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The abc Conjecture is False: The End of The Mystery

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Abstract In this note, I give the proof that the abc conjecture is false because, in the case $c > rad(abc)$, for $0 < \epsilon < 1$ presenting a counterexample that implies a contradiction for c very large.

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

1 Introduction

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

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Conjecture 1 Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{3}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a+b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]:

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

After studying the *abc* conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that $c < rad^2(abc)$ or $c < rad^{1.63}$ is true, the *abc* conjecture does not hold when $0 < \epsilon < 1$. Then the *abc* conjecture as it was defined is false. In this note, I give a counterexample that the *abc* conjecture is not true, in the case $rad(abc) < c$ taking $\epsilon \in]0, 1[$ without assuming one of the two open questions : $c < rad^2(abc)$ and $c < rad^{1.63}(abc)$ that was proposed in 1996 by A. Nitaj [4].

The paper is organized as follows: in the second section, we give a counterexample that *abc* conjecture is false in the case $rad(abc) < c$, choosing $\epsilon \in]0, 1[$.

2 Proof the *abc* Conjecture is False

We note $R = rad(ac)$ in the case $c = a + 1$ (respectively $R = rad(abc)$ if $c = a + b$).

2.1 Case $c < R$:

As $c < R \Longrightarrow c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since we choose $K(\epsilon) \geq 1$ and the conjecture (1) is verified.

2.2 Case $c = R$

Case to reject as a, b, c (respectively a, c) are relatively prime.

2.3 Case $R < c$

I will consider the case $c = a + 1$. I give the following counterexample:

$$8^n = 2^{3n} = (7+1)^n = 7^n + 7^{n-1}n + \ldots + 7n + 1 \Longrightarrow$$
$$2^{3n} = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \tag{5}$$

We consider that $n = 2m$ is even so that the condition $R < c$ is verified. In this case, $c = 2^{3n} = 2^{6m} \Longrightarrow a = c - 1 = 2^{6m} - 1$. As $2^3 \equiv -1 (\bmod 9) \Longrightarrow 2^{6m} \equiv (-1)^{2m} \equiv 1 \equiv 0 (\bmod 9) \Longrightarrow 3^2|a$, so we can write $a = 3a_1$ with $a_1 \geq rad(a)$, it follows $c > a \geq 3rad(a) > 2rad(a) \Longrightarrow c > R$. We suppose that for $n = 2m$ large, the *abc* conjecture holds taking $\epsilon = \epsilon_0 \in]0, 1[$. Then $\exists\ K(\epsilon_0) > 0$ and:

$$2^{6m} < K(\epsilon_0) R^{1+\epsilon_0} \tag{6}$$

We obtain $rad(c) = rad(2^{6m}) = 2$. As $a = \mu_a.rad(a)$ and $3^2|a \Longrightarrow \mu_a \geq 3$, we can write $\mu_a = \mu_{\mu_a} rad(\mu_a)$ and $rad(a) = rad(\mu_a).\prod_{i=1}^{i=I_1} a_i$.

But:

$$a = \mu_a rad(a) = \mu_{\mu_a} rad(\mu_a) rad(a) = \mu_{\mu_a}.\prod_{i=1}^{i=I_1} a_i.rad^2(\mu_a) \Longrightarrow$$
$$rad^2(\mu_a) = \frac{a}{\mu_{\mu_a}.\prod_{i=1}^{i=I_1} a_i} \Longrightarrow rad(\mu_a) = \frac{\sqrt{a}}{\sqrt{\mu_{\mu_a}.\prod_{i=1}^{i=I_1} a_i}} < \sqrt{a} \Longrightarrow$$
$$rad(\mu_a) < 2^{3m}.\left(1 - \frac{1}{2^{6m}}\right)^{1/2} \Longrightarrow rad(a) < \prod_{i=1}^{i=I_1} a_i.2^{3m}.\left(1 - \frac{1}{2^{6m}}\right)^{1/2} \tag{7}$$

We re-write the equation (6) in detail:

$$2^{6m} < K(\epsilon_0) 2^{1+\epsilon_0} rad^{1+\epsilon_0}(a) < K(\epsilon_0) 2^{1+\epsilon_0} \prod_{i=1}^{i=I_1} a_i^{1+\epsilon_0}.2^{3m(1+\epsilon_0)}.\left(1 - \frac{1}{2^{6m}}\right)^{\frac{1+\epsilon_0}{2}} \tag{8}$$

That we can write as:

$$2^{3m(1-\epsilon_0)}.\left(1 - \frac{1}{2^{6m}}\right)^{-\frac{1+\epsilon_0}{2}} < K(\epsilon_0) 2^{1+\epsilon_0}.\prod_{i=1}^{i=I_1} a_i^{1+\epsilon_0} \tag{9}$$

The left member of the above inequality depends of m, but the right member does not depend explicitly of m. Now we consider that m becomes m' very large ($m' \longrightarrow +\infty$), then we obtain:

$$+\infty \leq K(\epsilon_0) 2^{1+\epsilon_0}.\prod_{i=1}^{i=I_1} a_i^{1+\epsilon_0} \tag{10}$$

where the prime numbers a_i obtained for the case $2^{6m} = a + 1$. Hence the contradiction, and the *abc* conjecture is false for the value $\epsilon_0 \in]0, 1[$.

However, We can announce the following theorems that are very easy to prove:

Theorem 1 ***(The truncated abc conjecture:)*** *Let a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^2(abc)$ is true, then for each $\epsilon \geq 1$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{11}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad ,\epsilon \geq 1$$

and:

Theorem 2 ***(The truncated abc conjecture:)*** *Let a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^{1.63}(abc)$ is true, then for each $\epsilon \geq 0.63$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{12}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad ,\epsilon \geq 0.63$$

Ouf! The end of the mystery!

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