Stability of the Electroweak Scale from Hamiltonian Chaos

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Abstract

We have shown over recent years that the dynamics of quantum fields is likely to slide outside equilibrium above the Fermi scale of electroweak interactions. In proximity to this scale, spacetime dimensionality flows with the probing energy and leads to the concept of *minimal fractal manifold* (MFM). The goal of this brief report is to combine the MFM conjecture with the transition to chaos in nearly-integrable Hamiltonian systems. In doing so, we find that the KAM theorem can conceivably explain the stability of the Fermi scale in the low TeV sector.

Key words: Fermi scale, Standard Model, minimal fractal manifold, KAM theorem, Hamiltonian chaos.

The key property underlying the *minimal fractal manifold* (MFM) is a continuous and vanishingly small deviation from four spacetime dimensions $\varepsilon = 4 - D << 1$ [1]. This deviation may be configured as a large multivariable set that runs with the dimensionless energy scale μ as in [2]

$$\varepsilon = f(\mu)\ ,\ \ \varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)\ ,\ \ N >> 1 \tag{1}$$

A remarkable attribute of smooth phase-space trajectories is that they can be represented as groups of continuous transformations. As a result, the flow equations

$$\frac{d\varepsilon_i(\mu)}{d\mu} = \beta_i(\varepsilon_1(\mu), \varepsilon_2(\mu), ..., \varepsilon_N(\mu), \mu, q) \tag{2}$$

are equivalent to the group of transformations

$$\varepsilon_i(\mu) = f_i(\varepsilon_1(\mu_0), \varepsilon_2(\mu_0), ..., \varepsilon_N(\mu_0), \mu_0, q) \tag{3}$$

in which q is the order parameter associated with the multifractal description of (1). Introducing the notation

$$\dot{\varepsilon} = \frac{d\varepsilon(\mu)}{d\mu} = \beta(\mu) \tag{4}$$

enables a one-to-one mapping of the MFM to the pair of canonically conjugate variables of classical mechanics viz. [3]

$$\boxed{(\varepsilon(\mu), \beta(\mu)) \Rightarrow (Q(t), P(t))} \tag{5}$$

Alternatively, using the link between ε and the $f(\alpha)$ spectrum [4], (5) may be cast in the equivalent representation

$$(\varphi(\alpha), \pi(\alpha)) \Rightarrow (Q(t), P(t)) \tag{6}$$

$$\varphi(\alpha) = \int_{q_1}^{q_2} f(\alpha)\, dq \ , \ \ \pi(\alpha) = f(\alpha) \tag{7}$$

Assuming that (5) behaves as a Hamiltonian system of classical oscillators, the equations describing its dynamics are given by

$$\dot{\varepsilon} = \frac{\partial H}{\partial \beta}, \ \ \dot{\beta} = -\frac{\partial H}{\partial \varepsilon} \tag{8}$$

with the corresponding action defined by the contour integral

$$I = \int_C P\, dQ \Rightarrow \int_C \beta\, d\varepsilon \tag{9}$$

Consider next the case of a pair of harmonic uncoupled oscillators. It represents a two-degree-of-freedom integrable system with Hamiltonian

$$H = \frac{1}{2}(\beta_1^2 + \beta_2^2 + \omega_1^2\varepsilon_1^2 + \omega_2^2\varepsilon_2^2) \tag{10}$$

Introducing the action-angle variables (I_k, θ_k), $k = 1, 2$

$$\beta_k = (2I_k\omega_k)^{1/2} \cos(\theta_k) \tag{11a}$$

$$\varepsilon_k = (2I_k/\omega_k)^{1/2} \sin(\theta_k) \tag{11b}$$

turns (10) into

$$H = \omega_1 I_1 + \omega_2 I_2 \tag{12}$$

where

$$\theta_k = \omega_k \mu + \theta_{k,0} \tag{13}$$

The equations of motion read

$$\dot{I}_k = -\frac{\partial H}{\partial \theta_k} = 0 \tag{14a}$$

$$\dot{\theta}_k = \frac{\partial H}{\partial I_k} = \omega_k \tag{14b}$$

A basic theorem of topology states that the phase-space orbits of this system lie on a two-dimensional invariant torus having constant radii I_k. A discrete version of (10)-(14) based on difference equations is the *linear circle map* [5]

$$I_{n+1} = I_n \tag{15a}$$

$$\theta_{n+1} = \theta_n + 2\pi\rho \tag{15b}$$

where $n = 1, 2, \ldots$ denotes the number of iterated periods of rotation in the longitudinal cross-section of the torus and the winding number ρ represents the frequency ratio ω_1/ω_2. The motion is either *stable-periodic* or *stable-quasiperiodic* depending on whether ρ is either a rational or an irrational number. In the former case, the intersection of the orbit with a plane transverse to the torus (the *Poincaré section*) consists of a finite set of points. By contrast, in the latter case, the intersection is a dense point set whose closure is a circle as $n \to \infty$. Liouville's theorem states that (15) is an area preserving map in the (I_n, θ_n) plane defined by the unitary Jacobian

$$\frac{\partial(I_{n+1}, \theta_{n+1})}{\partial(I_n, \theta_n)} = 1 \tag{16}$$

Analysis shows that the onset of chaos in a Hamiltonian system of the type (12) and (15) amounts to a breakdown of invariant tori due to perturbations. Specifically, the *KAM theorem* describes the progressive disintegration of invariant tori, a process referred to as Hamiltonian or *soft chaos* [6]. The cross-section portrait of soft chaos is smooth if the tori are intact but is gradually filled with irregular regions associated with *phase-locking*, which occurs when ρ is closely approximated by a rational number.

To highlight the main point of the KAM theorem, consider a perturbation of (15) presented in the form [5]

$$I_{n+1} = I_n + \eta f(I_{n+1}, \theta_n) \tag{17a}$$

$$\theta_{n+1} = \theta_n + 2\pi\rho(I_{n+1}) + \eta g(I_{n+1}, \theta_n) \tag{17b}$$

Here, the additional terms account for the perturbation effects and, under certain conditions, turn (15) into a *nearly-integrable* or *non-integrable* system. KAM theorem is based on evaluating the generic irrational winding number ρ by a sequence of rational approximations (u_j/v_j) $j = 1, 2, ..., n, ...$ as in

$$\rho \to \{u_1/\mathrm{v}_1, u_2/\mathrm{v}_2, ..., u_n/\mathrm{v}_n, ...\} \tag{18}$$

If, as $\eta \to 0$ and $n \to \infty$, the following condition is satisfied

$$\left|\rho - \frac{u_n}{\mathrm{v}_n}\right| > \frac{K(\eta)}{\mathrm{v}_n^{5/2}} \tag{19}$$

then the orbit with irrational number ρ survives as a stable quasi-periodic orbit under the perturbation. The factor $K(\eta)$ is a positive constant dependent on the perturbation strength η. KAM theorem shows that the eventual decay of tori occurs in successive phases: new pairs of stable-periodic orbits emerge as older stable periodic orbits vanish via period-doubling bifurcations. An important outcome of the theorem is that the most resilient torus to perturbations is the one characterized by *golden-mean* winding number

$$\phi = \frac{\sqrt{5}-1}{2} \tag{20}$$

The often-cited reason for this outcome is that rational approximations of (20) are the most slowly convergent of all algorithmically computable winding numbers [5].

In summary, considering the wide range of connections between the MFM and the Standard Model of particle physics, it is conceivable that (19) can justify the stability of the Fermi scale in the low TeV sector. To this end, it is instructive to numerically evaluate the winding number and the condition (19) under the following couple of hypotheses:

a) the primary frequency ω_1 of (12) reflects the fluctuations of the cosmic microwave background (CMB) induced by variations of matter density in the early Universe. This hypothesis is motivated by the maximal observable dimensional deviation $|\varepsilon_{\max}| = O(10^{-5})$ obtained by fitting the black-body CMB spectrum [7].

b) the secondary frequency ω_2 of (12) reflects the temporal variations of the Hubble parameter as the Universe expands and cools off.

Results of this analysis will be reported elsewhere.

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