y(V\backslash F) = cl_Y(V')$ and as $x \in cl_Y(O \cap (U\backslash F)) = cl_Y((O\backslash F) \cap U) = cl_Y(O' \cap U)$, it follows that $x \in cl_Y(O' \cap U) \cap cl_Y(V')$ which means that $Y\backslash X$ cuts X at x relatively to U. A contradiction.

(4)⇒(10) Let $F = cl_Y(U \cup V) \subseteq X$. Then $x \notin F = F \cap X \in \sigma(X)$ Hence, $U' = U\backslash F$ and $V' = V\backslash F$ are two disjoint open sets of X and by (4), $x \notin \langle U' \cup V'\rangle\backslash(\langle U'\rangle \cup \langle V'\rangle)$. So, by 2.2.(2), $\langle U'\rangle = \langle U\rangle\backslash F$, $\langle V'\rangle = \langle V\rangle\backslash F$ and $\langle U' \cup V'\rangle = \langle U \cup V\rangle\backslash F$. Thus, $\langle U' \cup V'\rangle\backslash(\langle U'\rangle \cup \langle V'\rangle) = (\langle U \cup V\rangle\backslash(\langle U\rangle \cup \langle V\rangle))\backslash F$ and as $x \notin F$ this implies that $x \notin \langle U \cup V\rangle\backslash(\langle U\rangle \cup \langle V\rangle)$.

(6)⇒(11) It is obvious, because if F C-separates X in U and V, i.e. if $X\backslash(F \cup C) = U \cup V$ and $U \cap V = \emptyset$, by (6) it follows — in particular — that $x \in cl_Y(X\backslash(U \cup V)) \cup \langle U\rangle \cup \langle V\rangle$, i.e. that $x \in cl_Y(F) \cup C \cup \langle U\rangle \cup \langle V\rangle$.

(12)⇒(6) If $U \cap V = \emptyset$, it is clear that $F = X\backslash(U \cup V)$, F separates X in U and V and hence by (12), $x \in cl_Y(X\backslash(U \cup V)) \cup \langle U\rangle \cup \langle V\rangle$.

(6)⇒(13) Let $C \in \sigma(Y)$, $V \in \tau(X)$ such that $C \subseteq X$ and $(U \cup C) \cap V = \emptyset$. Let us suppose that $x \notin \langle U\backslash C\rangle \cup \langle V\rangle$. Since $U \cap V = \emptyset$, by (6) we have $x \in cl_Y(X\backslash(U \cup V)) \cup \langle U\rangle \cup \langle V\rangle$ and so that $x \in (cl_Y(X\backslash(U \cup V)) \cup \langle U\rangle \cup \langle V\rangle)\backslash(\langle U\backslash C\rangle \cup \langle V\rangle) =$ by

2.1.(1) $= ((Y\backslash\langle U\cup V\rangle)\cup\langle U\rangle\cup\langle V\rangle)\backslash(\langle U\backslash C\rangle\cup\langle V\rangle) = (Y\backslash\langle U\cup V\rangle)\cup(\langle U\rangle\backslash\langle U\backslash C\rangle) =$ by 2.2.(1) $= (Y\backslash\langle U\cup V\rangle)\cup(U\cap C)$. Hence, being $x\notin C$, it follows that $x\in(Y\backslash\langle U\cup V\rangle)\backslash C = Y\backslash(\langle U\cup V\rangle\backslash C) =$ by 2.2.(2) $= Y\backslash\langle(U\cup V)\backslash C\rangle =$ by 2.1.(1) $= cl_Y(X\backslash((U\cup V)\backslash C)) = cl_Y(X\backslash((U\backslash C)\cup(V\backslash C))) = cl_Y(X\backslash((U\backslash C)\cup V))$ which proves (13). □

Since, by definition, Y is a perfect extension of X relatively to $U\in\tau(X)$ if and only if for every $x\in Y\backslash X$ the pair (x,U) is perfect, from the correspondent points in 3.1., we have immediately the following characterization for a perfect extension of a space relatively to a fixed open set.

Proposition 3.2. *Let Y be an extension of X and $U\in\tau(X)$. The following are equivalent:*

(1) *Y is a perfect extension of X relatively to U;*
(2) *$Y\backslash X$ does not cut X at any point of $Y\backslash X$ relatively to U;*
(3) *for any $x\in Y\backslash X$ there is no neighbourhood O of x in Y such that $O\cap X = (O\cap U)\cup(O\cap(X\backslash cl_X(U)))$ and $x\in cl_Y(O\cap U)\cap cl_Y(O\cap(X\backslash cl_X(U)))$;*
(4) *for every $V\in\tau(X)$ such that $U\cap V=\emptyset$, $\langle U\cup V\rangle = \langle U\rangle\cup\langle V\rangle$;*
(5) *$\langle U\cup(X\backslash cl_X(U))\rangle = \langle U\rangle\cup\langle X\backslash cl_X(U)\rangle$;*
(6) *for every $V\in\tau(X)$ such that $U\cap V=\emptyset$, $cl_Y(X\backslash(U\cup V))$ separates Y in $\langle U\rangle$ and $\langle V\rangle$;*
(7) *$cl_Y(bd_X(U))$ separates Y in $\langle U\rangle$ and $\langle X\backslash cl_X(U)\rangle$;*
(8) *for every $F\in\sigma(Y)$ such that $F\subseteq X$, $Y\backslash X$ is a perfect extension of X relatively to $U\backslash F$;*
(9) *for every $F\in\sigma(Y)$ such that $F\subseteq X$, $Y\backslash X$ does not cut X at any point of $Y\backslash X$ relatively to $U\backslash F$;*
(10) *for every $V\in\tau(X)$ such that $cl_Y(U\cap V)\subseteq X$, $\langle U\cup V\rangle = \langle U\rangle\cup\langle V\rangle$;*
(11) *for every $F\in\sigma(X)$ and $C\in\sigma(Y)$ such that $C\subseteq X$ and F C-separates X in U and V, $cl_Y(F)$ C-separates Y in $\langle U\rangle$ and $\langle V\rangle$;*
(12) *for every $F\in\sigma(X)$ which separates X in U and V, $cl_Y(F)$ separates Y in $\langle U\rangle$ and $\langle V\rangle$;*
(13) *for every $C\in\sigma(Y)$ and $V\in\tau(X)$ such that $C\subseteq X$ and $(U\cup C)\cap V=\emptyset$, $cl_Y(X\backslash((U\backslash C)\cup V))$ separates Y in $\langle U\backslash C\rangle$ and $\langle V\rangle$.*

Definition [S_1]. Let Y be an extension of X and $x\in Y\backslash X$. We say that $Y\backslash X$ *cuts* (= *separates* in [S_1]) X *at* x if there exists some O neighbourhood of x in Y and a pair U,V of disjoint open sets of X such that $O\cap X = U\cup V$ and $x\in cl_Y(U)\cap cl_Y(V)$.

Lemma 3.3. *Let Y be an extension of X and $x\in Y\backslash X$, then $Y\backslash X$ does not cut X at x iff $Y\backslash X$ does not cut X at x relatively to any open set of X.*

Proof: ($\Longrightarrow$) If $Y\backslash X$ does not cut X at x and, by contradiction, $Y\backslash X$ cuts X at x relatively to some $U\in\tau(X)$, we have that there are some O neighbourhood of x in Y and some $V\in\tau(X)$ such that $O\cap X=(O\cap U)\cup V$, $(O\cap U)\cap V=\emptyset$ and $x\in cl_Y(O\cap U)\cap cl_Y(V)$. Since $U\in\tau(X)$, $U'=O\cap U\in\tau(U)\subseteq\tau(X)$. So, it results $O \cap X = U' \cup V$, $U' \cap V = \emptyset$ and $x \in cl_Y(U') \cap cl_Y(V)$, that is $Y \backslash X$ cuts X at x. A contradiction.

($\Longleftarrow$) Suppose that $Y \backslash X$ does not cut X at x relatively to any $U \in \tau(X)$. If, by contradiction, $Y \backslash X$ cuts X at x, i.e. there are a neighbourhood O of x in Y and $U, V \in \tau(X)$ such that $O \cap X = U \cup V$, $U \cap V = \emptyset$ and $x \in cl_Y(U) \cap cl_Y(V)$, it suffices to observe that $O \cap U = U$ to conclude that $Y \backslash X$ cuts X at x relatively to U obtaining a contradiction. □

Now, using 3.1. and 3.3. (only for the equivalence (1) $\Leftrightarrow$ (2)), we are able to give a characterization of a perfect extension of a space relatively to some point of its remainder.

Proposition 3.4. *Let Y be an extension of X and $x \in Y \backslash X$. The following are equivalent:*

(1) *$Y \backslash X$ is a perfect extension of X relatively to x;*
(2) *$Y \backslash X$ does not cut X at x;*
(3) *for any $U \in \tau(X)$ there is no neighbourhood O of x in Y such that $O \cap X = (O \cap U) \cup (O \cap (X \backslash cl_X(U)))$ and $x \in cl_Y(O \cap U) \cap cl_Y(O \cap (X \backslash cl_X(U)))$;*
(4) *for every pair U, V of disjoint open sets of X, $x \notin \langle U \cup V \rangle \backslash (\langle U \rangle \cup \langle V \rangle)$;*
(5) *for every $U \in \tau(X)$, $x \notin \langle U \cup (X \backslash cl_X(U)) \rangle \backslash (\langle U \rangle \cup \langle X \backslash cl_X(U) \rangle)$;*
(6) *for any pair U, V of disjoint open sets of X, $x \in cl_Y(X \backslash (U \cup V)) \cup \langle U \rangle \cup \langle V \rangle$;*
(7) *for every $U \in \tau(X)$, $x \in cl_Y(bd_X(U)) \cup \langle U \rangle \cup \langle X \backslash cl_X(U) \rangle$;*
(8) *for every $U \in \tau(X)$ and $F \in \sigma(Y)$ such that $F \subseteq X$, the pair $(x, U \backslash F)$ is perfect;*
(9) *for every $U \in \tau(X)$ and $F \in \sigma(Y)$ such that $F \subseteq X$, $Y \backslash X$ does not cut X at x relatively to $U \backslash F$;*
(10) *for every $U, V \in \tau(X)$ such that $cl_Y(U \cap V) \subseteq X$, $x \notin \langle U \cup V \rangle \backslash (\langle U \rangle \cup \langle V \rangle)$;*
(11) *for every $F \in \sigma(X)$ and $C \in \sigma(Y)$ such that $C \subseteq X$ and F C-separates X in U and V $x \in cl_Y(F) \cup C \cup \langle U \rangle \cup \langle V \rangle$;*
(12) *for every $F \in \sigma(X)$ which separates X in U and V, $x \in cl_Y(F) \cup \langle U \rangle \cup \langle V \rangle$;*
(13) *for every $C \in \sigma(Y)$ and $U, V \in \tau(X)$ such that $C \subseteq X$ and $(U \cup C) \cap V = \emptyset$, $x \in cl_Y(X \backslash ((U \backslash C) \cup V)) \cup \langle U \backslash C \rangle \cup \langle V \rangle$.*

The following characterization of a perfect extension of a space is again a direct consequence of the main Proposition 3.1. and of the Lemma 3.3.

Proposition 3.5. *Let Y be an extension of X. The following are equivalent:*

(1) *Y is a perfect extension of X;*
(2) *$Y \backslash X$ does not cut X at any point of $Y \backslash X$;*
(3) *for every $U \in \tau(X)$ and $x \in Y \backslash X$ there is no neighbourhood O of x in Y such that $O \cap X = (O \cap U) \cup (O \cap (X \backslash cl_X(U)))$ and $x \in cl_Y(O \cap U) \cap cl_Y(O \cap (X \backslash cl_X(U)))$;*
(4) *for every pair U, V of disjoint open sets of X, $\langle U \cup V \rangle = \langle U \rangle \cup \langle V \rangle$;*
(5) *for every $U \in \tau(X)$, $\langle U \cup (X \backslash cl_X(U)) \rangle = \langle U \rangle \cup \langle X \backslash cl_X(U) \rangle$;*

(6) *for every pair* U, V *of disjoint open sets of* X, $cl_Y(X\backslash(U \cup V))$ *separates* Y *in* $\langle U\rangle$ *and* $\langle V\rangle$;
(7) *for every* $U \in \tau(X)$, $cl_Y(bd_X(U))$ *separates* Y *in* $\langle U\rangle$ *and* $\langle X\backslash cl_X(U)\rangle$;
(8) *for every* $U \in \tau(X)$ *and* $F \in \sigma(Y)$ *such that* $F \subseteq X$, Y *is a perfect extension of* X *relatively to* $U\backslash F$;
(9) *for every* $U \in \tau(X)$ *and* $F \in \sigma(Y)$ *such that* $F \subseteq X$, $Y\backslash X$ *does not cut* X *at any point of* $Y\backslash X$ *relatively to* $U\backslash F$;
(10) *for every* $U, V \in \tau(X)$ *such that* $cl_Y(U \cap V) \subseteq X$, $\langle U \cup V\rangle = \langle U\rangle \cup \langle V\rangle$;
(11) *for every* $F \in \sigma(X)$ *and* $C \in \sigma(Y)$ *such that* $C \subseteq X$ *and* F C*-separates* X *in* U *and* V, $cl_Y(F)$ C*-separates* Y *in* $\langle U\rangle$ *and* $\langle V\rangle$;
(12) *for every* $F \in \sigma(X)$ *which separates* X *in* U *and* V, $cl_Y(F)$ *separates* Y *in* $\langle U\rangle$ *and* $\langle V\rangle$;
(13) *for every* $C \in \sigma(Y)$ *and* $U, V \in \tau(X)$ *such that* $C \subseteq X$ *and* $(U \cup C) \cap V = \emptyset$, $cl_Y(X\backslash((U\backslash C) \cup V))$ *separates* Y *in* $\langle U\backslash C\rangle$ *and* $\langle V\rangle$.

Remark. The last proposition improves some results found by Skljarenko in [S_1] and by Diamond in [D]. In particular, the equivalence (1) $\Leftrightarrow$ (4) was given by Skljarenko only for the Stone-Cěch compactification of a normal space and by Diamond only for a generic compactification of a Tychonoff space. Moreover, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (12) were obtained in [S_1] for compactifications of Tychonoff spaces by using proximities.

4. Applications and other properties

We conclude with some applications of the Propositions 3.2. and 3.5. Also, we establish a characterization for the T_2 perfect extensions which improves and generalizes an analogous result for the compactifications of Tychonoff spaces given by Diamond in [D].

Proposition 4.1. *If* Y *is a perfect extension of* X *and* Z *be a space such that* $X \subseteq Z \subseteq Y$, *then* Z *is a perfect extension of* X, *too.*

Proof: Obviously X is dense in Z, i.e. Z is an extension of X. Moreover, for every pair U, V of disjoint open sets of X, as Y is a perfect extension of X, by 2.1.(3) and 3.5.(4), we have that $\langle U \cup V\rangle_Z = \langle U \cup V\rangle_Y \cap Z = (\langle U\rangle_Y \cup \langle V\rangle_Y) \cap Z = (\langle U\rangle_Y \cap Z) \cup (\langle V\rangle_Y \cap Z) = \langle U\rangle_Z \cup \langle V\rangle_Z$ and so, by 3.5.(4), it follows that Z is a perfect extension of X. □

Proposition 4.2. *Let* Y *be an extension of a space* X *and* $U \in \tau(X)$. *The following are equivalent:*

(1) Y *is a perfect extension of* X *relatively to* U;
(2) *every* $V \in \tau(Y)$ *is a perfect extension of* $X \cap V$ *relatively to* $U \cap V$;
(3) *for every* $\mathcal{V}$ *open cover of* Y, *any* $V \in \mathcal{V}$ *is a perfect extension of* $X \cap V$ *relatively to* $U \cap V$;
(4) *there exists some* $\mathcal{V}$ *open cover of* Y *such that every* $V \in \mathcal{V}$ *is a perfect extension of* $X \cap V$ *relatively to* $U \cap V$.

Proof: (1)⇒(2) Suppose that Y is a perfect extension of X relatively to U and let $V \in \tau(Y)$. Then, for every $W \in \tau(X \cap V)$ such that $W \cap (U \cap V) = \emptyset$, it results $W = W' \cap V$ for some $W' \in \tau(X)$. Since $W' \cap U = \emptyset$, by 2.4. and 3.2.(4), we have that $\langle W \cup (U \cap V)\rangle_V = \langle (W' \cup U) \cap V\rangle_V = \langle W' \cup U\rangle_Y \cap V = (\langle W'\rangle_Y \cup \langle U\rangle_Y) \cap V = (\langle W'\rangle_Y \cap V) \cup (\langle U\rangle_Y \cap V) = \langle W' \cap V\rangle_V \cup \langle U \cap V\rangle_V = \langle W\rangle_V \cup \langle U \cap V\rangle_V$ and again by 3.2.(4), this means that V is a perfect extension of $X \cap V$ relatively to $U \cap V$.

(2)⇒(3) Trivial.

(3)⇒(4) It suffices to consider $\mathcal{V} = \{Y\}$.

(4)⇒(1) Let $\mathcal{V}$ be an open cover of Y such that every $V \in \mathcal{V}$ is a perfect extension of $X \cap V$ relatively to $U \cap V$. Then, for every $W \in \tau(X)$ such that $W \cap U = \emptyset$ it is clear that for any $V \in \mathcal{V}$, $W \cap V$ and $U \cap V$ are two disjoint open sets of V. So, by 2.5. and 3.2.(4), it results $\langle W \cup U\rangle_Y = \bigcup_{V \in \mathcal{V}} \langle (W \cup U) \cap V\rangle_V = \bigcup_{V \in \mathcal{V}} \langle (W \cap V) \cup (U \cap V)\rangle_V = \bigcup_{V \in \mathcal{V}} (\langle W \cap V\rangle_V \cup \langle U \cap V\rangle_V) = \left(\bigcup_{V \in \mathcal{V}} \langle W \cap V\rangle_V\right) \cup \left(\bigcup_{V \in \mathcal{V}} \langle U \cap V\rangle_V\right) = \langle W\rangle_Y \cup \langle U\rangle_Y$ and by 3.2.(4) we can conclude that Y is a perfect extension of X relatively to U. □

Corollary 4.3. *Let Y be an extension of a space X. The following are equivalent:*

1. *Y is a perfect extension of X;*
2. *every $V \in \tau(Y)$ is a perfect extension of $X \cap V$;*
3. *for every $\mathcal{V}$ open cover of Y, any $V \in \mathcal{V}$ is a perfect extension of $X \cap V$;*
4. *there exists some $\mathcal{V}$ open cover of Y such that every $V \in \mathcal{V}$ is a perfect extension of $X \cap V$.*

In order to obtain a stronger version of the Proposition 3.5. for the Hausdorff perfect extensions, we give the following:

Definition. Let Y be an extension of X and $x \in Y \setminus X$. We say that $Y \setminus X$ *c-cuts* ($\equiv$ *cuts by a compact set*) X *at* x if there exists some O neighbourhood of x in Y, a compact set $K \subseteq X$ and a pair of disjoint open sets U, V of X such that $(O \setminus K) \cap X = U \cup V$ and $x \in cl_Y(U) \cap cl_Y(V)$.

Remark. Obviously, if $Y \setminus X$ cuts X in some point $x \in Y \setminus X$, then $Y \setminus X$ c-cuts X in the same point x. The converse in general is false, but for Hausdorff extensions we have the following result:

Proposition 4.4. *Let Y be a Hausdorff extension of X and $x \in Y \setminus X$. Then $Y \setminus X$ cuts X at x iff $Y \setminus X$ c-cuts X at x.*

Proof: By the previous remark we need only to prove the second implication. Let us suppose that $Y \setminus X$ c-cuts X at x, i.e. that there exist a neighbourhood O of x in Y, a compact set $K \subseteq X$ and two disjoint open subsets U, V of X such that $(O \setminus K) \cap X = U \cup V$ and $x \in cl_Y(U) \cap cl_Y(V)$. Since Y is Hausdorff, $K \in \sigma(Y)$. So, being $K \subseteq X$ and $x \in Y \setminus X$, it is clear that $O' = O \setminus K$ is a neighbourhood of x in Y such that $O' \cap X = U \cup V$. This proves that $Y \setminus X$ cuts X at x. □

Now we can give a characterization of the Hausdorff perfect extensions.

Proposition 4.5. *Let Y be a Hausdorff extension of X. The following are equivalent:*

(1) *Y is a perfect extension of X;*
(2) *$Y \backslash X$ does not c-cut X at any point of $Y \backslash X$;*
(3) *for every pair U, V of open sets of X such that $cl_X(U \cap V)$ is compact, $\langle U \cup V \rangle = \langle U \rangle \cup \langle V \rangle$;*
(4) *for every closed set F of X and every compact set $K \subseteq X$ such that F K-separates X in U and V, $cl_Y(F)$ K-separates Y in $\langle U \rangle$ and $\langle V \rangle$.*

PROOF: (1)⇒(2) It is obvious by 3.5.(2) and 4.4.

(2)⇒(3) Let $U, V \in \tau(X)$ such that $cl_X(U \cap V)$ is compact. Since Y is Hausdorff, by 4.4. $Y \backslash X$ does not cut X at any point of $Y \backslash X$. Moreover, $cl_X(U \cap V) \in \sigma(Y)$ and it results $cl_Y(U \cap V) \subseteq cl_X(U \cap V) \subseteq X$ and so, by 3.5.(10), we have that $\langle U \cup V \rangle = \langle U \rangle \cup \langle V \rangle$.

(3)⇒(4) Let $F \in \sigma(X)$ and $K \subseteq X$ be a compact set such that F K-separates X in $U, V \in \tau(X)$. Since Y is Hausdorff, $K \in \sigma(Y)$ while $U \cap V = \emptyset$ implies obviously that $cl_Y(U \cap V)$ is a compact set. So, by hypothesis (3), it results $\langle U \cup V \rangle = \langle U \rangle \cup \langle V \rangle$ and by the equivalence (4)⇔(11) of 3.5., it follows that $cl_Y(F)$ K-separates Y in $\langle U \rangle$ and $\langle V \rangle$.

(4)⇒(1) In fact, for every $F \in \sigma(X)$ such that F separates X in $U, V \in \tau(X)$, it suffices to consider the compact set $\emptyset$ to have that F $\emptyset$-separates X in U and V and so by the hypothesis (4), it follows that $cl_Y(F)$ $\emptyset$-separates Y in $\langle U \rangle, \langle V \rangle$ that is $cl_Y(F)$ separates Y in $\langle U \rangle$ and $\langle V \rangle$. Thus, by 3.5.(12), Y is a perfect extension of X. □

Remark. The equivalence (1)⇔(3) of 4.5. generalizes to any Hausdorff extension of a (Hausdorff) space a result given by Diamond in [D] only for Hausdorff compactifications of Tychonoff spaces.

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