A Derivation of Flat Galactic Rotation Curve and Baryonic Tully-Fisher Relation

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Abstract

It has been illustrated how one can obtain a gravitational potential as a solution of Einstein Field Equations, that can explain flat rotation curves without any Dark Matter hypotheses. The derived potential agrees with MOND-type theories in outcome, but provides proof from first principles of General relativity.

1 Introduction

The problem of the anomaly observed in spiral galaxy rotation curves is a longstanding one. While prevalent scientific opinion weighs towards a Dark matter hypothesis that explains the 'mass-gap' required to reconcile rotation curves with newtonian gravity, there hasn't been any direct observation of the postulated dark matter yet. On the other hand, there are theories that try to modify newton's gravity formulation to achieve the same reconciliation. But they are empirical in nature and doesn't explain how they emerge from a more fundamental understanding of the spacetime.

In this work, we've gone back to the gravity formulations of General Relativity(GR). It is not necessary that newtonian gravity be the only low-energy solution of Einstein Field Equations. In fact, we only derive Newton's gravity as a limit assuming a spherical symmetry. However, consideration of the intrinsic structure of most of the spiral galaxies, should lead us more towards a cylindrically symmetric solution of Field equations.

We'll see that with the assumptions of cylindrical symmetry, one can arrive at a gravitational potential that yields a geodesic which allows flat tails of rotation curves.

2 Modified gravity potential

As usual we would start by applying field equations on the vacuum just outside of the baryonic mass range of a spiral galaxy. That means the stress energy tensor would be assumed to be zero, $T_{\mu\nu} = 0$.

To obtain the potential we would assume a familiar structure of the metric g,

$$g = \begin{pmatrix} \phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

We will denote ϕ as Disc potential. Usually ϕ is assumed to be spherically symmetric i.e. exclusively a function of r, distance from the center in spherical coordinates. However, in this particular case, we would consider the problem in cylindrical coordinates (t, r, θ, z). Here z denotes the direction along the central axis of the galactic disc, and r is the 2-D distance of the star from the central axis.

In this set up, cylindrical symmetry would give us, $\partial_z \phi = 0$ and $\partial_\theta = 0$. Usual calculations of the Ricci tensor and scalar curvature, would also produce the Laplace equation,

$$\triangle^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi \partial_z^2 \phi = 0 \tag{2}$$

However, since $\partial_z = 0$, this would boil down to 2-D laplacian,

$$\triangle^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi = 0 \tag{3}$$

Now if we change the coordinates to cylindrical, we know the 2-D laplacian can be written as $\triangle^2 \phi = \partial_r^2 \phi + \frac{1}{r} \partial_r \phi + \frac{1}{r^2} \partial_\theta^2 \phi$.

But from cylindrical symmetry we know, $\partial_\theta \phi = 0$. That'll yield the differential equation to solve for ϕ,

$$\partial_r^2 \phi + \frac{1}{r} \partial_r \phi = 0 \tag{4}$$

Solving the equation we get

$$\partial_r \phi = \frac{\lambda}{r} \tag{5}$$

where λ is a constant. Solving further we get,

$$\phi = \lambda \ln \frac{r}{r_0} \tag{6}$$

where r_0 is a constant distance.

To derive the constants of motion λ, r_0, we would need boundary conditions based on smoothness of the geodesic. More specifically, if we assume that r_0 is the effective radius of the galactic baryonic mass, and within that radius the field continues to be spherically symmetric dominated by the halo, we would require the geodesic to have a well-defined 2nd derivative at r_0 from both side.

From the geodesic equation, we know that at a low energy limit radial acceleration to be

$$\frac{d^2r}{dt^2} = \partial_r \phi \tag{7}$$

At r_0, from inside $\phi = \frac{GM}{r}$ where M is the baryonic mass of the galaxy (inside r_0) and G is the gravitational constant. Hence the inside limit for radial acceleration would be $-\frac{GM}{r_0^2}$. On the other hand, same limit from the outside would be, $\frac{\lambda}{r_0}$. Hence continuity would dictate that $\lambda = -\frac{GM}{r_0}$. This would yield the final form of the disc potential to be,

$$\phi = -\frac{GM}{r_0} \ln \frac{r}{r_0} \tag{8}$$

Finally we would investigate rotation curve required by the disc potential. As usual, we would counterbalance the inward radial acceleration by outward centrifugal acceleration.

$$\frac{v^2}{r} = \frac{GM}{r_0} \frac{1}{r} \tag{9}$$

where v is the rotation velocity of a star on the disc at a distance r from the centre.

This clearly produces a flat rotation curve outside of r_0

$$v_0 = [\frac{GM}{r_0}]^{\frac{1}{2}} \tag{10}$$

Here v_0 denotes the terminal velocity at the edges of the galaxy. It would be possible to establish a relationship between terminal velocity and the baryonic mass in the galaxy if we assume ρ to be the average density of matter across the galactic disc. Then total baryonic mass could be expressed as $M = \pi \rho r_0^2$, which when plugged into the expression for v_0 yields,

$$M = \frac{v_0^4}{\pi \rho G^2} \tag{11}$$

which is exactly the empirical Tully-Fisher relationship if one assumes ρ to be consistent across galaxies.

3 Conclusion

Through a very simple proof it has been shown that there exists a metric structure which is differentiable everywhere (except galactic centre), that satisfy the field equations of general relativity. It has also been shown that such metric structure can produce newtonian gravity within a certain radius, and flat rotation curves outside of it. There are structural similarities between the results presented here and Modified Newtonian Dynamics (MOND). However, unlike MOND, the potential presented here doesn't attempt to alter Newton's laws of general dynamics. Rather it provides a GR based spacetime structure from which flat rotation curves emerge naturally. Additionally the emergence of the flat rotation curve is shown to be consistent with Tully-Fisher relationship which has been proven empirically.