The Riemann Hypothesis

Shekhar Suman

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

$The\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ pole\ at$

$s = 1. The\ trivial\ zeros\ of\ \zeta(s)\ are\ -2, -4, -6, ... Its\ non\ trivial\ zeros\ lie\ in$

$the\ critical\ strip\ 0 < Re(s) < 1.$

$The\ Riemann\ Hypothesis\ states\ that\ all\ the\ non\ trivial\ zeros\ lie\ on\ the\ critical\ line$

$Re(s) = 1/2.$

2 Proof

Dirichlet eta function is defined as,

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \ Re(s) > 0. \tag{1}$$

Dirichlet eta function is convergent for $Re(s) > 0$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ Re(s) > 1. \tag{2}$$

Subtract equation (1) *from equation* (2) :

$$\zeta(s) - \eta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

$$\zeta(s) - \eta(s) = [\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...] - [\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ...]$$

$$\zeta(s) - \eta(s) = \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + ...$$

$$\zeta(s) - \eta(s) = \frac{2}{2^s}[1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...]$$

$$\zeta(s) - \eta(s) = 2^{1-s}\zeta(s)$$

$$\zeta(s) = \frac{1}{1-2^{1-s}} \ \eta(s)$$

$\frac{1}{1-2^{1-s}} \ \eta(s)$ *is analytic on* $1 \neq Re(s) > 0$

Thus, Riemann zeta function has an analytic continuation to

$1 \neq Re(s) > 0$ *defined as,*

$$Thus, \ \zeta(s) = \frac{1}{1-2^{1-s}} \ \eta(s), 1 \neq Re(s) > 0 \tag{3}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \ Re(s) > 0$$

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} - \frac{1}{10^s} + \frac{1}{11^s} - \frac{1}{12^s} + \frac{1}{13^s} - \frac{1}{14^s} + \frac{1}{15^s} - \frac{1}{16^s} + \frac{1}{17^s} - \frac{1}{18^s} + \frac{1}{19^s} - \frac{1}{20^s} + \frac{1}{21^s} - \frac{1}{22^s} + \frac{1}{23^s} - \frac{1}{24^s} + \frac{1}{25^s} - \frac{1}{26^s} + \ldots \quad (4)$$

$$\frac{\eta(s)}{2^s} = \frac{1}{2^s} - \frac{1}{4^s} + \frac{1}{6^s} - \frac{1}{8^s} + \frac{1}{10^s} - \frac{1}{12^s} + \frac{1}{14^s} - \frac{1}{16^s} + \frac{1}{18^s} - \frac{1}{20^s} + \frac{1}{22^s} - \frac{1}{24^s} + \frac{1}{26^s} - \ldots \quad (5)$$

$Subtract\ equation$ (5) $from\ equation$ (4) :

$$\eta(s)(1 - \frac{1}{2^s}) = 1 - \frac{2}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} - \frac{2}{6^s} + \frac{1}{7^s} + \frac{1}{9^s} - \frac{2}{10^s} + \frac{1}{11^s} + \frac{1}{13^s} - \frac{2}{14^s} + \frac{1}{15^s} + \frac{1}{17^s} - \frac{2}{18^s} + \frac{1}{19^s} + \frac{1}{21^s} - \frac{2}{22^s} + \frac{1}{23^s} + \frac{1}{25^s} - \frac{2}{26^s} + \ldots \quad (6)$$

$$\eta(s)(1 - \frac{1}{2^s})\frac{1}{3^s} = \frac{1}{3^s} - \frac{2}{6^s} + \frac{1}{9^s} + \frac{1}{15^s} - \frac{2}{18^s} + \frac{1}{21^s} + \ldots \quad (7)$$

$Subtract\ equation$ (7) $from\ equation$ (6) :

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s}) = 1 - \frac{2}{2^s} + \frac{1}{5^s} + \frac{1}{7^s} - \frac{2}{10^s} + \frac{1}{11^s} + \frac{1}{13^s} - \frac{2}{14^s} + \frac{1}{17^s} + \frac{1}{19^s} - \frac{2}{22^s} + \frac{1}{23^s} + \frac{1}{25^s} - \frac{2}{26^s} + \ldots \quad (8)$$

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\frac{1}{5^s} = \frac{1}{5^s} - \frac{2}{10^s} + \frac{1}{25^s} + \ldots \quad (9)$$

$Subtract\ equation$ (9) $from\ equation$ (8) :

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s}) = 1 - \frac{2}{2^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} - \frac{2}{14^s} + \frac{1}{17^s} + \frac{1}{19^s} - \frac{2}{22^s} + \frac{1}{23^s} - \frac{2}{26^s} + \ldots \quad (10)$$

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})\frac{1}{7^s} = \frac{1}{7^s} - \frac{2}{14^s} + \ldots \quad (11)$$

$Subtract\ equation$ (11) $from\ equation$ (10) :

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})(1 - \frac{1}{7^s}) = 1 - \frac{2}{2^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} - \frac{2}{22^s} + \frac{1}{23^s} - \frac{2}{26^s} + \ldots \quad (12)$$

$$\eta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})(1 - \frac{1}{7^s})\frac{1}{11^s} = \frac{1}{11^s} - \frac{2}{22^s} + \ldots \quad (13)$$

$Subtract\ equation\ (13)\ from\ equation\ (12):$

$$\eta(s)(1-\tfrac{1}{2^s})(1-\tfrac{1}{3^s})(1-\tfrac{1}{5^s})(1-\tfrac{1}{7^s})(1-\tfrac{1}{11^s}) = 1-\tfrac{2}{2^s}+\tfrac{1}{13^s}+\tfrac{1}{17^s}+\tfrac{1}{19^s}+\tfrac{1}{23^s}-\tfrac{2}{26^s}+... \tag{14}$$

$$\eta(s)(1-\tfrac{1}{2^s})(1-\tfrac{1}{3^s})(1-\tfrac{1}{5^s})(1-\tfrac{1}{7^s})(1-\tfrac{1}{11^s})\tfrac{1}{13^s} = \tfrac{1}{13^s}-\tfrac{2}{26^s}+... \tag{15}$$

$Subtract\ equation\ (15)\ from\ equation\ (14):$

$$\eta(s)(1-\tfrac{1}{2^s})(1-\tfrac{1}{3^s})(1-\tfrac{1}{5^s})(1-\tfrac{1}{7^s})(1-\tfrac{1}{11^s})(1-\tfrac{1}{13^s}) = 1-\tfrac{2}{2^s}+\tfrac{1}{17^s}+\tfrac{1}{19^s}+\tfrac{1}{23^s}+...$$

$Continuing\ in\ this\ way\ we\ get,$

$$\eta(s)\prod_{p\in Primes}(1-\frac{1}{p^s}) = 1-\frac{2}{2^s}.$$

$$\eta(s) = \frac{1-2^{1-s}}{\prod_{p\in Primes}(1-\frac{1}{p^s})}$$

$From\ equation\ (3),$

$$\zeta(s) = \frac{1}{1-2^{1-s}}\ \eta(s), 1\neq Re(s)>0$$

$$\zeta(s) = \frac{1}{1-2^{1-s}}\frac{1-2^{1-s}}{\prod_{p\in Primes}(1-\frac{1}{p^s})},\ 1\neq Re(s)>0$$

$$\zeta(s) = \frac{1}{\prod_{p\in Primes}(1-\frac{1}{p^s})},\ 1\neq Re(s)>0$$

$$\zeta(s) = \frac{1}{1-2^{1-s}}\ \eta(s) = \prod_{p\in Primes}(1-\frac{1}{p^s})^{-1},\ 1\neq Re(s)>0$$

$$\frac{1}{1-2^{1-s}}\ \eta(s)\ is\ convergent\ for\ 1\neq Re(s)>0.$$

$$\Rightarrow \prod_{p\in Primes}(1-\frac{1}{p^s})^{-1}\ is\ convergent\ for\ 1\neq Re(s)>0$$

$Value\ of\ a\ convergent\ infinite\ product\ is\ 0\ if\ and\ only\ if\ atleast\ one\ of\ the\ factors\ is\ 0.$

$$For, 1 \neq Re(s) > 0,\ \zeta(s) = \prod_{p \in Primes} (1 - \frac{1}{p^s})^{-1} \neq 0.$$

$$\Rightarrow \zeta(s) \neq 0, 1 \neq Re(s) > 0$$

$Thus,\ \zeta(s)\ has\ no\ zeros\ in\ the\ critical\ strip$ $0 < Re(s) < 1.$

$Thus,\ we\ have\ disproved\ the\ Riemann\ Hypothesis.$

3 References

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