Riemann Hypothesis

Shekhar Suman

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a pole at

$s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in*

the critical strip $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line

$Re(s) = 1/2$.

2 Proof

Analytic continuation of $\zeta(s)$ $is\ defined\ as$ $[see\ 1,\ p.14\ ,\ Eq.\ 2.1.4]$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2},\ 0 < Re(s) < 1 \qquad (1)$$

$Here\ [.]\ denotes\ the\ Greatest\ Integer\ Function.$

$Let, s = \sigma + it,\ 0 < \sigma < 1.$

$For\ 0 < \sigma < 1,\ [see\ 1, p.14],$

$$\frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx = \frac{1}{2}.$$

$So,\ using\ \frac{1}{2} = \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx\ in\ (1),$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + s\int_1^\infty \frac{1/2}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+1}{x^{s+1}}dx + \frac{1}{s-1}$$

$Let,\ \rho\ be\ a\ non\ trivial\ zero\ of\ the\ Riemann\ Zeta\ Function.$

$Let,\ \zeta(\rho) = 0\ ;\ 0 < Re(\rho) < 1,\ Im(\rho) \in (-\infty, -1/2) \cup (1/2, \infty)$

$$\zeta(\rho) = \rho\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx + \frac{1}{\rho-1} = 0$$

$$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \qquad (2)$$

$The\ functional\ equation\ of\ the\ Riemann\ Zeta\ function\ is$ $[see\ [1],\ p.22, 2.6.4],$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

$So,\ by\ functional\ equation\ if\ \rho\ is\ a\ zero\ of\ the\ Riemann\ Zeta\ function\ then$ $1-\rho\ is\ also\ a\ zero\ [see\ [1], p.30].$

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = (1-\rho)\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx - \frac{1}{\rho} = 0;$

$$\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \quad (3)$$

Equating left sides of equation (2) *and* (3),

$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx$

$$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\rho+1}} - \frac{1}{x^{2-\rho}}\right)dx = 0 \quad (4)$$

$Let, \rho = \sigma + it\ ; 0 < \sigma < 1,\ t \in (-\infty, -1/2) \cup (1/2, \infty)$

Since, $0 < \sigma < 1$ *so we discuss* 2 *cases*

$0 < \sigma \leq 1/2$ *and* $1/2 \leq \sigma < 1$.

Case $1 : 0 < \sigma \leq 1/2.$

Putting, $\rho = \sigma + it$ *in equation* (4),

$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\sigma+1+it}} - \frac{1}{x^{2-\sigma-it}}\right)dx = 0.$

$\int_1^\infty ([x]-x+1)\left(\frac{x^{-it}}{x^{\sigma+1}} - \frac{x^{it}}{x^{2-\sigma}}\right)dx = 0.$

$\int_1^\infty ([x]-x+1)\left(\frac{e^{-it(lnx)}}{x^{\sigma+1}} - \frac{e^{it(lnx)}}{x^{2-\sigma}}\right)dx = 0.$

$\int_1^\infty ([x]-x+1)\left(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}}\right)dx +$

$i\int_1^\infty ([x]-x+1)\left(\frac{-sin(tlnx)}{x^{\sigma+1}} - \frac{sin(t(lnx))}{x^{2-\sigma}}\right)dx = 0$

Equating Real part to zero,

$\int_1^\infty ([x]-x+1)\left(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}}\right)dx = 0$

$$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}\right)cos(tlnx)dx = 0 \quad (5)$$

$$Let, I = \int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\sigma+1}}-\frac{1}{x^{2-\sigma}}\right)cos(tlnx)dx = 0 \tag{6}$$

$Claim : For,\ 0 < \sigma \leq 1/2,\ \frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}} \geq 0.$

$\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}$

$= e^{-(\sigma+1)lnx} - e^{(\sigma-2)lnx}$

$= \frac{1-e^{(2\sigma-1)lnx}}{e^{(\sigma+1)lnx}}.$

$For,\ 0 < \sigma \leq 1/2,\ 2\sigma - 1 \leq 0.$

$for\ x \geq 1,\ lnx \geq 0.$

$(2\sigma - 1)lnx \leq 0.$

$e^{(2\sigma-1)lnx} \leq 1.$

$1 - e^{(2\sigma-1)lnx} \geq 0.$

$So,\ \frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}} = \frac{1-e^{(2\sigma-1)lnx}}{e^{(\sigma+1)lnx}}$

$$\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}} \geq 0. \tag{7}$$

which proves the claim.

Equation (6) *gives,*

$\int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx = 0.$

$0 = |\int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx|$

$0 \leq \int_1^\infty \mid ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx) \mid dx$

$$0 \leq \int_1^\infty \mid ([x]-x+1) \mid\mid (\frac{1}{x^{\sigma+1}}-\frac{1}{x^{2-\sigma}}) \mid\mid cos(tlnx) \mid dx \tag{8}$$

$0 \leq x - [x] < 1$

$0 < [x] - x + 1 \leq 1.$

Using the above inequality in (8),

$0 \leq \int_1^\infty \mid (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) \mid\mid cos(tlnx) \mid dx.$

Since, by (7)

$\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}} \geq 0.$

$0 \leq \int_1^\infty (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) \mid cos(tlnx) \mid dx.$

$Substitute,\ u = lnx \Rightarrow x = e^u$

$\Rightarrow dx = e^u du.$

$0 \leq \int_0^\infty (\frac{1}{e^{u(\sigma+1)}} - \frac{1}{e^{u(2-\sigma)}}) e^u \mid cos(tu) \mid du.$

$0 \leq \int_0^\infty (e^{-u(\sigma+1)} - e^{u(\sigma-2)}) e^u \mid cos(tu) \mid du.$

$0 \leq \int_0^\infty (e^{-\sigma u} - e^{-(1-\sigma)u}) \mid cos(tu) \mid du.$

$0 \leq \int_0^\infty e^{-\sigma u} \mid cos(tu) \mid du - \int_0^\infty e^{-(1-\sigma)u} \mid cos(tu) \mid du.$

$0 \leq \int_0^\infty e^{-\sigma u}\ \frac{|cos(tu)|}{costu}\ costu\ du - \int_0^\infty e^{-(1-\sigma)u}\ \frac{|cos(tu)|}{costu}\ costu\ du.$

$0 \leq \int_0^\infty e^{-\sigma u}\ sgn(costu)\ costu\ du - \int_0^\infty e^{-(1-\sigma)u}\ sgn(costu)\ costu\ du$

where sgn denotes the signum function

$0 \leq sgn(costu)[\int_0^\infty e^{-\sigma u}\ costu\ du - \int_0^\infty e^{-(1-\sigma)u}\ \ costu\ du]$

$\int e^{ax} cosbx\ dx = \frac{e^{ax}(acosbx+bsinbx)}{a^2+b^2}$

$0 \leq sgn(costu)[-\frac{e^{-\sigma u}[\sigma costu - tsintu]}{\sigma^2+t^2}\Bigg|_0^\infty + sgn(costu)\frac{e^{-(1-\sigma)u}[(1-\sigma)costu - tsintu]}{(1-\sigma)^2+t^2}\Bigg|_0^\infty]$

$0 \leq \frac{\sigma}{\sigma^2+t^2} - \frac{1-\sigma}{(1-\sigma)^2+t^2}$

$$\sigma(1-\sigma)^2 + \sigma t^2 - \sigma^2(1-\sigma) - t^2(1-\sigma) \geq 0$$

$$\sigma(1-\sigma)^2 - \sigma^2(1-\sigma) + \sigma t^2 - t^2(1-\sigma) \geq 0$$

$$\sigma(\sigma-1)(2\sigma-1) + t^2(2\sigma-1) \geq 0$$

$$(2\sigma-1)[\sigma^2 - \sigma + t^2] \geq 0$$

$$(2\sigma-1)[(\sigma-1/2)^2 + t^2 - 1/4] \geq 0 \tag{9}$$

Since, $t \in (-\infty, -1/2) \cup (1/2, \infty)$

$$t^2 - 1/4 > 0$$

$(\sigma - 1/2)^2 + t^2 - 1/4 > 0, (9)$ *implies*

$$(2\sigma - 1) \geq 0$$

$$\sigma \geq 1/2$$

Since by case (1), $0 < \sigma \leq 1/2$

$$1/2 \leq \sigma \leq 1/2$$

$$\sigma = 1/2$$

Now we proceed to Case 2

Case $2 : 1/2 \leq \sigma < 1$.

Let, $\rho = \sigma + it, 1/2 \leq \sigma < 1,\ t \in (-\infty, -1/2) \cup (1/2, \infty)$.

Let, $\zeta(\rho) = 0$

The functional equation of the Riemann Zeta function is [*see* [1], *p*.22, 2.6.4],

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s).$$

$So,\ by\ functional\ equation\ if\ \rho = \sigma + it\ is\ a\ zero\ of\ the\ Riemann\ Zeta\ function\ then$
$1-\rho = 1-\sigma-it \quad is\ also\ a\ zero\ and\ then$
$1-\bar{\rho} = 1-\sigma+it\ \ is\ also\ a\ zero\ [see\ [1], p.30].$

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = 0 \Rightarrow \zeta(1-\bar{\rho}) = 0$

$Since,\ \rho = \sigma + it.$

$\zeta(1-\bar{\rho}) = 0 \Rightarrow \zeta(1-\sigma+it) = 0.$

$1/2 \leq \sigma < 1 \Rightarrow 0 < 1-\sigma \leq 1/2.$

$\zeta(1-\sigma+it) = 0,\ 0 < 1-\sigma \leq 1/2.$

$Let,\ \sigma' = 1-\sigma.$

$\zeta(\sigma'+it) = 0,\ 0 < \sigma' \leq 1/2.$

$So,\ by\ case\ (1),$

$\sigma' = 1/2 \Rightarrow 1-\sigma = 1/2.$

$\sigma = 1/2.$

$So,\ by\ the\ above\ two\ cases\ we\ get\ that$

$\zeta(\rho) = 0\ ; 0 < Re(\rho) < 1;\ t \in (-\infty, -1/2) \cup (1/2, \infty)$

$\Rightarrow Re(\rho) = 1/2$

$, which\ proves\ the\ R.H.$

3 References

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