The Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s,\ \Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a pole at

$s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in*

the critical strip $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line

$Re(s) = 1/2$.

2 Proof

Define a function $\zeta^*(s)$ as,

$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \ \Re(s) > 0.$

$Let, \ s = \sigma + it, \ \sigma > 0.$

$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\sigma+it}}, \ \sigma > 0.$

$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n e^{-it \ lnn}}{n^{\sigma}}, \ \sigma > 0.$

$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n cos(t \ lnn)}{n^{\sigma}} - i\sum_{n=1}^{\infty} \frac{(-1)^n sin(t \ lnn)}{n^{\sigma}}, \ \sigma > 0.$

$Since \ , for \ \sigma > 0, \ \frac{1}{n^{\sigma}} \ is \ monotonic$

$and \ \frac{1}{n^{\sigma}} \to 0 \ as \ n \to \infty \ . \ Also \ |\sum_{n=1}^{N}(-1)^n cos(t \ lnn) \ |is \ bounded.$

$So, \ by \ Dirichlet's \ test(see \ [6]), \sum_{n=1}^{\infty} \frac{(-1)^n cos(t \ lnn)}{n^{\sigma}} is \ convergent$

$for \ \sigma > 0$

$Similarly, \ by \ Dirichlet's \ test \ (see \ [6]), \sum_{n=1}^{\infty} \frac{(-1)^n sin(t \ lnn)}{n^{\sigma}} is \ convergent$

$for \ \sigma > 0$

$Thus, \ \zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n cos(t \ lnn)}{n^{\sigma}} - i\sum_{n=1}^{\infty} \frac{(-1)^n sin(t \ lnn)}{n^{\sigma}}, \sigma > 0 \ is \ convergent.$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \Re(s) > 1$

$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \ \Re(s) > 0.$

$\zeta(s) + \zeta^*(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \Re(s) > 0.$

$\zeta(s) + \zeta^*(s) = [1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...] + [-1 + \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} - ...]$

$\zeta(s) + \zeta^*(s) = \frac{2}{2^s} + \frac{2}{4^s} + ...$

$\zeta(s) + \zeta^*(s) = \frac{2}{2^s}(\frac{1}{1^s} + \frac{1}{2^s} + ...)$

$\zeta(s) + \zeta^*(s) = \frac{2}{2^s}\zeta(s)$

$\zeta(s) = \frac{\zeta^*(s)}{2^{1-s}-1}.$

$The\ Right\ hand\ side\ is\ convergent\ for\ \Re(s) > 0.$

$Thus,\ \zeta(s)\ has\ an\ analytic\ continuation\ to\ \Re(s) > 0\ defined\ as,$

$\zeta(s) = \frac{\zeta^*(s)}{2^{1-s}-1}, \Re(s) > 0$

$Let,\ us\ assume,\ \zeta(s) = 0,\ 0 < \Re(s) < 1$

$\Rightarrow \zeta^*(s) = 0,\ 0 < \Re(s) < 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\sigma+it}} = 0,\ 0 < \sigma < 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-it\ lnn}}{n^\sigma} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^n cos(t\ lnn)}{n^\sigma} - i\sum_{n=1}^{\infty} \frac{(-1)^n sin(t\ lnn)}{n^\sigma} = 0,\ 0 < \sigma < 1.$

$Equating\ real\ and\ imaginary\ parts\ to\ 0,$

$\sum_{n=1}^{\infty} \frac{(-1)^n cos(t\ lnn)}{n^\sigma} = 0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n sin(t\ lnn)}{n^\sigma} = 0 \tag{1}$$

$sin(t\ lnn) \leq 1, so,\ Equation\ (1)\ gives$

$0 = \sum_{n=1}^{\infty} \frac{(-1)^n sin(t\ lnn)}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma}$

$$0 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma} \tag{2}$$

$0 < \sigma < 1 \Rightarrow 1 < n^\sigma < n.$

$\frac{1}{n^\sigma} < 1.$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma} < \sum_{n=1}^{\infty} (-1)^n$ (3)

$\sum_{n=1}^{\infty} (-1)^n = 0, \ if \ n \ is \ even$

$\sum_{n=1}^{\infty} (-1)^n = -1, \ if \ n \ is \ odd$

$Equation \ (3) \ gives,$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma} < 0.$

$which \ contradicts \ (2), \ since \ by \ (2), \sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma} \geq 0$

$So, \ our \ assumption \ that \ \zeta(s) = 0, \ 0 < \Re(s) < 1 \ is \ wrong.$

$Thus, \ \zeta(s) \neq 0, \ 0 < \Re(s) < 1$

$Thus, \ we \ have \ disproved \ the \ Riemann \ Hypothesis.$

3 References

1. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

2. Kevin Broughan - Equivalents of the Riemann Hypothesis: Arithmetic Equivalents Cambridge University Press (2017).

3. Kevin Broughan - Equivalents of the Riemann Hypothesis: Analytic Equvalents Cambridge University Press (2017) .

4. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

4. H.M Edwards - Riemann's Zeta function- Academic Press (1974).

5. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

6. https://en.wikipedia.org/wiki/Dirichlet